Nonstandard a posteriori error bounds in Euler-Galerkin schemes for parabolic problems with elliptic reconstruction techniques

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New Directions in Computational Partial Differential Equations

Warwick Mathematical Institute
Outline

1 Motivation
   - Introduction
   - A posteriori analysis
   - Before reconstruction

2 Elliptic reconstruction technique
   - A world without reconstruction is possible, but...
   - A user’s guide to the elliptic reconstruction
   - Remarks
   - Reconstruction vs. direct approach

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   - Duality estimates
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   - Allen–Cahn equation
   - Energy only estimates with reconstruction
   - Gradient recovery
   - Recovery energy estimates
   - Recovery energy estimates
   - Recovery energy estimates
   - Nonconforming methods (dG)
   - Nonconforming methods (DG)

Closing remarks
Aim

Derive a posteriori error estimates for

Problem (general linear diffusion)

Find $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ satisfying

(linear parabolic diffusion PDE) $\partial_t u + \mathcal{A} u = f$ in $\Omega \times (0, T] \subset \mathbb{R}^d \times \mathbb{R},$

(with initial Cauchy condition) $u(\cdot, 0) = u_0,$

(and Dirichlet boundary value) $u|_{\partial \Omega} = 0$ on $(0, T].$
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Derive \textit{a posteriori} error estimates for

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(linear parabolic diffusion PDE) \quad \partial_t u + \mathcal{A} u = f \text{ in } \Omega \times (0, T] \subset \mathbb{R}^d \times \mathbb{R},
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with elliptic operator $\mathcal{A}(t) : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ satisfying

$$\langle \mathcal{A}(t)v | \phi \rangle := a(v, \phi) := \int_{\Omega} \nabla \phi(x)^T a(x, t) \nabla v(x) \forall \phi \in H^1_0(\Omega),$$

$$\alpha_b(t) |w|^2 \leq w^T a(t) w \leq \alpha_#(t) |w|^2 \forall w \in \mathbb{R}^d$$

(e.g., $\mathcal{A}(t) = -\Delta$ and 0-Dirichlet BC).
Problem (weak formulation)

Find $u : [0, T] \rightarrow H^1_0(\Omega)$ such that

$$\langle \partial_t u, \phi \rangle + a(u, \phi) = \langle f, \phi \rangle \quad \forall \phi \in H^1_0(\Omega)$$

and

$u(0) = u_0 \in L^2(\Omega)$. 

Remarks

conforming method $\Rightarrow V_h \subseteq H^1_0(\Omega)$,

consistent method $\Rightarrow B(V, \Phi) = a(V, \Phi)$.

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Discrete Model Problem

weak form and spatial semidiscretization

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Problem (spatially discrete problem)

Find $U : [0, T] \rightarrow V_h$ such that

$$\langle \partial_t U, \Phi \rangle + B(U, \Phi) = \langle f, \Phi \rangle \quad \forall \Phi \in V^h \quad \text{and} \quad U(0) = \Pi^h u_0$$

($\Pi^h = L^2$-projection)
Problem (weak formulation)

Find $u : [0, T] \rightarrow H_0^1(\Omega)$ such that

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Remarks

- **conforming** method $\Rightarrow\ V_h \subseteq H_0^1(\Omega)$. 
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Problem (spatially discrete problem)

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Remarks

- **conforming** method \( \Rightarrow \quad V_h \subseteq H^1_0(\Omega), \)
- **consistent** method \( \Rightarrow \quad B(V, \Phi) = a(V, \Phi) \),
Find $U : [0, T] \rightarrow \mathbb{V}_h$ such that

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Problem (Fully discrete implicit Euler-Galéarkin FEM)

\((\mathcal{V}^n)_{n=0,...,N} \text{ a sequence of FE spaces, find } (U^n)_{n=0,...,N} \text{ such that} \)

\[
U^0 = \Pi^0 u_0 \quad \text{and} \quad \forall \, n \in [1 : N] : \]

\[
\langle \frac{U^n - U^{n-1}}{\tau_n}, \Phi \rangle + B(U^n, \Phi) = \langle f^n, \Phi \rangle , \quad \forall \, \Phi \in \mathcal{V}^n.
\]
Finite element spaces $\mathcal{V}^0, \mathcal{V}^1, \ldots, \mathcal{V}^N$ ...and their interaction
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- for each $n$, $\mathcal{T}_n$ be a partition of (aka mesh on) $\Omega$, into elements (simplexes/quadrilaterals/...),

