Conditional sampling for jump processes with Lévy copulas

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Conditional sampling for Lévy copulas

- Series representation for a $d-$dimensional positive Lévy process with finite variation (Tankov (2003)):

$$X_t^k = \sum_{i=1}^{\infty} U_k^{-1}(\Gamma_i^k)\mathbf{1}_{\{V_i \in [0,t]\}}, \ k = 1, \dots, d, \ t \in [0,1] \qquad (1)$$

  where $\{\Gamma_i^1\}, \dots, \{\Gamma_i^d\}$ are independent r.s. $\{\Gamma_i^1\}$ jump times of a SPP. F the Lévy copula. $(\Gamma_i^2, \dots, \Gamma_i^d)$ conditionally on Γ_i^1 has distribution $\partial_{x_1} F(x_1, \dots, x_d)|_{x_1=\Gamma_i^1}$. $\{V_i\}$ i.i.d. $\mathcal{U}_{[0,1]}$.
- To sample from the joint distribution $\partial_{x_1} F(x_1, \dots, x_d)|_{x_1=\Gamma_i^1}$, use the general conditional sampling approach.
- When conditional sampling or numerical inversion of U_ks is too computationally expensive, use acceptance-rejection methods.

Overview

Part I: Simulating dependent jump processes with Lévy copulas

- Conditional sampling with inverse Lévy method.
- Acceptance-rejection methods.

Part II: Archimedean Lévy copulas

- General results.
- Some parametric copulas.

Part III: Examples and applications

- Gamma processes with Gumbel Lévy copula.
- A stochastic default intensity model with dependent jumps.

Lévy processes

Definition

A *positive pure jump* Lévy process $(X_t)_{(t>0)}$ has stationary and independent positive jumps and is of finite variation, i.e. $\lim_{\Delta t_k \to 0} \sum_{t_k \le t} |X_{t_{k+1}} - X_{t_k}| < \infty$ where $0 = t_1 < t_2 < \cdots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. The distribution of X_t for any time $t > 0$ is infinitely divisible and its characteristic function satisfies the Lévy-Khintchine formula:

$$\mathbb{E}\left[e^{i\langle z, X_t\rangle}\right] = e^{-t\Psi(z)}, \; z \in \mathbb{R}^d \quad (2)$$

$$\Psi(z) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle z, x\rangle}\right) \nu(dx) \quad (3)$$

where ν is a measure on $\mathbb{R}^d \backslash \{0\}$ such that $\int (1 \wedge |x|)\nu(dx) < \infty$.

Lévy tail mass integrals and copulas

Definition

The tail mass of the Lévy measure is $U(x_1, \ldots, x_d) = \nu([x_1, \infty) \times \cdots \times [x_d, \infty))$ for $(x_1, \ldots, x_d) \in \mathbb{R}^d \backslash \{0\}$ and such that $U(x_1, \ldots, x_d) = 0$ if $x_j = \infty$ for some $j \in 1, \ldots, d$ and U is finite everywhere except at zero, $U(0, \ldots, 0) = \infty$. The marginal Lévy measures have as their tail masses the margins $U_k(x_k) = U(0, \ldots, 0, x_k, 0, \ldots, 0)$.

Definition

A $d-$dimensional Lévy copula is a $d-$increasing grounded function $F : [0, \infty]^d \rightarrow [0, \infty]$ with margins $F_k, k = 1 \ldots d$, which satisfy $F_k(u) = u, \forall u \in [0, \infty]$.

A multivariate Lévy process = Lévy copula + marginal tail masses

Theorem (Tankov (2003))

Any $d-$dimensional tail mass U can be constructed as a Lévy copula taking the margins of U as arguments. Conversely, if F is a Lévy copula and $U_1, \ldots, U_d$ are one-dimensional tail masses, then $U(x_1, \ldots, x_d) = F(U_1(x_1), \ldots, U_d(x_d))$ defines a $d-$dimensional tail mass.

