Flexing the Default Barrier

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Model value of assets as stochastic process $y_t$

When asset value hits barrier the corporation defaults

Quantity of interest is the survival probability
Related Models

- Black and Cox (1976)
- Brigo and Tarenghi (2005)

- Asset value follows Geometric Brownian Motion
- Default Barrier has specific functional form for tractability
- Closed-form expressions for survival probabilities even with time-dependent drift and diffusion coefficients
Our Contribution to Literature

Our model grants:

✔ Complete **freedom** in the shape of the default barrier
✔ **Consistency** with observable quantities (equity vol)
✔ Numerical **stability** in efficient numerical procedure
✔ **Sequential calibration** to leverage – equity – interest rate – credit
✔ **Extendibility** to inaccessible model

Our model is relevant for:

✔ Determination of market-implied expected default barrier
✔ Pricing under counterparty risk
From SDE:

\[ \frac{dy_t}{y_t} = (r(t) - q(t)) \, dt + \sigma(t) \, dB_t, \quad y_0 = 1, \]

We look at the first passage time of \( y(t) \) to some time-dependent barrier \( b(t) \):

\[ \tau = \inf \{ t \geq 0 : y_t \leq b(t) \}. \]

Survival probability is not known for general \( b(t) \)
Survival probability is known for natural barrier

\[
b^\circ(s) = \exp \left\{ \frac{\tilde{\alpha}(t, T) \tilde{\beta}(t, s) - \tilde{\alpha}(t, s) \tilde{\beta}(t, T)}{\tilde{\alpha}(t, T)} \right\} \left( \frac{d}{c} \right) \frac{\tilde{\alpha}(t, s)}{\tilde{\alpha}(t, T)} c
\]

for \( t \leq s \leq T \), however, where we define

\[
\tilde{\alpha}(t, T) := \int_t^T \alpha(s) \, ds, \quad \tilde{\beta}(t, T) := \int_t^T \beta(s) \, ds
\]

and \( \tilde{r}(t, T) := \int_t^T r(s) \, ds \)

for \( \beta(s) := r(s) - q(s) \), \( \alpha(s) = \sigma^2(s)/2 \), \( c = b^\circ(t) \), \( d = b^\circ(T) \)
Green function $g^+$ solves the survival probability

$$Q(T|y_0, 0) := Q(\tau > T) = \int_0^\infty g^+(x, T|y_0, 0) \, dx$$
Approximation to General Boundary

- $g^+$ available in closed-form
- **Idea:** develop approximation of general $b(t)$ using

$$b(t) \approx \sum_{i=1}^{n} b_i^\triangledown(t) \mathbb{1}_{[t_{i-1}, t_i]}(t) \quad \text{for} \quad 0 \leq t \leq T$$

- Solve a sequence of feasible $b_i^\triangledown$ problems instead of the infeasible $b$ problem

**Proposition 1**

$$\mathbb{Q}[y_t > b(t), \forall t \in [0, T]] = \mathbb{Q}[\bar{B}_t > \bar{b}(t), \forall t \in [0, T]], \quad \text{where}$$

$$\bar{b}(t) = \frac{1}{\sqrt{\frac{2}{t} \tilde{\alpha}(0, t)}} \left( \ln \frac{b(t)}{y_0} - \tilde{\beta}(0, t) + \tilde{\alpha}(0, t) \right), \quad 0 \leq t \leq T$$
Proposition 2 \textit{The approximation quality can be bounded}

1. \textit{for the boundary itself}:
\[
|b - b^\diamond|_2^2 \leq \sum_{i=1}^{n} (t_i - t_{i-1})^2 |b' - (b_i^\diamond)'|_2^2,
\]

2. \textit{for the survival probability over } [0, T]:
\[
|Q[y_t > b(t), \forall t \in [0, T]] - Q[y_t > b^\diamond(t), \forall t \in [0, T]]| \\
\leq \sqrt{\frac{2}{\pi}} |b' - (b^\diamond)'|_2.
\]
Fix start and end points at prescribed times $t_i$, i.e.

$$b^\diamond_i(t_{i-1}) = b(t_{i-1}) \text{ and } b^\diamond_i(t_i) = b(t_i) \text{ such that}$$

$$\int_{t_{i-1}}^{t_i} (b^\diamond_i(t) - b(t))^2 \, dt \leq \varepsilon^2, \ i = 1, \ldots, n$$
We can approximate the $b$ problem to arbitrary precision backward-propagating the sequence of $b^\diamond_i$ problems.

At an intermediate point in time the survival probability up to time $t_n$ is given by

$$Q(t_n|y, t_i) = \int_{b(t_{i+1})}^{y_{i+1}} Q(t_n|x, t_{i+1}) g^+(x, t_{i+1}|y, t_i) \, dx$$

approximates

$$Q(t_n|y, t_i) \approx \int_{b(t_{i+1})}^{y_{i+1}} Q(t_n|x, t_{i+1}) g^+(x, t_{i+1}|y, t_i) \, dx + G^+(y_{i+1}^*, t_{i+1}|y, t_i), \text{ where}$$

$$G^+(y_{i+1}^*, t_{i+1}|y, t_i) := \int_{y_{i+1}^*}^{\infty} g^+(x, t_{i+1}|y, t_i) \, dx$$
Everything available in closed-from $\Rightarrow$ very fast, algorithm converges
Through CDS protection against default can be bought/sold

\[
V_{\text{fix}}(T) = \sum_{i=1}^{N} (T_i - T_{i-1})P(T_i|0)Q[\tau > T_i] \\
+ \int_{0}^{T} (s - T_{I(s)})P(s|0)dQ[\tau \leq s]
\]

\[
V_{\text{def}}(T) = (1 - R^Q) \int_{0}^{T} P(s|0)dQ[\tau \leq s].
\]

\[
s(T) = \frac{V_{\text{def}}(T)}{V_{\text{fix}}(T)}.
\]
We observe equity implied volatilities and estimates of the unconditional equity volatility

We observe term-structure of zero yields

We observe estimates of debt-to-asset ratios

Putting everything together we get estimates for instantaneous asset volatilities and instantaneous short rates as functions of time

We fit Nelson-Siegel polynomials to market data
## Implied Default Barrier

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### Model

- There are 5 canonical maturities for CDS contracts, 1y, 3y, 5y, 7y, 10y
- Barrier can be specified as function of 5 parameters
- Barrier is smooth function of time, and specific functional form admits sequential calibration, starting from 1y CDS
- All 5 liquid CDS contracts are fitted exactly
Daimler Chrysler Implied Default Barrier

Motivation

Model

Market Calibration

CDS
Input from the Market
Default Barrier

DCX Barrier
DCX Probs
TIT Barrier

Conclusion
Daimler Chrysler Survival Probabilities

Motivation

Model

Market Calibration
CDS
Input from the Market
Default Barrier
DCX Barrier
DCX Probs
TIT Barrier

Conclusion
What we have done so far:

- General structural model that nests Black and Cox (1976) as well as Brigo and Tarenghi (2005)
- Green function approach to solve the model; fast and efficient numerical procedure
- Prices are fitted via changing the shape of the default barrier

What is possible (future research):

- Extension to jump-to-default model
- Explanation of CDS premia written on investment-grade obligors