Methods for Pricing Strongly Path-Dependent Options in Libor Market Models without Simulation

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Workshop on Computational Methods for Pricing and Hedging Exotic Options

W_MI

July 9, 2008

Introduction

Option scope = "path dependent" $\Longrightarrow$ what can be done *without* simulation?

- Discrete observation — typical in market models.
- Linear/logical, e.g. TARN, Snowball (non-callable), Snowblade, LPI.

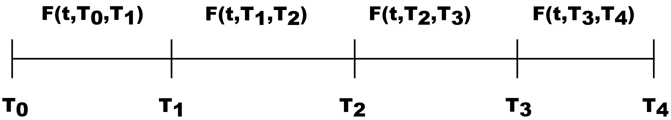
Model scope = discrete Market Models (dMM)

- Libor Forward Models (BGM type).
- Other LFM-like models, e.g. Swap MM [Jam97], Inflation MM [Mer05, Ken08].
- Volatility smiles.

Challenges

- dMMs describe the dynamics of a curve not a point (unlike stock price).
- Volatility smiles.

Reminder: LFM, LSM



- A Libor Forward Model (LFM) is based on a discrete set of spanning forward rates $F(t, T_{i-1}, T_i)$.
- Directly describe the dynamics of observable market quotes of tradeables.

In the T_i-forward measure we have for the LFM model:

$$dF(t, T_{i-1}, T_i) = \sigma_i(t)\ F(t, T_{i-1}, T_i)\ dW_i(t)$$

where $t \leq T_{i-1}$, with instantaneous correlation $dW_i(t)dW_j(t) = \rho_{i,j}dt$.

Similarly under the Libor Swap Model (LSM) the forward swap rate is a martingale under the $C_{\alpha,\beta}$-annuity measure:

$$dS(t)_{\alpha,\beta} = S(t)_{\alpha,\beta}\sigma_{\alpha,\beta}(t)dW_{\alpha,\beta}(t)$$

Example Payoffs

TARN Target Note, e.g.

- Target 20%; maturity 20Y; annual; observes 1Y Euribor
- Coupon is the 1Y forward, until 20% is reached, then the note redeems.

Snowball (non-callable) e.g.

- Maturity 5Y; quarterly; observes 3M Euribor
- Coupon is 3M Euribor + previous coupon.

Snowblade = Snowball + TARN.

LPI Limited Price Indexation, e.g.

- Maturity 20Y; annual; observes YoY RPI, collared [0%, 5%].
- Final coupon is sum of all observations.

Related Work

- [HW04] CDO pricing without simulation based on factor model. Either use Fourier Transforms for computing return distributions or bucketing method.
- [dIO06] latest of several papers pricing discretely observed products where the main assumption is independence of the returns per period. Payoffs: Asian; Guaranteed Return.
- [HJJ01] Predictor-corrector method allowing high accuracy for single steps up to 20 years in LFM.

Our contributions:

- Development of Fixed Income adaptation of [HW04].
- Pseudo-analytic TARN and Snowblade pricing in dMM without simulation.
- Inclusion of a volatility smile in Fixed Income context.

Basic Method . . . in words

1. Express dynamics of the observables in a common measure (e.g. of the last payoff).
2. Approximate the drifts by freezing-the-forwards with their $t = 0$ values.
3. Condition the joint discrete observation distributions on their most significant driving factors.
4. Calculate the conditional option price given that the conditional observation distributions are independent.
5. Integrate out the driving factors.

