Utility-based Valuation of Employee Stock Options

Tim Leung

Department of Operations Research & Financial Engineering
Princeton University

Workshop on Computational Methods for Exotic Options

University of Warwick, 12 July, 2008

Employee Stock Options (ESOs)

- ESOs are call options (with non-standard features) given to employees as a form of compensation.
- Idea is to align the interests of employees and shareholders.
- Huge debate 1993-2004 about whether ESOs should be expensed. Required by FASB to do so since 2005.

Employee Stock Options (ESOs)

- ESOs are call options (with non-standard features) given to employees as a form of compensation.
- Idea is to align the interests of employees and shareholders.
- Huge debate 1993-2004 about whether ESOs should be expensed. Required by FASB to do so since 2005.
- Next debate: How to value them?

Main Features of ESOs

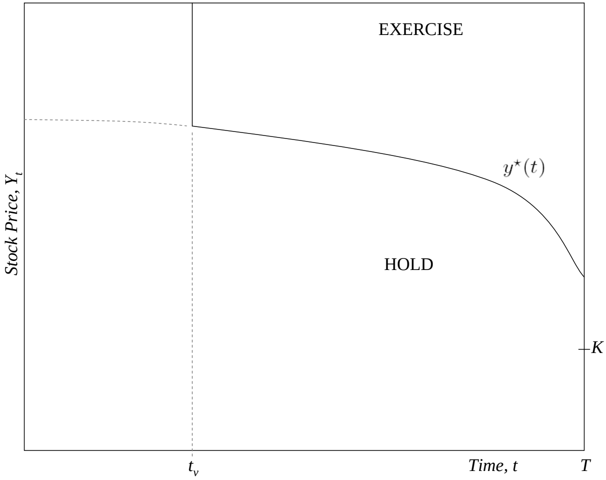
- American call: firm's stock Y, strike K $\rightarrow$ payoff $(Y_t - K)^+$.
- Vesting period: length $t_v \approx 4$ years; expiration date $T \approx 10$ years.
- Non-tradability: employee can't sell/transfer ESO.
- Short-sale constraint: employee can't short own firm's stock.
- Job termination risk: possible departure at a random time τ^λ, ESO is either forfeited or exercised immediately.

Main Features of ESOs

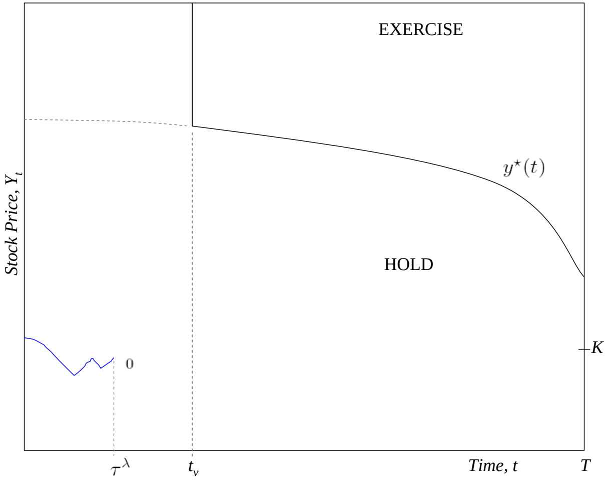
- American call: firm's stock Y, strike K $\rightarrow$ payoff $(Y_t - K)^+$.
- Vesting period: length $t_v \approx 4$ years; expiration date $T \approx 10$ years.
- Non-tradability: employee can't sell/transfer ESO.
- Short-sale constraint: employee can't short own firm's stock.
- Job termination risk: possible departure at a random time τ^λ, ESO is either forfeited or exercised immediately.

- Early Exercise Phenomenon: empirically, employees tend to exercise early/suboptimally, often right after vesting.