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Nonstandard a posteriori parabolic estimates

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- for each $n$, $\mathcal{T}_n$ be a partition of (aka mesh on) $\Omega$, into elements (simplexes/quadrilaterals/$\ldots$),
- denote by $h_n$ the meshsize function of $\mathcal{T}_n$, 

\[ \forall K \in \mathcal{T}_n, K' \in \mathcal{T}_{n-1}, \quad K \cap K' = \emptyset \text{ or } K \subseteq K' \text{ or } K' \subseteq K, \]
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Example of (conforming) finite element space $p \in N$ and $V^n := \{\Phi \in C(\Omega) : \Phi |_K \text{poly of deg } p\}$. 

O Lakkis (Sussex) Nonstandard a posteriori parabolic estimates Warwick, 15 Jan 2009
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- many constants depend on shape-regularity
  $$\mu(\mathcal{T}_n) := \inf_{K \in \mathcal{T}_n} \sup_{B_\rho(x) \subseteq K} \rho / \text{diam } K,$$
Finite element spaces $V^0, V^1, \ldots, V^N$ … and their interaction

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- example of (conforming) finite element space
  \[ p \in \mathbb{N} \text{ and } V^n := \{ \Phi \in C(\Omega) : \Phi|_K \text{ poly of deg } p \}. \]
a posteriori estimates in general
first used in linear algebra 1960’s

Exact problem

Given $f$ find $u \in \mathcal{V}$ ($\dim \mathcal{V} = \infty$) such that $\lambda[u] = f$. 
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Approximate problem
Let $F \approx f$, $\Lambda \approx \lambda$ find $U \in \mathcal{V}$ ($\dim \mathcal{V} < \infty$) s.t. $\Lambda[U] = F$.

“Model” theorem
Error bound: There exists a computable estimator functional $E$ such that
\[ \|U - u\| \leq E[U; f, F; \lambda, \Lambda] \] (upper bound)
and
\[ E[U; f, F, \lambda, \Lambda] = O(\|U - u\|) \] (optimal order)

Main point: “estimator” $E$ is independent of exact solution $u$.

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a posteriori estimates in general
first used in linear algebra 1960's

**Exact problem**

Given \( f \) find \( u \in \mathcal{V} \) (\( \dim \mathcal{V} = \infty \)) such that \( \lambda[u] = f \).

**Approximate problem**

Let \( F \approx f, \Lambda \approx \lambda \) find \( U \in \mathcal{V} \) (\( \dim \mathcal{V} < \infty \)) s.t. \( \Lambda[U] = F \).

"Model" theorem

Error bound: There exists a **computable estimator functional** \( \mathcal{E} \) such that

(upper bound) \( \|U - u\| \leq \mathcal{E}[U; f, F; \lambda, \Lambda] \)

(optimal order) and \( \mathcal{E}[U; f, F, \lambda, \Lambda] = O(\|U - u\|) \).

Main point: "estimator" \( \mathcal{E} \) is **independent of exact solution** \( u \).
Subtract exact

\[ \langle \partial_t u, \phi \rangle + a(u, \phi) = \langle f, \phi \rangle \quad \forall \; \phi \in H^1_0(\Omega) \]

from residual, i.e., apply exact weak PDO on discrete solution,

\[ \langle \partial_t U, \phi \rangle + a(U, \phi) \].
Direct (violent) FEM a posteriori analysis

Error-Residual PDE

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\[ \langle \partial_t U, \phi \rangle + a(U, \phi) \]

2. Obtain error \((e = U - u)\) relation

\[ \langle \partial_t e, \phi \rangle + a(e, \phi) = \langle \partial_t U - f, \phi \rangle + a(U, \phi) =: \langle r | \phi \rangle \]

(weak) PDO on error

for all \( \phi \in H^1_0(\Omega) \).
Direct (violent) FEM a posteriori analysis

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(weak) PDO on error

for all \(\phi \in H^1_0(\Omega)\).