Basic Method: Comments

- In general, in Fixed Income "path-dependent" products are not path-dependent in the Equity sense of discrete observations of a single underlying.
  - Usually they look at different underlyings at different discrete times.
  - E.g. coupon(n) = coupon(n-1) + 6mLibor, the 6mLibor is a different product at each observation.
- The correlation between the observation distributions is the **terminal correlation** of the underlyings **not** their *instantaneous correlation*.
- Accuracy depends on two approximations:
  1. change of measure;
  2. number of driving factors.
- Speed depends on:
  1. number of driving factors;
  2. build-up method for payoff distribution.
  3. implementation details (not covered but see later)

Common Forward Measure in LFM

Standard machinery, e.g. Brigo & Mercurio 2006 Chapter 2, gives for LFM (Chapter 6), change measure by multiplying by ratio of old numeraire / new numeraire:

$$i > k, \quad dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dW^k$$

We approximate the drift by freezing-the-forwards at $t = 0$ value to obtain:

$$i > k, \quad dF_k(t) = -\sigma_k(t)F_k(0)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(0)}{1+\tau_jF_j(0)}dt + \sigma_k(t)F_k(t)dW^k$$

This means that we can take a single step to the maturity of each observation distribution, and the joint distribution is multivariate Lognormal (however recall [HJJ01]).

Common Swap Measure in LSM

Proposition

$S(t)_{\alpha,\beta}$**-forward-swap-rate dynamics in the** $S_{\gamma,\beta}$**-forward-swap-rate measure,** $\gamma > \alpha$**.** *The dynamics of the forward swap rate* $S(t)_{\alpha,\beta}$ *under the numeraire* $C_{\gamma,\beta}$*,* $\gamma > \alpha$ *is given by:*

$$
\begin{aligned}
dS(t)_{\alpha,\beta} &= m^{\gamma,\beta} S(t)_{\alpha,\beta} dt + \sigma_{\alpha,\beta}(t) S(t)_{\alpha,\beta} dW, \\
m^{\gamma,\beta} &= \sum_{h,k=\alpha+1}^{\beta} \mu_h \mu_k \tau_h \tau_k FP_h FP_k \rho_{h,k} \sigma_h \sigma_k F_h F_k, \\
\mu_h &= \frac{FP_{\alpha,\beta} \sum_{i=\alpha+1}^{h-1} \tau_i FP_{\alpha,i} + \sum_{i=h}^{\beta} \tau_i FP_{\alpha,i}}{(1 - FP_{\alpha,\beta}) \left(\sum_{i=\alpha+1}^{\beta} \tau_i \; FP_{\alpha,i}\right)^2} \\
\mu_k &= \frac{\left(\sum_{i=k}^{\beta} \tau_i FP_{\alpha,i}\right)\left(\sum_{i=\gamma+1}^{\beta} \tau_i FP_{\alpha,i}\right) - \left(\sum_{i=\max(k,\gamma+1)}^{\beta} FP_{\alpha,i}\right)\left(\sum_{i=\alpha+1}^{\beta} \tau_i FP_{\alpha,i}\right)}{\sum_{i=\gamma+1}^{\beta} \tau_i \; FP_{\alpha,i}}
\end{aligned}
$$

where W is a $Q^{\gamma\beta}$ *standard Brownian motion.*

Where

$$
\begin{aligned}
FP_{\alpha,i} &:= \frac{P(t,T_i)}{P(t,T_\alpha)} = \prod_{j=\alpha+1}^{i} FP_j \\
FP_j &:= \frac{1}{1+\tau_j F_j}
\end{aligned}
$$

Terminal Correlation

This is the relevant correlation for pricing, and potentially totally different from the instantaneous correlation (as emphasized by Rebonato). Consider two lognormal processes G_i, G_j in a common measure:

$$dG_k = G_k\mu_k + G_k\sigma_k(t)dW_k \qquad k = i, j$$

for pricing we want $\rho_{\text{Terminal}} := \rho(G_i(T_i), G_j(T_j))$. Then their distributions at times t, s are:

$$\begin{aligned} \mathcal{X}(t) &= e^{W(t)\int_0^t \sigma + \mu t} \\ \mathcal{Y}(s) &= e^{\rho W(s)\int_0^s \nu + Z(t)\sqrt{1-\rho^2}\int_0^s \nu + \eta t} \end{aligned}$$

where W, Z are Standard Normals; and ρ is the instantaneous correlation. Hence, elementary considerations and Ito's isometry lead to:

$$\rho_{\text{Terminal}}(\mathcal{X}(t), \mathcal{Y}(s)) = \frac{e^{\rho\int_0^{\min(s,t)} \sigma\nu} - 1}{\sqrt{(e^{\int_0^t \sigma^2} - 1)}\sqrt{(e^{\int_0^s \nu^2} - 1)}}$$

Pricing with Independent Observations: Asian & GRR

If the observation distributions are independent **and** the payoff underlying is essentially linear . . .