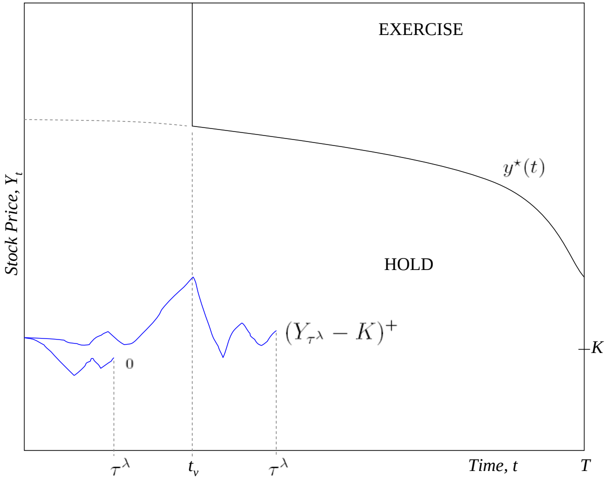
ESO Payoff Structure



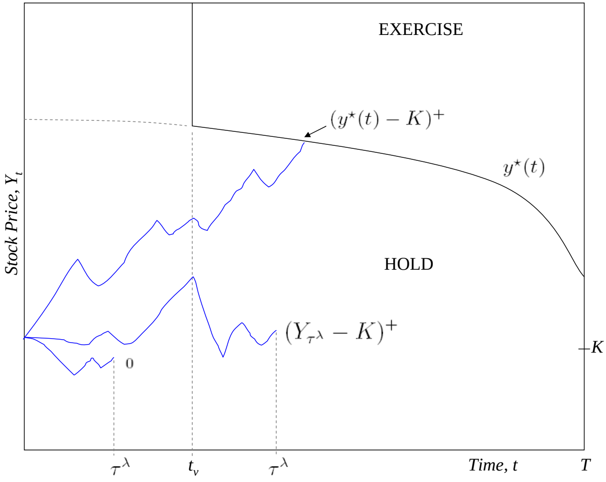
ESO Payoff Structure



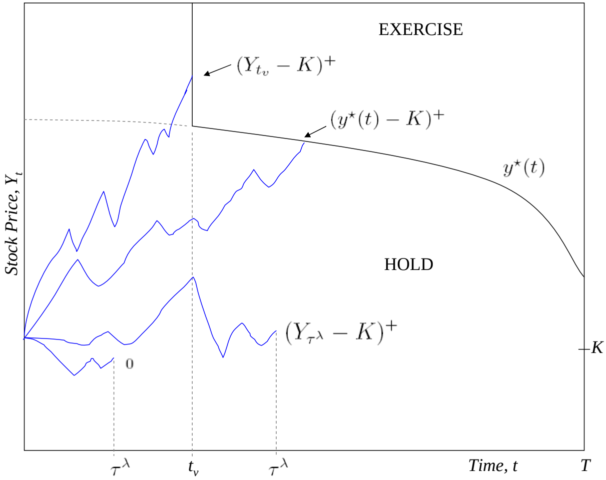
ESO Payoff Structure



ESO Payoff Structure



ESO Payoff Structure



Some Existing Models

- Exercise at a given barrier:
  Hull-White (2004), Cvitanic-Wiener-Zapatero (2004)
- Exogenous exercises:
  Jennergen-Naslund (1993), Carpenter (1998), Carr-Linetsky (2000)
- Indifference pricing approach for American options:
  Oberman-Zariphopoulou (2003), Henderson (2005)
- Multiple American ESOs:
  Grasselli (2005), Grasselli-Henderson (2006), Scheinkman-Rogers (2006)

Valuation Methodology

- The employee's investment problem:
  - Account for risk aversion, optimal hedging, & job termination.
  - Solve for the optimal exercise policy (boundary $y^\star(t)$).
  - Analyze contributors to early exercises.
- The firm's cost calculation:
  - Firm is allowed to hedge their liability.
  - Determine ESO cost by no-arbitrage (risk-neutral) pricing theory, with $y^\star(t)$ as an input.
  - Study the impact of factors on ESO cost.

Model Formulation

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with price processes:

$$dY_t = (\nu - q)Y_t\,dt + \eta Y_t\,dW_t, \quad \text{(Firm, nontraded)}$$
$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t, \quad \text{(Index, traded)}$$

  where $\mathbb{E}\{dW_t \cdot dB_t\} = \rho\,dt$.

- A dynamic trading strategy $(\theta_t)_{0 \leq t \leq T}$ is the cash amount invested in the index, with $\mathbb{E}\{\int_0^T \theta_t^2 dt\} < \infty$. The trading wealth follows

$$dX_t = \theta_t \frac{dS_t}{S_t} + (X_t - \theta_t) r\,dt$$
$$= [\theta_t(\mu - r) + rX_t]\,dt + \theta_t \sigma\,dB_t.$$

- Employee's utility function: $U(x) = -e^{-\gamma x}$.

Stochastic Control Problem

- Job termination time: $\tau^{\lambda} \sim \exp(\lambda)$, independent of W and B.
- Exercise time is a stopping time $\tau \in [0, T]$, and let $\hat{\tau} = \tau \wedge \tau^{\lambda}$.

Stochastic Control Problem

- Job termination time: $\tau^\lambda \sim \exp(\lambda)$, independent of W and B.
- Exercise time is a stopping time $\tau \in [0, T]$, and let $\hat{\tau} = \tau \wedge \tau^\lambda$.
- After exercise, the employee will face the classical Merton problem:

$$
\begin{aligned}
M(t,x) &= \sup_\theta \mathbb{E}\left\{ -e^{-\gamma X_T} \,|\, X_t = x \right\} \\
&= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}.
\end{aligned}
$$

Stochastic Control Problem

- Job termination time: $\tau^\lambda \sim \exp(\lambda)$, independent of W and B.
- Exercise time is a stopping time $\tau \in [0, T]$, and let $\hat{\tau} = \tau \wedge \tau^\lambda$.
- After exercise, the employee will face the classical Merton problem:

$$M(t,x) = \sup_{\theta} \mathbb{E}\left\{ -e^{-\gamma X_T} \,|\, X_t = x \right\} = -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu - r)^2}{2\sigma^2}(T-t)}.$$

- The employee's value function at time t is

$$V(t,x,y) = \sup_{\tau,\theta} \mathbb{E}_{t,x,y}\left\{ M(\hat{\tau}, X_{\hat{\tau}} + (Y_{\hat{\tau}} - K)^+) \right\}.$$