Series representation with conditional sampling

A $d-$dimensional positive Lévy process with finite variation admits the series representation:

$$X_t^k = \sum_{i=1}^{\infty} U_k^{-1}(\Gamma_i^k)\mathbf{1}_{\{V_i\in[0,t]\}},\ k=1,\dots,d,\ t\in[0,1] \tag{4}$$

where $\{\Gamma_i^1\}$ are jump times of a SPP and for each $j=2,\dots,d$:

$$\Gamma_k^j = f_j^{-1}\left(u_k^j \times \partial_{x_1,\dots,x_{j-1}} F(x_1,\dots,x_{j-1},\infty,\dots,\infty)\big|_{x_1=\Gamma_k^1,\dots,x_{j-1}=\Gamma_k^{j-1}}\right) \tag{5}$$

The function f_j is given by:

$$f_j(y) = \partial_{x_1,\dots,x_{j-1}} F(x_1,\dots,x_{j-1},y,\infty,\dots,\infty)\big|_{x_1=\Gamma_k^1,\dots,x_{j-1}=\Gamma_k^{j-1}}$$

where u_k^js are i.i.d. $\mathcal{U}_{[0,1]}$ $\forall j=2,\dots,d$ and $\forall k\geq 1$.

Series representation with rejection method

Given a Lévy measure ν' such that $\frac{\nu}{\nu'} \leq 1$, another series representation is:

$$X_t^k = \sum_{i=1}^{\infty} J_i^k \mathbf{1}_{\{\frac{\nu}{\nu'}(J_i) \geq W_i\}} \mathbf{1}_{\{V_i \in [0,t]\}}, \; k = 1, \ldots, d, \; t \in [0,1] \qquad (6)$$

where $J_i = (J_i^1, \ldots, J_i^d)$ are jumps of the processes with measure ν' and $\{W_i\}$ are i.i.d. $\mathcal{U}_{[0,1]}$.
Lévy densities can be obtained by differentiation:

$$\nu(x_1, \ldots, x_d) = \frac{\partial^d F(y_1, \ldots, y_d)}{\partial y_1 \ldots \partial y_d}\bigg|_{y_1 = U_1(x_1), \ldots, y_d = U_d(x_d)} \nu_1(x_1) \ldots \nu_d(x_d) \qquad (7)$$

Archimedean Lévy copulas

Definition

Archimedean Lévy copulas are Lévy copulas satisfying:

$$F(x_1, \ldots, x_d) = \phi^{-1}\left(\phi(x_1) + \cdots + \phi(x_d)\right) \tag{8}$$

where ϕ is strictly decreasing from $[0, \infty]$ to $[0, \infty]$ with $\phi(0) = \infty$, $\phi(\infty) = 0$ and such that its inverse ϕ^{-1} has derivatives up to the order d on $(0, \infty)$ satisfying $(-1)^k \frac{d^k}{dt^k}\phi^{-1}(t) > 0$ for all $k = 1, \ldots, d$.

Conditional sampling

Result

Let F be a $d-$dimensional Archimedean Lévy copula with generator ϕ. Then its marginal tail integrals can be simulated as follows:

- $\{\Gamma_j^1\}_{(j>0)}$ *are jump times of a SPP.*
- *For each* $j > 0$, *simulate* $(d-1)$ *i.i.d.* $\mathcal{U}_{[0,1]}$: $u_j^2, \ldots, u_j^d$
- *For each* $j > 0$, Γ_j^k *for* $k = 2, \ldots, d$ *are given by:*

$$\Gamma_j^k = \phi^{-1}\left(g_{k-1}^{-1}\left(u_j^k * g_{k-1}\left(\sum_{l=1}^{k-1}\phi(\Gamma_j^l)\right)\right) - \left(\sum_{l=1}^{k-1}\phi(\Gamma_j^l)\right)\right) \quad (9)$$

where $g_{k-1}(x) := (\phi^{-1})^{(k-1)}(x)$

Clayton-Lévy copula

- $F(x_1,\dots,x_d) = \left(\sum_{j=1}^{d} x_j^{-\gamma}\right)^{-\frac{1}{\gamma}}$, $\phi(u) = u^{-\gamma}$ and $\phi^{-1}(t) = t^{-\frac{1}{\gamma}}$
- The derivatives of ϕ^{-1} are given by:

$$g_k(x) = \frac{(-1)^k \prod_{j=1}^{k-1}(1+j\gamma)}{\gamma^k x^{k+\frac{1}{\gamma}}} \tag{10}$$

$$g_k^{-1}(x) = \frac{(-1)^{\frac{k\gamma}{1+k\gamma}} \prod_{j=1}^{k-1}(1+j\gamma)^{\frac{\gamma}{1+k\gamma}}}{\gamma^{\frac{k\gamma}{1+k\gamma}} x^{\frac{\gamma}{1+k\gamma}}} \tag{11}$$

