Parametrix approximations in finance

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Workshop on Computational Methods for Pricing and Hedging Exotic Options

Option pricing and fundamental solutions

A simple example: local volatility

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$\Gamma =$ **transition density of S / fundamental solution**

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- $L_{(T,\xi)}$ is a **heat operator (Black&Scholes)**
- $\Pi(t,S;T,\xi)$ is a **Gaussian function in** (t,S)

The backward parametrix

$\Pi(z;\zeta)$ is a Gaussian function <u>as a function of z</u> but

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$$\Gamma(z;\zeta) \;=\; P(z;\zeta) \;+\; \text{“correction term”}$$

$P(z;\zeta)$ is a **Gaussian function in ζ**

The backward parametrix, II

<u>**Second idea**</u>: look for Γ in the form

$$\Gamma(z;\zeta) = P(z;\zeta) + \int P(z;\cdot)\Phi(\cdot\,;\zeta)$$

here $z = (t, S)$ and $\zeta = (T, \xi)$

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- recursive formula

$$\Phi(z;\zeta) = LP(z;\zeta) + \int LP(z;\cdot)LP(\cdot\,;\zeta) + \dots$$

Option price expansion

C option price with payoff φ:

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- k-th term:
  B&S price with $\sigma = \sigma(t,S)$ and transaction cost LC_{k-1}

$$C_k(t,S) = \int LC_{k-1}(\zeta)P(t,S;\zeta)d\zeta$$

Global error estimates

$$|\Gamma(t,x) - P_{2n}(t,x)| \leq \delta \, \frac{t^n}{n!} \, \Gamma_{\text{heat}}(t,x)$$

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- asymptotically exact as $t \to 0$
- rate of convergence independent on dimension

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Call option with strike K:

$$C(t, S) \simeq C_{\mathrm{BS}}(t, S) + \frac{K(T-t)}{2}\left(a(T, K) - a(t, S)\right)P(t, S; T, K)$$

Numerical test: CEV model

$$\frac{dS_t}{S_t} = r dt + \sigma_0 S_t^{-\alpha} dW_t, \qquad \alpha \in [0, 1]$$

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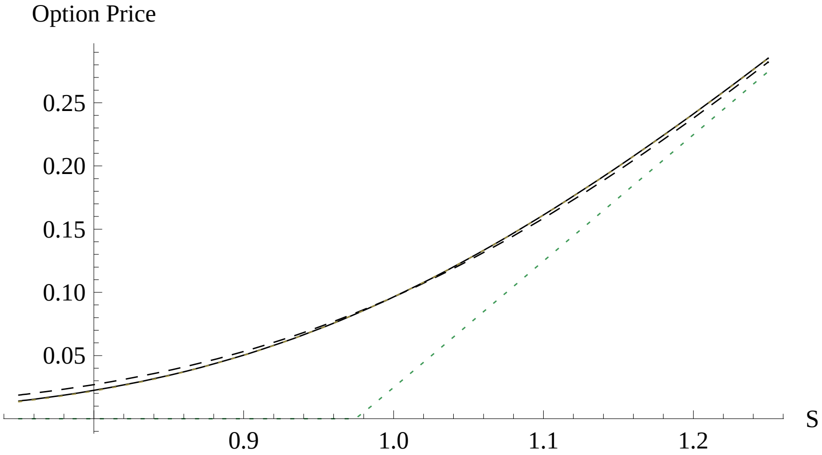
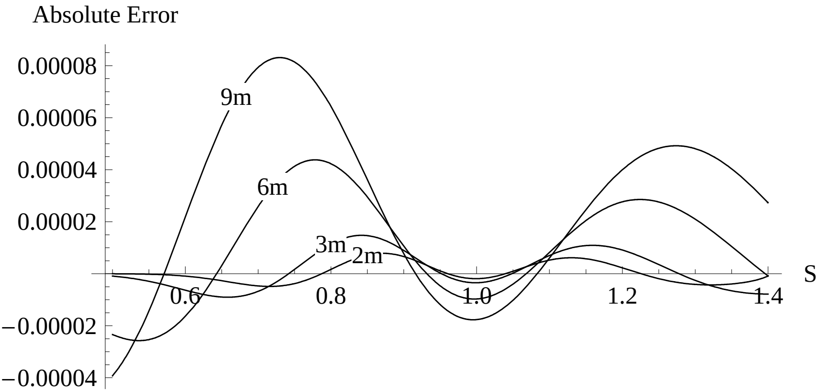
Parameters:

- $\alpha = \frac{1}{2}, \frac{3}{4}$
- $T = 2, 3, 6, 9, 12$ months
- $K = 1$
- $\sigma_0 = 30\%$, $r = 5\%$

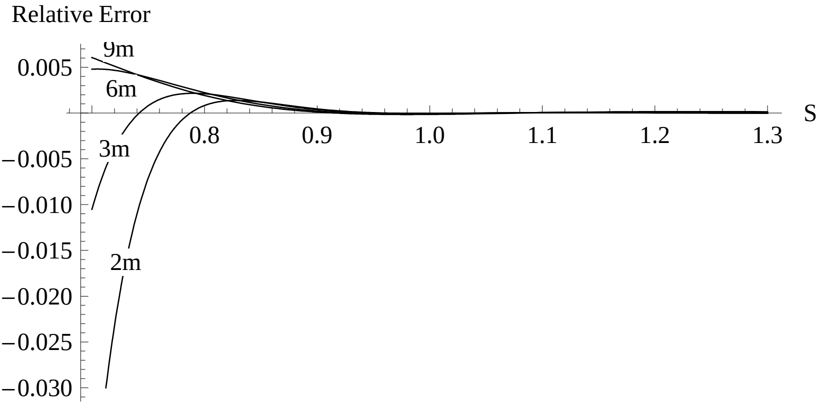
Call in CEV with $\alpha = \frac{3}{4}$, $T = 6\text{m}$, $K = 1$