3. Briefly, error-residual PDE (generalized sense)

\[ \partial_t e + A e = r . \]
Remark (Galërkin orthogonality of residual)

Key property in analysis:

\[ \langle r \mid \Phi \rangle = \langle \partial_t U - f, \Phi \rangle + a(U, \Phi) = 0 \quad \forall \Phi \in V_h. \]
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Combined with error-residual PDE \((\partial_t e + \mathcal{A} e = r)\)

$$\langle \partial_t e, \phi \rangle + a(e, \phi) = \langle r | \phi - \Pi_h \phi \rangle \quad \forall \phi \in \mathcal{H}_0^1(\Omega),$$
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- \(\Pi_h : \mathbb{H}_0^1(\Omega) \rightarrow \mathbb{V}_h\) Clément-type interpolant:

\[ \|(\phi - \Pi_h \phi)/h\| \leq C_1 |\phi|_a \quad \text{and} \quad \|(\phi - \Pi_h \phi)/\sqrt{h}\|_\Sigma \leq C_2 |\phi|_a, \]
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\[ |\phi|_a = \| \nabla \phi \|. \]
Test with $\phi = e$ relation

\[
\langle \partial_t e, \phi \rangle + a(e, \phi) = \langle r | \phi - \Pi_h \phi \rangle, \quad \forall \phi \in H^1_0(\Omega).
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Test with $\phi = e$ relation

$$\langle \partial_t e, \phi \rangle + a(e, \phi) = \langle r | \phi - \Pi_h \phi \rangle, \quad \forall \phi \in H^1_0(\Omega).$$

Obtain

$$\frac{1}{2} d_t \|e\|^2 + |e|_a^2 = \langle r | e - \Pi_h e \rangle$$

$$= \langle R, e - \Pi_h e \rangle + \langle J, e - \Pi_h e \rangle_{\Sigma},$$

$$= \langle Rh, (e - \Pi_h e)/h \rangle + \langle J \sqrt{h}, (e - \Pi_h e)/\sqrt{h} \rangle_{\Sigma},$$
Test with $\phi = e$ relation

$$\langle \partial_t e, \phi \rangle + a(e, \phi) = \langle r | \phi - \Pi_h \phi \rangle, \ \forall \ \phi \in H^1_0(\Omega).$$

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Residual decomposition $r := J|_\Sigma + R$ where

$$R = R[U; f, \Pi^h f, \mathcal{A}] \text{ internal (regular) part of distribution } r$$

$$J = J[U; \mathcal{A}] \text{ jump (singular } \Sigma \text{-concentrated) part of } r$$
Energy-residual, interpolation and CBS inequalities yield

\[
\frac{1}{2} \frac{d}{dt} \| e \|^2 + |e|^2_a \leq \left( C_1 \| R h \| + C_2 \| J \sqrt{h} \|_{\Sigma} \right) |e|_a.
\]

\[
=: \mathcal{E}[U; f, \Pi^h f, A, \Sigma] = \mathcal{E}[U]
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Energy-residual, interpolation and CBS inequalities yield

\[ \frac{1}{2} \frac{d}{dt} \| e \|^2 + |e|^2_a \leq \left( C_1 \| Rh \| + C_2 \| J \sqrt{h} \|_{\Sigma} \right) |e|^a_a. \]

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Integrate in time to obtain final a posteriori error estimate

\[ \left( \| e(t) \|^2 + \int_0^t |e|^2_a \right)^{1/2} \leq C \int_0^t \mathcal{E}[U]. \]
Energy-residual, interpolation and CBS inequalities yield

$$\frac{1}{2} \Delta t \| e \|^2 + |e|^2_a \leq \left( C_1 \| Rh \| + C_2 \| J \sqrt{h} \|_{\Sigma} \right) |e|^a_a$$

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Pros: simple, straightforward, natural, optimal rate for $|e|^a_a$
Direct (violent) FEM a posteriori analysis (continued)

Heat energy estimate

- Energy-residual, interpolation and CBS inequalities yield
  \[ \frac{1}{2} \frac{d}{dt} \| e \|^2 + \| e \|_a^2 \leq \left( C_1 \| R h \| + C_2 \| J \sqrt{h} \|_{\Sigma} \right) \| e \|_a. \]
  \[ =: \mathcal{E}[U; f, \Pi^h f, \mathcal{A}, \Sigma] = \mathcal{E}[U] \]

- Integrate in time to obtain final a posteriori error estimate
  \[ \left( \| e(t) \|^2 + \int_0^t \| e \|_a^2 \right)^{1/2} \leq C \int_0^t \mathcal{E}[U]. \]

Pros: simple, straightforward, natural, optimal rate for \(| e |_a\)

Cons: limited to residual, limited to energy, inflexible, mixes the norms and the error indicators, suboptimal rate for \(\| e \|\), reinventing the wheel.
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Integrate in time to obtain final a posteriori error estimate
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**Pros**
- Simple, straightforward, natural, optimal rate for \( |e|_a \)