- Given Γ_k^1, $(\Gamma_k^2,\dots,\Gamma_k^d)$ are sampled according to:

$$\Gamma_k^j = \left[\left((\Gamma_k^1)^{-\gamma} + \cdots + (\Gamma_k^{j-1})^{-\gamma}\right)\left(u_k^{\frac{\gamma}{\gamma(1-k)-1}} - 1\right)\right]^{-\frac{1}{\gamma}} \tag{12}$$

Gumbel-Lévy copula

- $F(x_1, \ldots, x_d) = \exp\left[\left(\sum_{j=1}^{d} (\log(x_j + 1))^{-\gamma}\right)^{-\frac{1}{\gamma}}\right] - 1,$

  $\phi(u) = (\log(u+1))^{-\gamma}$ and $\phi^{-1}(t) = e^{t^{-1/\gamma}} - 1$
- The derivatives of ϕ^{-1} are given by:

$$g_k(x) = e^{w(x)} \sum \frac{k!}{1!^{m_1} \ldots k!^{m_k} m_1! \ldots m_k!} \prod_{j:m_j \neq 0} \left(w^{(j)}(x)\right)^{m_j} \quad (13)$$

$$w(x) = x^{-1/\gamma} \quad (14)$$

$$w^{(l)}(x) = (-1)w^{(l-1)}(x)\left(\frac{1}{\gamma} + l - 1\right)x^{-1} \quad (15)$$

where the summation in equation (13) is over all $k-$tuples $(m_1, \ldots, m_k)$ of non-negative integers satisfying the constraint $m_1 + 2m_2 + 3m_3 + \cdots + km_k = k$. The coefficients represent the number of partitions of a size-k set into m_j parts of size j, for $j = 1, \ldots, k$.

Frank-Lévy copula

- $F(x_1,\ldots,x_d) = -\frac{1}{\gamma}\log\left[1-\prod_{j=1}^{d}(1-e^{-\gamma x_j})\right]$,
  $\phi(u) = -\log(1-e^{-\gamma u})$ and $\phi^{-1}(t) = -\frac{1}{\gamma}\log(1-e^{-t})$
- The derivatives of ϕ^{-1} are given by:

$$g_k(x) = \frac{(-1)^k}{\gamma}\sum_{j=1}^{k}\frac{(j-1)!}{(1-e^{-x})^j}e^{-jx}S(k,j) \quad (16)$$

where $S(k,j)$ is a Stirling number of the second kind:

$$S(k,j) = \frac{1}{j!}\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}l^k$$

Tankov-Lévy copula

- $F(x_1,\dots,x_d) = \frac{1}{\gamma}\log\left[1+\left(\sum_{i=1}^{d}\frac{e^{-\gamma x_i}}{1-e^{-\gamma x_i}}\right)^{-1}\right]$, $\phi(u) = \frac{e^{-\gamma u}}{1-e^{-\gamma u}}$ and $\phi^{-1}(t) = \frac{1}{\gamma}\log\frac{1+t}{t}$
- The derivatives of ϕ^{-1} are given by:

$$g_k(x) == (-1)^k\frac{1}{\gamma}\frac{(k-1)!}{x^k}(1-w^k) \text{ where } w = \frac{x}{1+x} \quad (17)$$