Call in CEV: **absolute errors** of II parametrix



Call in CEV: **relative errors** of II parametrix



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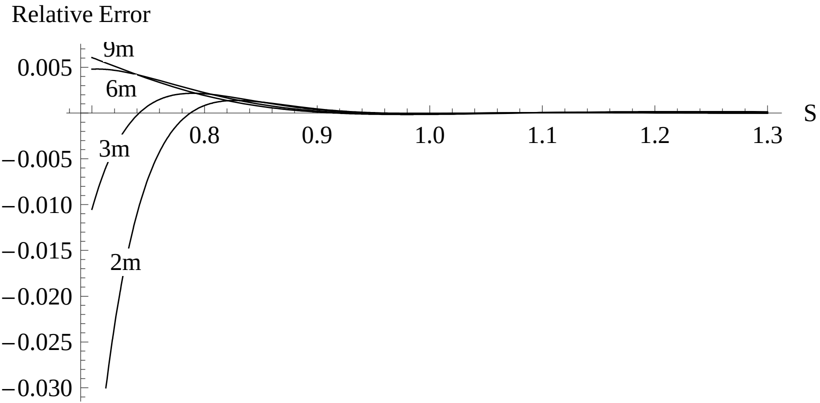

Option price:

$$\boxed{C(t,S) \simeq C_{\mathrm{BS}}(t,S) + \frac{K(T-t)}{2}(a(T,K) - a(t,S))P(t,S;T,K)}$$

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Z_t = log-price

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D_t = deviation from the normal trend

$$D_t = Z_t - \int_0^{+\infty} e^{-s} \, Z_{t-s} \, ds$$

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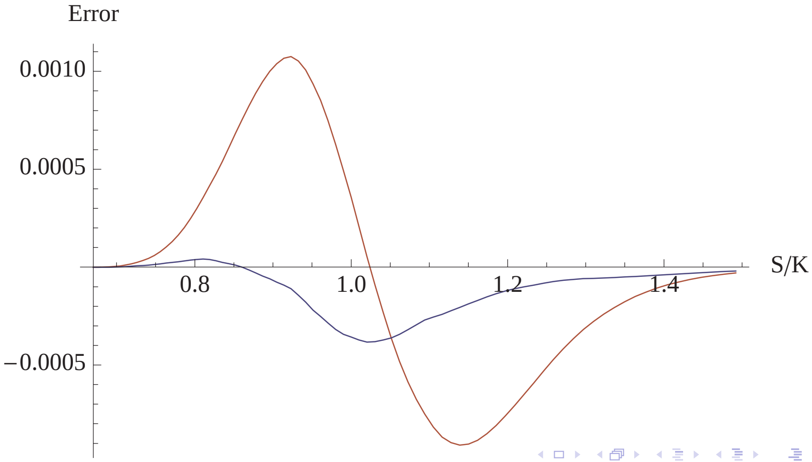
- Complete model

Path dependent volatility: call option

Monte Carlo vs **I (red) and II (blue) parametrix**:
absolute errors

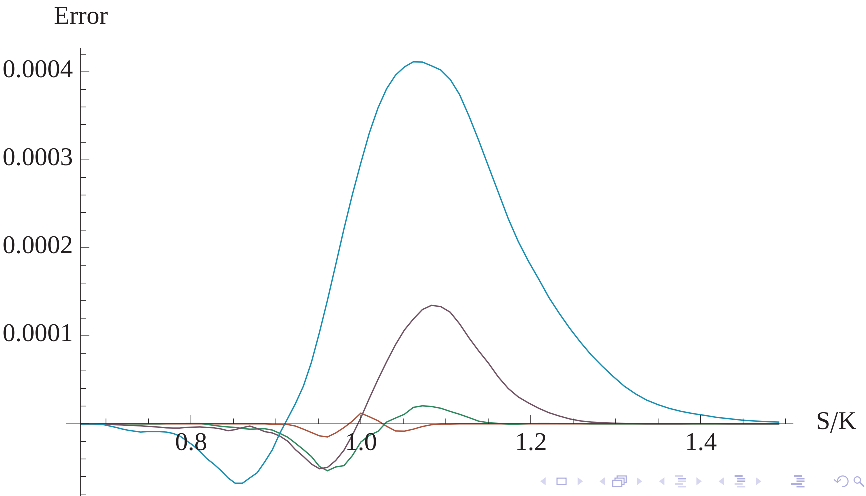
$\sigma(d) = 0.2 * \sqrt{1 + d^2}, \quad T = \frac{1}{4},$



Path dependent volatility: call option

Monte Carlo vs **II parametrix**: absolute errors

T = 1m (Red), 3m (Green), 9m (Purple), 12m (Cyan)



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- **explicit global** error estimates

- **Calibration:** analytic formulas for plain vanilla options (computationally cheap and simple as the Black&Scholes formula)

- **Pricing and hedging:** potentially useful in **high dimension** $\longrightarrow$ Monte Carlo
  further investigation and tests needed!

Call in CEV: Cox-Ross approximations