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Heat energy estimate

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Integrate in time to obtain final a posteriori error estimate

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**Pros**
- simple, straightforward, natural, optimal rate for $|e|_a$

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**Elliptic reconstruction’s purpose:**

Get rid of cons (and retain pros :-)

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Explicit a posteriori estimates for the heat equation
A one-slide (incomplete!) history

Duality techniques

- [Eriksson and Johnson, 1991] and many others,
- optimal order in $L_\infty(0, T; L_2(\Omega))$,
- serious mesh restrictions in many cases,
- no estimates for gradients.

Energy techniques

- [Picasso, 1998], [Chen and Jia, 2004], [Verfürth, 2003], [Bergam et al., 2005], [Bernardi and Süli, 2005] . . .
- optimal order in $L_2(0, T; H^1_0)$,
- suboptimal order in $L_2(\Omega)$ spaces,
- mesh change effects either absent or implicit (i.e., hidden in constants).
Anything that works for the **PDE stability analysis** is worth trying (especially for nonlinear problems) in time:

- Heat kernel estimates, e.g., to derive pointwise estimates [Demlow et al., vorg],
Other possible techniques

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3. Semigroup techniques, e.g., [Whalbin et al]
Anything that works for the **PDE stability analysis** is worth trying (especially for nonlinear problems) in time:

1. Heat kernel estimates, e.g., to derive pointwise estimates [Demlow et al., vorg],
2. Continuous dependence, useful in nonlinear problems [Feng and Wu, 2005]
3. Semigroup techniques, e.g., [Walbin et al]
4. Monotonicity estimates, e.g., [Nochetto et al., 2000]
Ultimate goal of a posteriori error estimates are: **error control** via **mesh-adaptive** methods.
Mesh adaption and a posteriori

- Ultimate goal of a posteriori error estimates are: **error control** via **mesh-adaptive** methods.

- Elliptic convergence thoroughly understood for linear problems [Nochetto, 2008, Binev et al., 2004, ?] and some nonlinear problems [Veeser 2007].
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Mesh adaptation and a posteriori

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- Tolerance reached **under termination assumption** [Chen and Jia, 2004].
- Proof of termination [Siebert et al].
- Convergence for wavelets on tensor meshes in space-time formulation [Schwab and Stevenson, 2008].
A world without reconstruction is possible, but one with it is better.

- Most work on parabolic a posteriori estimates is based on elliptic estimates in one way or another.
- Each elliptic technique (mainly residuals) is rederived at each attempt to derive parabolic estimates.
- Estimators are divided into “elliptic”, “time” and “mixed”.
- Fully discrete estimates can be very complicated.
- Why re-invent the wheel when elliptic a posteriori estimates can be read off the book, e.g., [Ainsworth and Oden, 2000]?
Energy **Optimal-rate** $L_2(\Omega)$-norm estimates via energy techniques

[McKridakis and Nochetto, 2003],

[Lakkis and Makridakis, 2006]. (Application is the use of $L_2$ estimates for Allen–Cahn simulations.)
What can Reconstruction buy us, that the direct approach wouldn’t?

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A wider range of problems and estimates via duality, following [Eriksson and Johnson, 1991] [Lakkis and Makridakis, 2007].
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Non-conforming FEMs  New **simple estimates in non-conforming methods**, e.g., [Georgoulis and Lakkis, vorg] for fully discrete...
A User’s Guide to the Elliptic Reconstruction

\[ u \in H^1_0(\Omega) \] exact solution

\[ H^1_0(\Omega) \]

\[ u \]

\[ V_h \]

parabolic

\[ V_h \text{-FE} \] approximation of \( u \).