Gumbel-Lévy copula for gamma processes

- $(Z^1, \dots, Z^d)$ gamma processes with $\nu_i(x) = \theta^i x^{-1} e^{-\alpha^i x} \mathbf{1}_{\{x>0\}}$. $U_i(x) = \theta^i \Gamma(\alpha^i x)$, where $\Gamma(a) = \int_a^\infty t^{-1} e^{-t} dt$
- Jump dependence specified by a Gumbel Lévy copula.
- To avoid numerical inversions of $\Gamma(a)$, we can use the rejection method. $\nu_i'(x) = \frac{\theta^i}{x(1+\alpha^i x)} \mathbf{1}_{\{x>0\}}$. $U_i'(x) = \theta^i \log \frac{1+\alpha^i x}{\alpha^i x}$ and $U_i'^{-1}(x) = \frac{1}{\alpha^i (e^{x/\theta^i} - 1)}$.
- The univariate processes Z_t^i for $i = 1, \dots, d$ are given by:

$$Z_t^i = \sum_{k=1}^{\infty} J_k^i \mathbf{1}_{\{\frac{\nu}{\nu'}(J_k^1, \dots, J_k^d) \geq W_k\}} \tag{18}$$

where $J_k^i = U_i'^{-1}\left(\frac{\Gamma_k^i}{t}\right)$.

Finite series approximations

- Finite series approximation for $(Z_t^{\prime i})$ with density $\nu_i'(x)$:

$$Z_{\tau^i,t}^{\prime i} := \sum_{\{k:\Gamma_k^i<\tau^i\}} U_i'^{-1}(\Gamma_k^i) \tag{19}$$

- Expected error of the approximation on one path:

$$\mathbb{E}[\epsilon_t^i] = t\int_0^{U_i'^{-1}(\tau^i)} x\nu_i'(dx) \tag{20}$$

- $\int_0^y x\nu_i(dx) = \frac{\theta^i}{\alpha^i}(1-e^{-\alpha^i y})$. For a target expected error of 10^{-2}, truncate the jumps that are smaller than $y = -\frac{1}{\alpha^i}\log(1-\frac{\alpha^i}{\theta^i}10^{-2})$

Finite series approximations

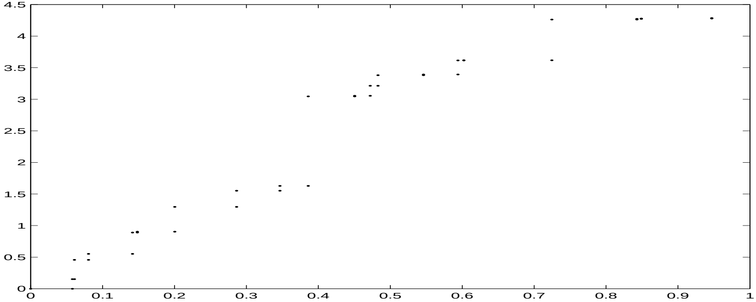

Figure: A simulated trajectory of a Gamma subordinator with Lévy density $\nu(x) = \theta x^{-1} e^{-\alpha x} \mathbf{1}_{\{x>0\}}$ on the unit interval with parameters $\alpha = 1$ and $\theta = 3$. The path was generated by series approximation and rejection method using the auxiliary Lévy density $\nu'(x) = \frac{\theta}{x(1+\alpha x)} \mathbf{1}_{\{x>0\}}$ neglecting all $\Gamma_k \geq \tau = 17.15$. This is equivalent to omitting jumps smaller than 3.3×10^{-3} and amounts to an average of 15.42 terms per path.

Implementing the rejection method for Gumbel-Lévy copula

- For ν', we can either use the same Lévy copula F or a different Lévy copula F' so that

$$\frac{\nu}{\nu'}(x_1,\dots,x_d) = \frac{\partial^d_{y_1\dots y_d} F(y_1,\dots,y_d)|_{y_1=U_1(x_1),\dots,y_d=U_d(x_d)}}{\partial^d_{y_1\dots y_d} F'(y_1,\dots,y_d)|_{y_1=U'_1(x_1),\dots,y_d=U'_d(x_d)}} \prod_{i=1}^{d} \frac{\nu_i}{\nu'_i}(x_i)$$

- In the multivariate case, finding a good candidate for ν' can be a difficult task. For example, figure (2) presents the plot of the ratio $\frac{\nu}{\nu'}$ with $F = F'$ in a bivariate case.
- Figure (3) presents a plot for the same problem where this time F' is Clayton Lévy copula and the marginals ν'_i have a larger parameter θ'.