- Asian: payoff underlying $P = \sum_{i=1}^{n} X_{T_i}$; payoff $\max(P - K, 0)$.
- GRR: payoff underlying $P = \sum_{i=1}^{n-1} q_i \frac{X_{T_n}}{X_{T_i}}$; payoff $P + \max(K - P, 0)$

. . . then many techniques are available; essentially these are variations on convolution.

- Fourier Transform: convolution is multiplication in Fourier-space.
  - Numerically fastest when number of points representing distributions is a power of two.
- Laplace Transform: ditto
- Hull & White bucketing: avoids transform/inverse-transform cost; potentially slower; allows for non-linear transformations (without going into distribution theory).

Pricing with Independent Observations: TARN

- Targets introduce another layer of complexity because there is now logic at each coupon (a trigger), that is **not only** a convolution.
  - When you reach the trigger level redemption occurs.
- To calculate the payoff of a coupon it is necessary and sufficient to know the state trigger underlying and the coupon underlying.

observation	coupon underlying	state trigger underlying
a	a	0
b	b	a
c	c	$a+b$
d	d	$a+b+c$

- Coupon underlying and state trigger underlying are independent.
- a, b, c, and d are also independent.

Pricing with Independent Observations: TARN

- We can represent the previous argument as follows, let:

$$\begin{aligned}\hat{X}_{n-1} &= \sum_{i=1}^{i=n-1} X_i \sim X_1 \otimes \ldots \otimes X_{n-1} \\ Y &\sim X_n \otimes (I_K \times \hat{X}_{n-1})\end{aligned}$$

  Then:

$$P_n = \begin{cases} K - Y + 100, & Y \geq K \\ X_n, & \text{otherwise} \end{cases}$$

  where

$$I_K(u) = \begin{cases} 1 & u < K \\ 0 & \text{otherwise} \end{cases}$$

- This depends on the independence of $\hat{X}_{n-1}$ and X_n.
- We can calculate this by adapting the bucketing algorithm of [HW04] (with stochastic recovery rates), or by using transforms/inverse-transforms plus arithmetic operations.

Pricing with Independent Observations: Snowball→Snowblade

- From the proceeding discussion a (non-callable) Snowball is just a repeated Asian underlying:

$$P_n = \hat{X}_n \sim X_1 \otimes \ldots \otimes X_n$$

- Direct to calculate in any of the standard methods for independent observations.

…so how about a Snowblade, i.e. a Snowball with a target return? Consider:

observation	coupon underlying	state trigger underlying
a	a	0
b	$a+b$	a
c	$a+b+c$	$2a+b$
d	$a+b+c+d$	$3a+2b+c$

- Coupon underlying and state trigger underlying are no longer independent.
- Hence we require a two-dimensional state rather than the 1-dimensional one we used for the TARN.

Pricing with Independent Observations: Snowblade

- Requires the joint distribution of $\sum_1^n X_i$ and $\sum_1^{n-1}(n-i)X_i$
- We can easily create this recursively. Let $J(a,b)$ stand for the joint distribution of a and b.

  Let $\hat{\hat{X}}_{n-1} = \sum_1^{n-1}(n-i)X_i$, and recall $\hat{X}_{n-1} = \sum_1^{n-1} X_i$. Define:

  $$J_n := J(\hat{X}_n, \hat{\hat{X}}_{n-1})$$

  Note that: $\hat{\hat{X}}_{n-1} = \hat{\hat{X}}_{n-2} + \hat{X}_{n-1}$ and we have $J(\hat{\hat{X}}_{n-2}, \hat{X}_{n-1})$ hence

  $$J_n(x,y) = J_{n-1}(x, y-x) \otimes J(X_n, \delta(0))$$

  because the Jacobian is unity, X_n is independent of $\hat{X}_{n-1}$ as before, and X_n is independent of $\hat{\hat{X}}_{n-1}$.
- We can now apply the same steps as for the TARN.