HJB Variational Inequality

- Look for a solution of the fully nonlinear Variational Inequality

$$\lambda\,(\Lambda - V) + V_t + \sup_{\theta} \mathcal{L}V \le 0\,,$$

$$V \ge \Lambda\,,$$

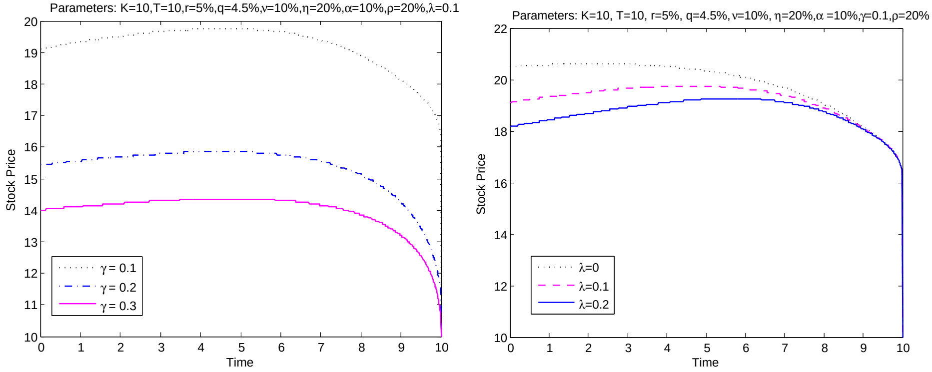
$$\left(\lambda\,(\Lambda - V) + V_t + \sup_{\theta} \mathcal{L}V\right)\cdot\left(\Lambda - V\right) = 0\,,$$

for $(t, x, y) \in [0, T) \times \mathbb{R} \times (0, +\infty)$, with $\Lambda(t, x, y) = M\,(t, x + (y - K)^+)$.

- Transformation: $V(t, x, y) = M(t, x) \cdot H(t, y)^{\frac{1}{(1-\rho^2)}}$.

- Solve for the optimal exercise boundary $y^\star(t)$, so that optimal exercise time: $\tau^\star = \inf\{\, 0 \le t \le T \,:\, Y_t = y^\star(t)\}$.

Optimal Exercise Boundary



- Numerical solution using standard finite-differences with obstacle constraint enforced by PSOR.

Analytical Results

- Risk aversion increases ($\gamma \uparrow$) / Job termination risk rises ($\lambda \uparrow$)
  $\Rightarrow$ optimal exercise boundary shifts downward.
- Firm's stock growth rate increases ($\nu - q \uparrow$)
  $\Rightarrow$ optimal exercise boundary shifts upward.

$\longrightarrow$ These follow from comparison principle for the VI.

- Connection with indifference price (p): $M(t, x + p(t, y)) = V(t, x, y)$.
  $\hookrightarrow$ The employee demands $\$p$ to forgo the ESO.
  $\hookrightarrow$ $\tau^\star = \inf\{\, t \le T \,:\, p(t, Y_t) = (Y_t - K)^+ \,\}$.

Cost to the Firm

- With $y^\star(t)$ known, the ESO cost is the expected discounted payoff under the risk-neutral measure $\mathbb{Q}$.
- Under $\mathbb{Q}$, the firm's stock evolves according to

$$dY_t = (r - q)Y_t\,dt + \eta Y_t\,dW_t^{\mathbb{Q}},$$

  where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion.
- Vested ESO Cost:

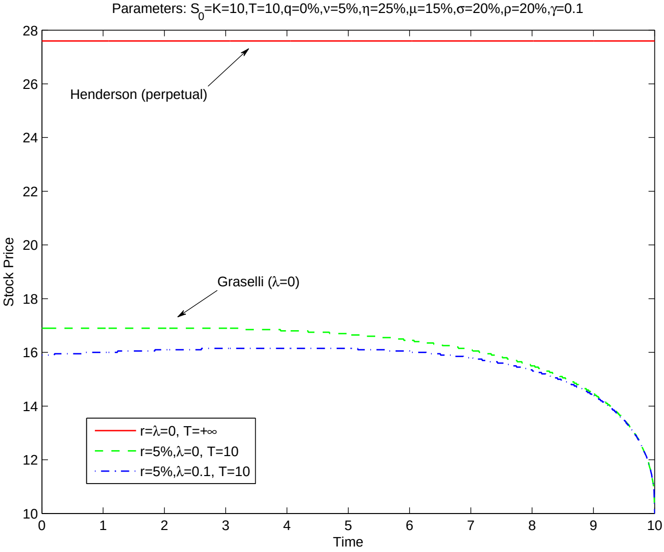
$$C(t,y) = \mathbb{E}_{t,y}^{\mathbb{Q}}\left\{ e^{-r(\tau^\star \wedge \tau^\lambda - t)}(Y_{\tau^\star \wedge \tau^\lambda} - K)^+ \right\}.$$

  Unvested ESO Cost:

$$\tilde{C}(t,y) = \mathbb{E}_{t,y}^{\mathbb{Q}}\left\{ e^{-r(t_v - t)} C(t_v, Y_{t_v}) \mathbf{1}_{\{\tau^\lambda > t_v\}} \right\}.$$

- We assume λ to be identical under both measures $\mathbb{P}$ and $\mathbb{Q}$.