Want an intermediate object \( w \) s.t. \( U \in V_h \) an elliptic \( V_h \text{-FE} \) approximation of \( w \), error \( \epsilon = U - u \), Galerkin orthogonality \( \Rightarrow \) a posteriori bounds on \( \|\epsilon\| \) are available off the shelf, \( w \) not computable, but parabolic error \( \rho = w - U \) satisfies parabolic equation with computable data.
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[Makridakis and Nochetto, 2003] and [Lakkis and Makridakis, 2006]

- $u \in H^1_0(\Omega)$ exact solution
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want $w$ intermediate object between $u$ and $U$ s.t.
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- $U \in V_h$ an elliptic $V_h$-FE approximation of $w$, error $\epsilon = U - u$.
\[ u \in H^1_0(\Omega) \text{ exact solution} \]
\[ U \in V_h \text{ parabolic } V_h\text{-FE approximation of } u. \]
\[ \text{want } w \text{ intermediate object between } u \text{ and } U \text{ s.t.} \]
\[ U \in V_h \text{ an elliptic } V_h\text{-FE approximation of } w, \text{ error } \epsilon = U - w, \]
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[ Makridakis and Nochetto, 2003] and [Lakkis and Makridakis, 2006]

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\[ U \in \mathbb{V}_h \] parabolic \( \mathbb{V}_h \)-FE approximation of \( u \).

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\[ H^1_0(\Omega) \]

\[ \mathbb{V}_h \]

Galerkin \( \perp \)
Crucial parabolic-elliptic $\rho$-$\epsilon$ error relation

Lemma (elliptic-parabolic error relation)

For each time-slab $I_n$, $n \in [1 : N]$, and $\phi \in H_0^1(\Omega)$,

$$
\langle \partial_t \rho \mid \phi \rangle + a(\rho, \phi) = \langle \partial_t \epsilon, \phi \rangle + a((w - w^n), \phi) 
+ \langle \Pi^nf^n - f, \phi \rangle + \tau_n^{-1} \langle \Pi^n U^{n-1} - U^{n-1}, \phi \rangle
$$

\[\iff\]

$$
\partial_t \rho + \mathcal{A} \rho = \partial_t \epsilon - \mathcal{A}(w - w^n) + (\Pi^n f^n - f) + \frac{\Pi^n U^{n-1} - U^{n-1}}{\tau_n}
$$

\(w^n := \mathcal{R}^n U\) elliptic reconstruction,

\(w(t)\) p.w.linear extension,

\(e := U - u\) total error = \(\begin{cases} 
\rho & \text{parabolic error}, \\
-\epsilon & \text{elliptic error}, 
\end{cases}\)

\(\Pi^n := L_2(\Omega)\)-projection onto \(\hat{\mathcal{V}}^n\).
Elliptic reconstruction: a user’s guide (controlling $\rho$)

$$\partial_t \rho + A \rho = \partial_t \epsilon + A(w - w^n) + \text{controlled terms}$$

control of the spatial error $\partial_t \epsilon$:

Use PDE for $\rho$ with $\partial_t \epsilon$ as data to obtain bound on $\|\rho\| < C$ $\|\partial_t \epsilon\|$.

$\partial_t \epsilon = \partial_t w - \partial_t U = R \partial_t U - \partial_t U$.

$\partial_t U$ elliptic $V_h$-FE solution with exact $R \partial_t U = \partial_t w \in H^1_0(\Omega)$.

$\Rightarrow \|\partial_t \epsilon\|$ elliptic error controlled a posteriori by estimator $E[\partial_t U, \partial_t f, V_h]$.

control of the time error $A(w - w^n)$:

Variety of methods depending on parabolic technique used. Example: use relation $\partial_t U + A w^n = \Pi f$ leads to explicit a posteriori representation $A w^n = \Pi f - \partial_t U$. 
Elliptic reconstruction: a user’s guide (controlling $\rho$)

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- Variety of methods depending on parabolic technique used.
- Example: use relation $\partial_t U + A w_n = \Pi_n f_n$ leads to explicit a posteriori representation $A w_n = \Pi_n f_n - \partial_t U$. 

O Lakkis (Sussex)
Elliptic reconstruction: a user’s guide (controlling $\rho$)

$\partial_t \rho + A \rho = \partial_t \epsilon + A(w - w^n) + \text{controlled terms}$

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control of the time error $A(w - w^n)$: Variety of methods depending on parabolic technique used. Example: use relation

$$\partial_t U + A w^n = \Pi^n f^n$$

leads to explicit a posteriori representation

$$A w^n := \Pi^n f^n - \partial_t U.$$
Towards an Energy estimate

- If interested only in energy estimates.
- Semidiscrete case
  \[
  \partial_t e + A \rho = 0.
  \]
- Fully discrete case
  \[
  \partial_t e + A \rho = \frac{(I^n U^{n-1} - U^{n-1})}{\tau_n} + f^n - f + \underbrace{A w - A^n w^n}_{\text{time, operator & mesh}}.
  \]

  data approximation/interpolation error
Comparison with direct approach

Error relation via reconstruction

\[ \langle \partial_t \rho \mid \phi \rangle + a (\rho, \phi) = \langle \partial_t \epsilon, \phi \rangle + a (w - w^n, \phi) + \langle \Pi^n f^n - f, \phi \rangle + \tau_n^{-1} \langle \Pi^n U^{n-1} - U^{n-1}, \phi \rangle \]