Implementing the rejection method for Gumbel-Lévy copula

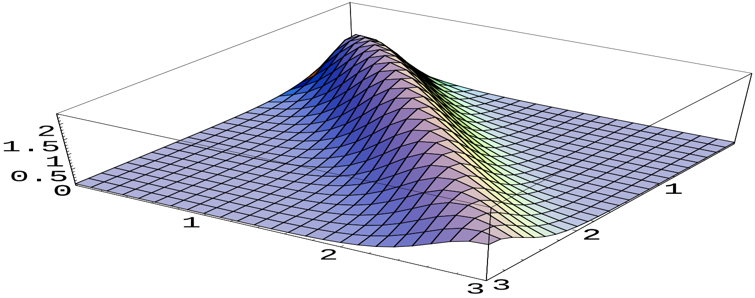

Figure: Graph of the ratio of Lévy densities $\frac{\nu}{\nu'}(x_1, x_2)$ where ν and ν' are generated with the same Gumbel Lévy copula with parameter $\gamma = 3$. The marginal Lévy densities satisfy $\nu_1(x) = \nu_2(x) = \theta x^{-1} e^{-\alpha x}$ and $\nu_1'(x) = \nu_2'(x) = \frac{\theta}{x(1+\alpha x)}$ with parameters $\alpha = 1$ and $\theta = 3$.

Implementing the rejection method for Gumbel-Lévy copula

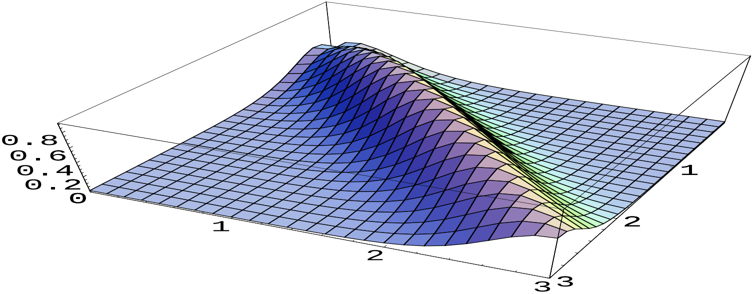

Figure: Graph of the ratio of Lévy densities $\frac{\nu}{\nu'}(x_1, x_2)$ where ν and ν' are generated with a Gumbel Lévy copula with parameter $\gamma = 3$ and a Clayton Lévy copula with parameter $\gamma = 2$ respectively. The marginal Lévy densities satisfy $\nu_1(x) = \nu_2(x) = \theta x^{-1} e^{-\alpha x}$ and $\nu_1'(x) = \nu_2'(x) = \frac{\theta'}{x(1+\alpha' x)}$ with parameters $\alpha = \alpha' = 1$, $\theta = 3$ and $\theta' = 7$.

Default intensities with O-U Gamma processes

- Individual default intensities $\lambda^1, \dots, \lambda^d$ follow:

$$d\lambda_t^i = \kappa^i(\mu^i - \lambda_t^i)dt + dZ_t^i \tag{21}$$

- To simulate the integrated O-U processes $\int_0^t \lambda_s^i ds$, note that:

$$\int_0^t \lambda_s^i ds = \kappa^i \mu^i t + \frac{\lambda_0^i}{\kappa^i}\left(1 - e^{-\kappa^i t}\right) + \frac{1}{\kappa^i}\left(Z_t^i - e^{-\kappa^i t}\int_0^t e^{\kappa^i s} dZ_s^i\right) \tag{22}$$

- $\int_0^t e^{\kappa^i s} dZ_s^i$ for $i = 1, \dots, d$ are given by:

$$\int_0^t e^{\kappa^i s} dZ_s^i = \sum_{k=1}^{\infty} J_k^i e^{\kappa^i t u_k} \mathbf{1}_{\{\frac{\nu}{\nu'}(J_k^1, \dots, J_k^d) \geq W_k\}} \tag{23}$$