Other Utility-based models

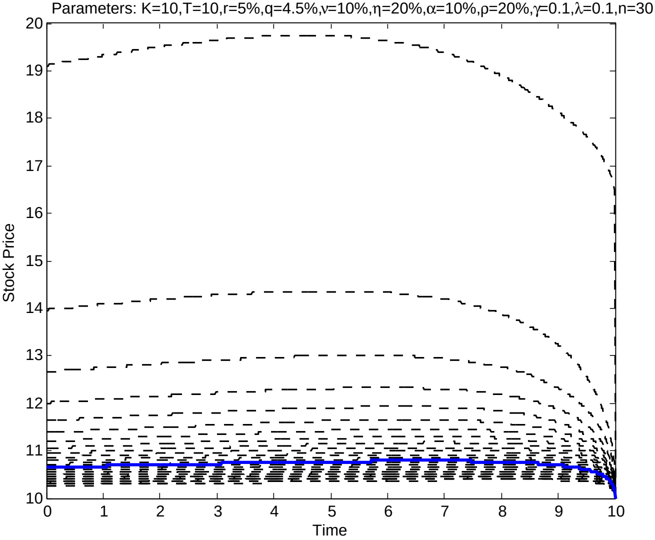


Impacts of Various Features

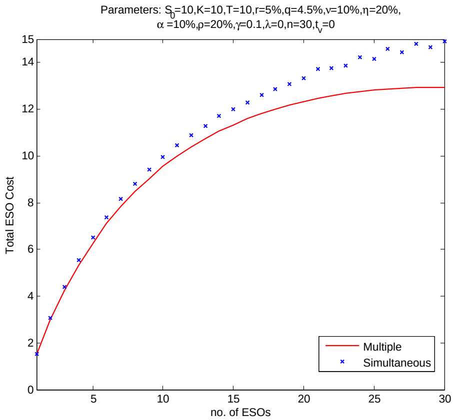
Black-Scholes	Henderson	Grasselli	$+\lambda = 0.1$	3yr vesting
4.879	4.510	3.412	2.597	2.491

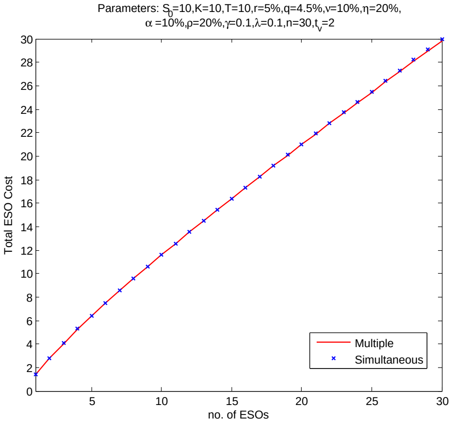
- Risk-aversion lowers the cost by about 8% in the perpetual approximation, or by about 30% when we retain finite maturity.
- Job termination risk reduces the cost by a further 17% of the Black-Scholes value.
- Vesting reduces by yet another 2%.

Optimal Multiple Exercise Boundaries



Impact of Multiple Grants on ESO Cost





Static-Dynamic Hedge for ESOs

- Call and put options on the firm's stock are traded in the market.
- While employee cannot short calls, he/she can purchase puts.
- Payoff of a put: $(K - Y_\tau)^+$.

Static-Dynamic Hedge for ESOs

- Call and put options on the firm's stock are traded in the market.
- While employee cannot short calls, he/she can purchase puts.
- Payoff of a put: $(K - Y_\tau)^+$.
- Augment the hedging strategy:
  - Dynamic Hedge: market index S and bank account.
  - Static Hedge: buy and hold α units of identical American puts.