Direct (reconstructionless) error relation

\[ \langle \partial_t e, \phi \rangle + a (e, \phi) = \langle \partial_t U, \phi \rangle + a (U^n, \phi) + a (U - U^n, \phi) + \langle \Pi^n f^n - f, \phi \rangle + \tau_n^{-1} \langle \Pi^n U^{n-1} - U^{n-1}, \phi \rangle . \]
A useful analogy with the a priori

- ER is to **a posteriori** what Ritz/elliptic projection is to **a priori** analysis.
- **Optimal-order yielding properties** of ER can interpreted as an a posteriori analog of the similar phenomena of **superconvergence** observed in the a priori analysis with the Ritz projection.
- In spatially discrete (or very small timestep) case $\rho$ **converges with a higher order error than** $\epsilon$.
- However, $\rho$ **plays an important role** when time error is comparable to spatial error.
Energy-residual estimates in $L_2(H^1)$ and $L_\infty(L_2)$

**Lemma (optimal-order elliptic residual a posteriori estimates)**

*For all $V \in V^n$ we have*

$$|\mathcal{R}V - V|_{H^1(\Omega)} \leq \mathcal{E}[V, H^1(\Omega)] = O(h^n) \text{ Residual-type estimator}$$

$$\|\mathcal{R}V - V\|_{L_2(\Omega)} \leq \mathcal{E}[V, L_2(\Omega)] O(h^{2n}) \text{ Residual-type estimator on convex } \Omega$$

**Lemma (parabolic energy a posteriori estimate)**

*There are estimators $\mathcal{E}_1$ and $\mathcal{E}_2$ such that*

$$\left( \max_{[0,t_m]} \|\rho(t)\|_{L_2(\Omega)}^2 + 2 \int_0^{t_m} |\rho(t)|_a^2 \, dt \right)^{1/2} \leq \|\rho^0\| + C \left( \mathcal{E}_1,m + \mathcal{E}_2,m \right) .$$

$O(h_n)^2 \text{ (small } \tau_n)$
Remarks on \( \rho \)'s order of convergence

The space estimate of \( \rho \) derives from estimating the term

\[
\langle \partial_t \epsilon, \rho \rangle \leq \| \partial_t \epsilon \| \| \rho \| \\
\text{or } ( \| \partial_t \epsilon \|_{H^{-1}(\Omega)} |\rho|_a )
\]

Remark (superconvergence of \( \rho \))

The fact that energy norm \( |\rho|_a = \mathcal{O}(h^2_n) \) leads to optimal-order estimates for norms \( \| e \|_X = L_2(\Omega), L_\infty(\Omega) \)

Higher order energy-residual estimates
Norms of $H^1(L_2)$ and $L_\infty(H^1)$

Similar elliptic results, but higher norm (test by $\partial_t \rho$) yield

**Lemma (higher-order parabolic energy a posteriori estimate)**

There are estimators $\tilde{E}_i$, $i = 1, 2$, such that

$$\left( \max_{t \in [0,t_m]} |\rho(t)|^2_a + 2 \int_0^{t_m} \|\partial_t \rho\|^2 \right)^{1/2} \leq |\rho^0|_a + 4 \left( \tilde{E}^2_{1,m} + \tilde{E}^2_{2,m} \right)^{1/2},$$

Lead to higher order norms optimal-order energy estimates.
[Lakkis and Makridakis, 2006].
Duality estimates

**Definition (Error estimators)**

Suppose an a posteriori elliptic error estimator function $\mathcal{E}[\cdot, \cdot, \cdot]$ is available. Let

$$
\begin{align*}
\varepsilon_n & := \mathcal{E}[U^n, V^n, L^2(\Omega)], \\
\eta_n & := \mathcal{E}[\partial_t U^n, V^n \cap V^{n-1}, L^2(\Omega)], \\
\theta_n & := \begin{cases} \\
\frac{1}{2} \left\| \Pi^1 f^1 - \overline{\partial U^1} - A^0 U^0 \right\| & \text{for } n = 1, \\
\frac{1}{2} \left\| \partial \left( \Pi^n f^n - \overline{\partial U^n} \right) \right\| \tau_n & \text{for } n \in [2 : N],
\end{cases}
\end{align*}
$$
Duality estimates