Basic Method ... in equations

The price of a path dependent instrument $\mathcal{I}$ of the types described is:

- For N coupons and M factors we have:

$$\begin{aligned}
\mathcal{I} &= \sum_{i=1}^{i=N} \mathbb{E}^{\mathbb{Q}_i}\left[df(T_i)\ {}_iP_i\right] \\
&= \sum_{i=1}^{i=N} \mathbb{E}^{\mathbb{Q}_N}\left[df(T_N)\ {}_NP_i\right] \\
&= \sum_{i=1}^{i=N} \int_{e_1} \dots \int_{e_M} df(T_N)\ \mathbb{E}^{\mathbb{Q}_N|e_1\dots e_M}\left[{}_NP_i|e_m, m=1,\dots M\right]\ de_1 \dots de_M \\
&= \int_{e_1} \dots \int_{e_M} \sum_{i=1}^{i=N} df(T_N)\ \mathbb{E}^{\mathbb{Q}_N|e_1\dots e_M}\left[{}_NP_i|e_m, m=1,\dots M\right]\ de_1 \dots de_M
\end{aligned}$$

  in the case of Forwards, where $\mathbb{Q}_i$ is the T_i-Forward measure of the i^{th}payment, $df()$ is the discount factor, and ${}_jP_i$ is the payoff P_i with the T_j-Forward numeraire.

- For every value of the factors the individual observation distributions (of the underlying) are independent.

Smile Extension Example — Mixture Model: TARN

- Generic mixture distribution M:

$$
\begin{aligned}
M &= \sum_{j=1}^{j=m} \lambda_j G_j, \quad \text{s.t.} \\
\sum_{j=1}^{j=m} \lambda_j &= 1, \quad \lambda_j \geq 0 \quad \forall j
\end{aligned}
$$

  It is possible to create a process corresponding exactly to any given mixture.

- TARN

obs	coupon underlying	state trigger	state trigger distribution
a	a	0	none
b	b	a	$M_{T_1} = \sum_j \lambda_j X_{1,j}$
c	c	$a+b$	$M_{T_1} + M_{T_2} = \sum_{i=1}^{i=2} \sum_j \lambda_j X_{i,j}$
d	d	$a+b+c$	$M_{T_1} + M_{T_2} + M_{T_3} = \sum_{i=1}^{i=3} \sum_j \lambda_j X_i$

- Relies on independence of M_{T_i}, the conditional mixture distributions.
- Direct extension *assuming* that the mixture components have a common correlation structure (usual assumption for mixtures).

Smile Extensions

- Mixture distributions have direct analytic extension.
  - N.B. Mixture distributions are not generally positively regarded for path dependent options on a *single underlying*.
  - However in Fixed Income, path dependent options do not rely on the path of a single underlying.
- Uncertain parameter models with splitting scenario structure are generally unsuitable for path dependent options because of scenario separation, i.e. only one possible past per future.

  $\Longrightarrow$ If make scenarios independent at each maturity, then cost is exponential number of scenarios . . . intractable.
- If the conditional analytic distributions or conditional Fourier Transforms of the terminal distributions are available, then any stochastic volatility model can be used.
  - N.B. The number of integrating factors must increase to take account of the volatility drivers.

Discussion and Conclusions

So far:

- Pseudo-analytic method for pricing strongly path-dependent options in discrete Market Models (e.g. LFM, LSM).
- In general applicable when the joint distribution of the coupon underlying and the state underlying are available.
- Examples of TARN, Snowball (non-callable), and Snowblade.
- Extension to include smiles.

Next steps:

- Numerical tests to identify best drift approximations and accuracy.
- Speed? Method dependent on mask + 1d/2d convolution + arithmetic operations . . . ideal for GPU implementation.

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