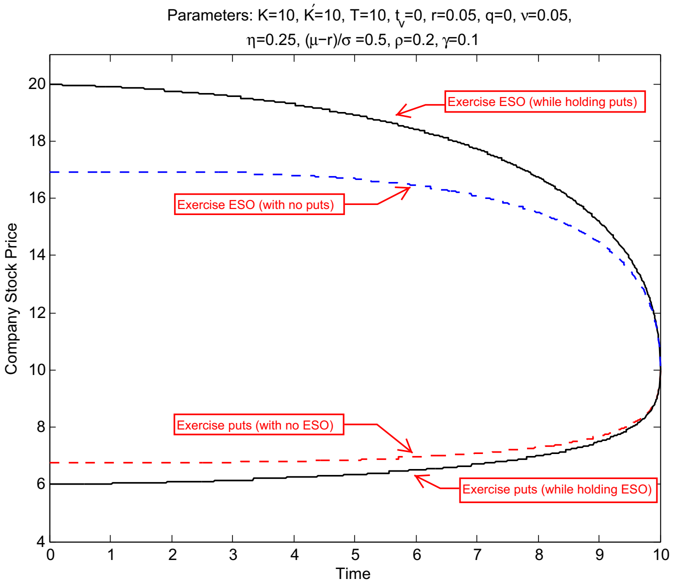
Static-Dynamic Hedge for ESOs

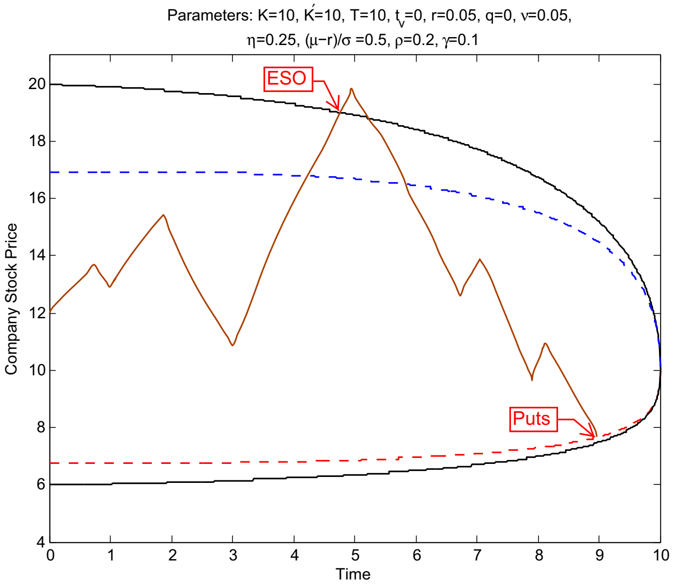
- Call and put options on the firm's stock are traded in the market.
- While employee cannot short calls, he/she can purchase puts.
- Payoff of a put: $(K - Y_\tau)^+$.
- Augment the hedging strategy:
  - Dynamic Hedge: market index S and bank account.
  - Static Hedge: buy and hold α units of identical American puts.
- Employee's investment problem:
  - Order of Exercises: ESO-puts, or puts-ESO? When?
  - Optimal Static Hedge: optimal number of puts that maximizes the value function.

Static-Dynamic Hedge for ESOs

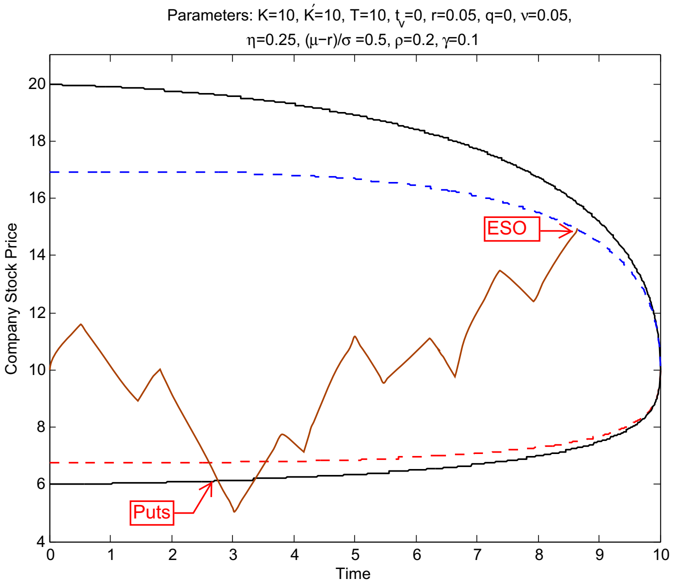
- Call and put options on the firm's stock are traded in the market.
- While employee cannot short calls, he/she can purchase puts.
- Payoff of a put: $(K - Y_\tau)^+$.
- Augment the hedging strategy:
  - Dynamic Hedge: market index S and bank account.
  - Static Hedge: buy and hold α units of identical American puts.
- Employee's investment problem:
  - Order of Exercises: ESO-puts, or puts-ESO? When?
  - Optimal Static Hedge: optimal number of puts that maximizes the value function.
- For simplicity, assume no job termination risk $(\lambda = 0)$ here.

Optimal Exercise Boundaries




Parameters: K=10, K´=10, T=10, t_v=0, r=0.05, q=0, ν=0.05,
η=0.25, (μ−r)/σ =0.5, ρ=0.2, γ=0.1
ESO
Puts
Company Stock Price
Time
20
18
16
14
12
10
8
6
4
0
1
2
3
4
5
6
7
8
9
10

Optimal Exercise Scenario II



The Impact of Static-Dynamic Hedge

- The optimal number of puts is found from the Fenchel-Legendre transform of p^* as a function of α, evaluated at the market price π.

$$\alpha^* = \arg\max_{\alpha \geq 0} p^*(t, y; \alpha) - \alpha\pi.$$

- ESO Cost Comparison:

Black-Scholes	Dynamic Hedge only	Static-Dynamic Hedge
4.879	3.412	3.831 $(\alpha^* = 2.6)$

  - Risk-aversion (with dynamic hedge) lowers the costs by 30%, compared to the Black-Scholes value.
  - When American puts are used, the cost increases by 8%, but still 22% lower than the Black-Scholes value.

Concluding Remarks

- Analytical and computationally tractable model for ESO valuation:
  - Risk aversion and job termination risk lead to early exercises.
  - Static hedges delay ESO exercises, and lead to higher costs.
- Some major challenges:
  - Inference of risk aversion from empirical exercises
    $\hookrightarrow$ Data segmentation based on employees' attributes.
  - Non-exponential/stochastic utility functions
    $\hookrightarrow$ Analyticity and tractability issues.
- General semimartingale framework:
  - Duality relationship between exponential utility maximization and relative entropy minimization with optimal stopping.
  - Characterization of optimal exercise times via indifference prices.