Definition (Logarithmic factors)

\[ b_n = \begin{cases} 
\frac{1}{4} \log \left( \frac{T-t_{n-1}}{T-t_n} \right) & \text{for } n \in [1 : N - 1], \\
\frac{1}{8} & \text{for } n = N;
\end{cases} \]

\[ a_n = \begin{cases} 
\lambda \left( \frac{\tau_n}{T-t_n} \right) - \lambda \left( -\frac{\tau_{n+1}}{T-t_n} \right), & \text{for } n \in [0 : n - 2], \\
\lambda \left( \frac{\tau_{N-1}}{\tau_N} \right) - 1, & \text{for } n = N - 1,
\end{cases} \]

where

\[ \lambda(x) := \begin{cases} 
(1 + 1/x) \log(1 + x) & \text{for } |x| \in (0, 1), \\
1 & \text{for } x = 0.
\end{cases} \]
Duality estimates

Theorem (General explicit duality-based a posteriori error estimates)

\[
\| \rho(T) \| \leq \| e(0) \| + \left( \sum_{n=0}^{N-1} a_n \varepsilon_n^2 \right)^{1/2} + \eta_N \\
+ \left( \sum_{n=1}^{N} b_n \left( \| U^{n-1} - U^n \| + \eta_n \right)^2 \right)^{1/2} + \sqrt{\frac{\tau_N}{2}} \| \delta_N h_N \| \\
+ \left( \sum_{n=1}^{N-1} b_n \left\| \delta_n h_n^2 \right\|^2 \right)^{1/2} + \sum_{n=1}^{N} \tau_n \left\| \partial_t f \right\|_{L_1(I_n; L_2(\Omega))}
\]

[Lakkis and Makridakis, 2007]
We take $\mathcal{A} = -\Delta$ thus study:

$$\partial_t u - \Delta u = f, \ u(0) = u_0 \quad \text{and} \quad u\big|_{\Omega} = 0.$$  

Direct approach is messy and hard, especially for fully discrete case [Boman 2000, cf.].

Our result based on elliptic a posteriori estimates [Nochetto, 1995, Nochetto et al., 2006]

$$\|V - \mathcal{R}V\| \leq C (\log h_n)^2 \mathcal{E}_{\infty,0}[V, g, V^n] = O \left( (\log h_n h_n)^2 \right)$$

for residual-based a posteriori estimator functional $\mathcal{E}_{\infty,0}$. 
**L∞ estimates**

Main idea [Demlow et al., vorg]

### Theorem (semidiscrete estimates)

**Recalling** $e = U - u$, $\rho = R U - u$ and $\epsilon = R U - U$:

$$
\|e(t)\|_{L^\infty(\Omega)} \leq \|\epsilon(0)\|_{L^\infty(\Omega)} + \|\epsilon(t)\|_{L^\infty(\Omega)} + \int_0^t \|\partial_t \epsilon(s)\| \, ds,
$$

$$
\|e(t)\|_{L^\infty(\Omega)} \leq \|\epsilon(0)\|_{L^\infty(\Omega)} + \|\epsilon(t)\|_{L^\infty(\Omega)} + C_{p,q}(t) \|\partial_t \epsilon\|_{L^q(0,t; W_p^{-1}(\Omega))}
$$

for $p, q \in (2, \infty]$ and $d/p + 2/q < 1$.

### Proof’s idea.

Use heat kernel estimates to bound $\rho (= e + \epsilon)$ in terms of $\epsilon$ knowing

$$
\partial_t \rho - \Delta \rho = \partial_t \epsilon.
$$

Negative Sobolev useful for convex domains and $P^2$ or higher elements.
Theorem (fully discrete estimates)

\[ \|e(t_n)\|_{L_\infty(\Omega)} \leq \|e(0)\|_{L_\infty(\Omega)} + \text{data} + \sum_{n=1}^{N} \tau_n \|g^n - g^{n-1}\|_{L_\infty(\Omega)} \]

\[ + C \log(t_n/\tau_n) (\log h_n)^2 \max_{1 \leq n \leq N} \mathcal{E}_{\infty,0}[U^n, g^n, V^n] \]

\[ g^n := (U^n - U^{n-1}) / \tau_n - f^n = A^n - A^{n-1} + f^n - \Pi^n f^n. \]