Asymmetric clustering of large jumps

- $\mathbb{P}\left[\Gamma_j^2 \leq v / \Gamma_j^1 \leq v\right] = \frac{\int_0^v e^{-y} \frac{y^{j-1}}{(j-1)!} \partial_x F(x,v)|_{x=y} dy}{\int_0^v e^{-y} \frac{y^{j-1}}{(j-1)!} dy}$
- $\partial_x F(x,v)|_{x=y} = \mathbb{P}\left[\Gamma_j^2 \leq v / \Gamma_j^1 = y\right]$ is non-increasing in y and $\forall y \in [0,v]$ we can deduce:

$$\partial_{x_1} F(v,v) \leq \partial_{x_1} F(y,v) \leq \partial_{x_1} F(0,v) \tag{24}$$

- Thus for “small" v, $\mathbb{P}\left[\Gamma_j^2 \leq v / \Gamma_j^1 \leq v\right] \approx \partial_{x_1} F(v,v)$
- $\partial_{x_1} F(v,v) = \frac{1}{2^{\frac{1+\gamma}{\gamma}} (v+1)^{1-2^{-1/\gamma}}}$ which is decreasing in v and increasing in γ with limit $\frac{1}{2^{\frac{1+\gamma}{\gamma}}}$ when $v \downarrow 0$

Portfolio default loss distribution

- $L_t = L_{GD} \sum_{k=1}^{d} \mathbf{1}_{\{\tau^k \leq t\}}$
- $\mathbb{E}\left[e^{-\eta L_t}\right] = \mathbb{E}\left[\prod_{k=1}^{d}\left[e^{-\eta L_{GD}}(1 - e^{-\int_0^t \lambda_s^k ds}) + e^{-\int_0^t \lambda_s^k ds}\right]\right]$
- Simulate a number of paths for $\int_0^t \lambda_s^k ds$ for $k = 1, \ldots, d$ and compute the M-C estimator of the Laplace transform for different values of η. Loss distribution obtained by Laplace transform inversion.

Portfolio total P&L distribution

- $PnL_t = \sum_{k=1}^{d} (V_t^k - V_0^k)$ where V_t^k and V_0^k represent the values of the k^{th} zero-coupon at time t and time 0 respectively.
- $V_t^k = \mathbf{1}_{\{\tau^k > t\}} \widetilde{V}_t^k + \mathbf{1}_{\{\tau^k \leq t\}} R_{ec}$
- $\widetilde{V}_t^k = \mathbb{E}\left[e^{-\int_t^{T_k} \lambda_s^k ds} | \sigma(\lambda_s^k : s \leq t) \right]$
- $\mathbb{E}\left[e^{-\eta PnL_t}\right] =$
  $\mathbb{E}\left[\prod_{k=1}^{d} \left[e^{-\eta(\widetilde{V}_t^k - V_0^k)} e^{-\int_0^t \lambda_s^k ds} + e^{-\eta(R_{ec} - V_0^k)} (1 - e^{-\int_0^t \lambda_s^k ds}) \right] \right]$
- The pre-default value of the defaultable ZCB is given by:

$$\widetilde{V}_t^k = \exp\left(-\frac{\lambda_t^k}{\kappa^k}(1 - e^{-\kappa^k (T_k - t)}) - \mu^k \left[(T_k - t) + \frac{e^{-\kappa^k (T_k - t)} - 1}{\kappa^k} \right] + \int_t^{T_k} \left(1 + \frac{1}{\kappa^k \alpha^k} (1 - e^{-\kappa^k (T_k - s)}) \right)^{-\theta^k} ds \right)$$

Concluding remarks: alternative to Lévy copulas

- If $\{\Gamma_k^i\}$ are jump times of a SPP then $\{U_i^{-1}(\Gamma_k^i)\}$ are distributed as ordered jumps of the process (X_t^i).
- We can simply simulate $\{\Gamma_k^i\}$s with dependent inter-arrival times using ordinary copulas.
- We loose the path information (times of jumps, simultaneous jumps) but can gain in efficiency and tractability.