Appendix

Transformation to Reaction-Diffusion VI

- The free boundary problem for H is of *reaction-diffusion* type.

$$H_t + \tilde{\mathcal{L}}\, H - (1-\rho^2)\lambda[H - b(t,y)H^{-\hat{\rho}}] \geq 0,$$

$$H(t,y) \leq \kappa(t,y),$$

$$\Big(H_t + \tilde{\mathcal{L}}\, H - (1-\rho^2)\lambda[H - b(t,y)H^{-\hat{\rho}}]\Big) \cdot \Big(\kappa(t,y) - H(t,y)\Big) = 0,$$

for $(t,y) \in [0,T] \times (0,+\infty)$, where

$$\hat{\rho} = \frac{\rho^2}{1-\rho^2}, \quad \tilde{\mathcal{L}} = \frac{\eta^2 y^2}{2}\frac{\partial^2}{\partial y^2} + (\nu - q - \rho\frac{\mu - r}{\sigma}\eta)y\frac{\partial}{\partial y}.$$

- **Optimal exercise boundary**:

$$y^\star(t) = \inf\{\, y \geq 0 \,:\, H(t,y) = \kappa(t,y)\}\,,$$

so that $\tau^\star = \inf\{\, 0 \leq t \leq T \,:\, Y_t = y^\star(t)\}$.

VI for H

$$\hat{\rho} = \frac{\rho^2}{1-\rho^2}\,,$$

$$b(t,y) = e^{-\gamma(y-K)^+ e^{r(T-t)}},$$

$$\kappa(t,y) = e^{-\gamma(1-\rho^2)(y-K)^+ e^{r(T-t)}}.$$

- The boundary conditions are

$$H(T,y) = \kappa(T,y), \quad H(t,0) = 1.$$

Connection with Indifference Price

Definition

The ESO holder's indifference price is defined by

$$M(t,x) = V(t, x - p, y).$$

The indifference price satisfies

$$V(t,x,y) = M(t,x) \cdot e^{-\gamma p(t,y) e^{r(T-t)}} .$$

$\Rightarrow$

Optimal hedge: $$\theta^\star = \underbrace{\frac{\mu - r}{\gamma\sigma^2} e^{-r(T-t)}}_{\text{Merton}} \underbrace{- \rho\frac{\eta}{\sigma} y p_y(t,y)}_{\text{due to ESO}} .$$

Optimal exercise time: $$\tau^\star = \inf\left\{ t \le u \le T \,:\, p(u, Y_u) = (Y_u - K)^+ \right\} .$$

Free Boundary Problem for the Indifference Price

The indifference price solves the free boundary problem:

$$p_t + \tilde{\mathcal{L}}\, p - rp - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 + \frac{\lambda}{\gamma}\left(1 - b(t,y)e^{\gamma p e^{r(T-t)}}\right) \le 0\,,$$

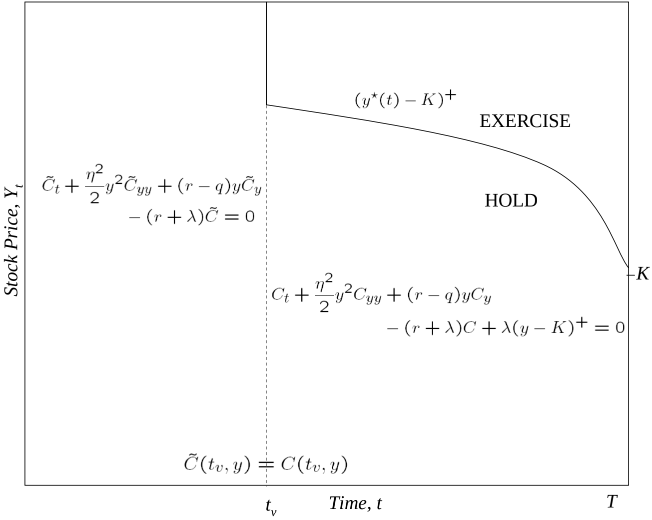
$$p \ge (y-K)^+,$$

$$\left(p_t + \tilde{\mathcal{L}}\, p - rp - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 + \frac{\lambda}{\gamma}\left(1 - b(t,y)e^{\gamma p e^{r(T-t)}}\right)\right) \cdot \left((y-K)^+ - p\right) = 0,$$

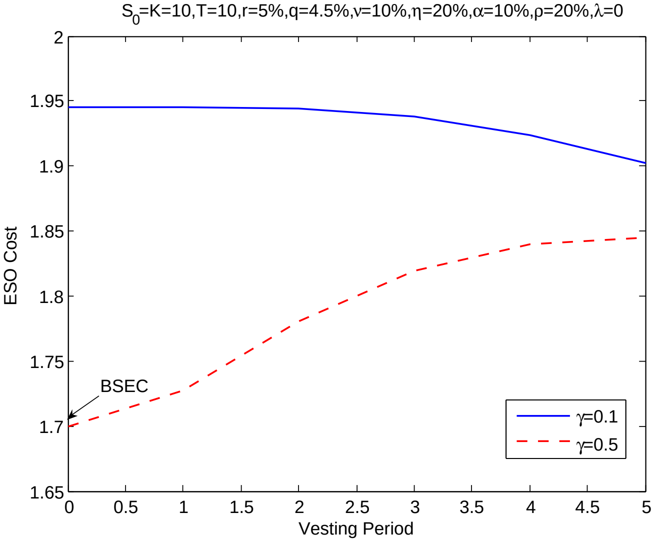
for $(t,y) \in [0,T] \times (0,+\infty)$, with $b(t,y) = e^{-\gamma(y-K)^+ e^{r(T-t)}}$, and

$$p(T,y) = (y-K)^+\,,$$
$$p(t,0) = 0\,.$$

Cost to the Firm



Effect of Risk Aversion & Vesting



ESOs with Multiple Exercises

- Employee is granted n ESOs with the same strike and maturity.
- Let τ_i be the exercise time when $i \leq n$ options remain unexercised. $\tau_n \leq \tau_{n-1} \leq \cdots \leq \tau_2 \leq \tau_1$.
- Employee's value function of holding i ESOs is defined recursively by

$$V^{(i)}(t,x,y) = \sup_{\tau_i,\,\theta} \mathbb{E}_{t,x,y}\Big\{ V^{(i-1)}\left(\tau_i, X_{\tau_i} + (Y_{\tau_i} - K)^+, Y_{\tau_i}\right) \cdot \mathbf{1}_{\{\tau_i < \tau^\lambda\}} + M\left(\tau^\lambda, X_{\tau^\lambda} + i\,(Y_{\tau^\lambda} - K)^+\right) \cdot \mathbf{1}_{\{\tau_i \geq \tau^\lambda\}} \Big\}$$

- This stochastic control problem with optimal sequential stopping leads to a system of free boundary problems of reaction-diffusion type.

ESOs With Multiple Exercises

- Solve the system of VIs

$$\lambda\left(M\left(t,x+i(y-K)^+\right)-V^{(i)}\right)+V_t^{(i)}+\sup_\theta \mathcal{L}\,V^{(i)}\leq 0\,,$$

$$V^{(i)}(t,x,y)\geq V^{(i-1)}(t,x+(y-K)^+,y)\,,$$

$$\left(\lambda\left(M\left(t,x+i(y-K)^+\right)-V^{(i)}\right)+V_t^{(i)}+\sup_\theta \mathcal{L}\,V^{(i)}\right)$$
$$\cdot\left(V^{(i-1)}\left(t,x+(y-K)^+,y\right)-V^{(i)}(t,x,y)\right)=0\,,$$

for $(t,x,y)\in[0,T)\times\mathbb{R}\times(0,+\infty)$, with boundary conditions

$$V^{(i)}(T,x,y)=-e^{-\gamma(x+i(y-K)^+)}\,,$$
$$V^{(i)}(t,x,0)=-e^{-\gamma x e^{r(T-t)}}e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}\,.$$

Connection with Indifference Price

Definition

The employee's indifference price for holding $i \leq n$ ESOs with multiple exercises is defined by

$$M(t,x) = V^{(i)}(t, x - p^{(i)}, y)\,.$$

The indifference price $p^{(i)}$ satisfies

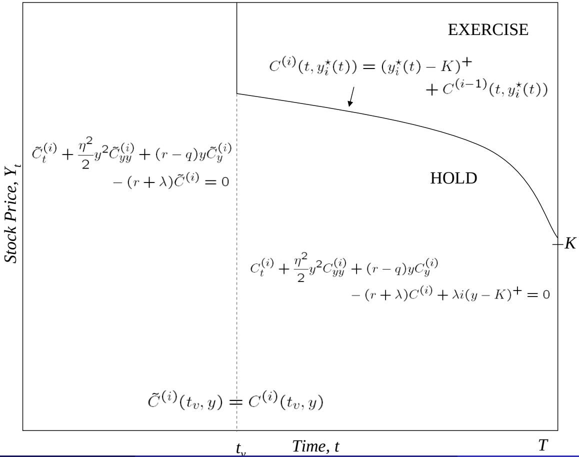
$$V^{(i)}(t,x,y) = M(t,x) \cdot e^{-\gamma p^{(i)}(t,y) e^{r(T-t)}} \tag{1}$$

$\Rightarrow$

$$\begin{aligned} \tau_i^\star &= \inf\left\{t \leq T \,:\, V^{(i)}(u, X_u^{\theta^\star}, Y_u) = V^{(i-1)}(u, X_u^{\theta^\star} + (Y_u - K)^+, Y_u)\right\} \\ &= \inf\{t \leq T \,:\, \underbrace{p^{(i)}(u,Y_u) - p^{(i-1)}(u,Y_u)}_{\text{premium for the } i\text{th ESO}} = (Y_u - K)^+\}. \end{aligned} \tag{2}$$

ESO Costs

$$C^{(i)}(t,y) = \mathbb{E}^{\mathbb{Q}}_{t,y}\Big\{ e^{-r(\tau^{\lambda}-t)}\, i\left(Y_{\tau^{\lambda}} - K\right)^{+} \mathbf{1}_{\{\tau^{\lambda} \le \tau_i^{\star}\}} + e^{-r(\tau_i^{\star}-t)}\left[\left(Y_{\tau_i^{\star}} - K\right)^{+} + C^{(i-1)}\left(\tau_i^{\star}, Y_{\tau_i^{\star}}\right)\right]\mathbf{1}_{\{\tau^{\lambda} > \tau_i^{\star}\}}\Big\}.$$



The Optimal Second Exercises

- There are two orders of exercises: ESO–Puts, or Puts–ESO. Consider the second exercises here.
- If the employee holds an ESO only:

$$\begin{aligned} V(t,x,y) &:= \sup_{\tau,\theta} \mathbb{E}_{t,x,y}\left\{ M(\tau, X_\tau + (Y_\tau - K)^+) \right\} \\ &= M(t, x - p(t,y)). \end{aligned}$$

- The value function for holding α puts:

$$\begin{aligned} \hat{V}(t,x,y;\alpha) &:= \sup_{\tau,\theta} \mathbb{E}_{t,x,y}\left\{ M(\tau, X_\tau + \alpha(K' - Y_\tau)^+) \right\} \\ &= M(t, x - \hat{p}(t,y;\alpha)). \end{aligned}$$

- Solving the VIs associated with V and $\hat{V}$, we obtain the optimal exercise boundaries for the second exercises.

The Optimal First Exercises

- The employee's value function is

$$V^*(t,x,y;\alpha) := \sup_{\tau,\,\theta} \mathbb{E}_{t,x,y}\{\max\{V(\tau, X_\tau + \alpha(K' - Y_\tau)^+),$$

$$\hat{V}(\tau, X_\tau + (Y_\tau - K)^+;\alpha)\}\}$$

$$= \sup_{\tau,\,\theta} \mathbb{E}_{t,x,y}\left\{M(\tau, X_\tau + R^\alpha_\tau)\right\},$$

where $R^\alpha_\tau = \max\{\alpha(K' - Y_\tau)^+ + p(\tau, Y_\tau), (Y_\tau - K)^+ + \hat{p}(\tau, Y_\tau;\alpha)\}$.

- Optimal first exercise time (of either ESO or puts) is

$$\tau^* = \inf\{t \le T : p^*(t, Y_t;\alpha) = R^\alpha_t\}$$

$$= \min(\tau^E, \tau^P),$$

where

$$\tau^E := \inf\{\, t \le T \,:\, p^*(t, Y_t;\alpha) = (Y_t - K)^+ + \hat{p}(\tau, Y_t;\alpha)\},$$

$$\tau^P := \inf\{\, t \le T \,:\, p^*(t, Y_t;\alpha) = \alpha(K' - Y_t)^+ + p(t, Y_t)\,\}.$$

The Optimal Static Hedge

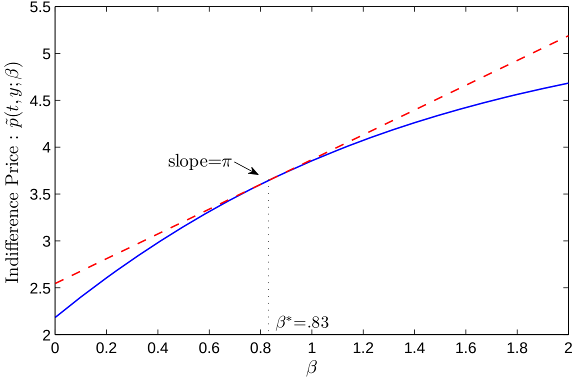
- Recall that indifference price is defined by the equation:

$$V^*(t, x, y; \alpha) = M\left(t, x + p^*(t, y; \alpha)\right)$$

- The employee chooses the optimal α to maximize the value function.

$$\begin{aligned}\alpha^* &= \arg\max_{\alpha \geq 0} V^*(t, x - \alpha\pi, y; \alpha) \\ &= \arg\max_{\alpha \geq 0} M\left(t, x - \alpha\pi + p^*(t, y; \alpha)\right) \\ &= \arg\max_{\alpha \geq 0} p^*(t, y; \alpha) - \alpha\pi.\end{aligned}$$

The Optimal Static Hedge



General Semimartingale Framework

- Consider the utility maximization (primal) problem

$$V(t, X_t) := \operatorname*{ess\,sup}_{\tau,\theta} \mathbb{E}\{M(\tau, X_\tau + (Y_\tau - K)^+) \,|\mathcal{F}_t\}.$$

- Derive the dual for V, and deduce from $V(t, X_t) = M(t, X_t + p_t)$ the indifference price

$$p_t = \operatorname*{ess\,sup}_{\tau} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left\{(Y_\tau - K)^+ + \phi_t(\tau, Q) \,|\mathcal{F}_t\right\},$$

where ϕ_t is a conditional entropic penalty.

- Have two stochastic games, with the same optimal exercise time

$$\tau^\star = \inf\{0 \le t \le T \,:\, p_t = (Y_t - K)^+\}.$$