Elliptic error estimators [Nochetto, 1995]

\[ \mathcal{E}_{\infty,0}[U^n, g^n, V^n] := \max_{K \in \mathcal{T}_n} \left( \| h_n^2 (g^n - \Delta U^n) \|_{L_\infty(K)} + \| h_n \|_{L_\infty(\partial K)} \right) \]
Known results by [Kessler et al., 2004] and [Feng and Wu, 2005] on the Allen-Cahn equation

\[ \partial_t u - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \]

yield a posteriori error estimates in energy norm with a polynomial (instead of exponential!) dependence on $1/\varepsilon$ exploiting special spectral properties of the linearised operator. (Un)fortunately rate is optimal only for energy norm in both results. Using the ER we provide optimal-order estimates for the $L_\infty(0, T; L_2(\Omega))$ norm based on identity

\[ \frac{1}{2} \, \text{d}_t \| \rho \|^2 - \lambda_0 \| \rho \|^2 = \langle \partial_t \varepsilon, \rho \rangle + \frac{1}{\varepsilon^2} \langle f(w) - f(U), \rho \rangle \]

Obtain an $\varepsilon$-free error.
Towards an Energy estimate

- If interested **only in energy estimates**.
- Semidiscrete case
  \[ \partial_t e + A \rho = 0. \]
- Fully discrete case
  \[ \partial_t e + A \rho = \left( \frac{I^n U^{n-1} - U^{n-1}}{\tau_n} \right) + f^n - f + A w - A^n w^n. \]

  - Data approximation/interpolation error
  - Time, operator & mesh
1. The parabolic-elliptic error relation \( \partial_t \rho + \mathcal{A} \rho = \partial_t \epsilon \) is sometime more useful written as

\[
\partial_t e + \mathcal{A} \rho = 0.
\]

2. Testing with \( e \) we obtain

\[
\frac{1}{2} \| e \|^2 + \| \rho \|_a^2 = a'(\rho, \epsilon).
\]

Continuity of \( a \) and integration in time yield estimate

\[
\int_0^t \| \rho \|_a^2 \leq C_a \int_0^t \| \epsilon \|_a^2.
\]

3. Note that superconvergence properties of \( \| \rho \|_a \) are lost (but not needed) in this context.

4. These simple observations and some elliptic technical work, lead to interesting nonstandard results: recovery estimators and nonconforming methods.
Theorem (Zienkiewicz-Zhou)

Let $G$ be the Zienkiewicz–Zhou gradient recovery operator. Then

$$\| V - RV \|_{H^1(\Omega)} \leq C \| \nabla V - GV \|.$$

Recovery-based estimates

[O Lakkis and Pryer, 2008]

Theorem (Zienkiewicz-Zhou)

Let $G$ be the Zienkiewicz–Zhou gradient recovery operator. Then

$$\|V - R V\|_{H^1(\Omega)} \leq C \|\nabla V - GV\|.$$

Lemma ((semidiscrete) energy norm parabolic-elliptic estimate)

$$\|e\|^2 + \int_0^t \|\rho\|^2_a \leq C \left( \|e(0)\|^2_a + \int_0^t \|\epsilon\|^2_a \right).$$
Recovery-based estimates

[Lakkis and Pryer, 2008]

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Let $G$ be the Zienkiewicz–Zhou gradient recovery operator. Then

$$\|V - \mathcal{R}V\|_{H^1(\Omega)} \leq C \|\nabla V - GV\|.$$

Lemma ((semidiscrete) energy norm parabolic-elliptic estimate)

$$\|e\|^2 + \int_0^t \|\rho\|^2_a \leq C \left(\|e(0)\|^2_a + \int_0^t \|\epsilon\|^2_a \right).$$

In combination lead to recovery based estimates for parabolic equations. (Related results by [Leykekhman and Wahlbin, 2006].)
Recovery estimates
Fully discrete estimators

- elliptic (gradient recovery error) estimator \( \varepsilon_n := \| \nabla U^n - G^n[U^n] \| \),
Recovery estimates
Fully discrete estimators

- elliptic (gradient recovery error) estimator $\varepsilon_n := \|\nabla U^n - G^n[U^n]\|$, 
- mesh-change (coarsening) error indicator $\gamma_n := \tau_n^{-1} \|I^n U^{n-1} - U^{n-1}\|$, 

Lemma (gradient recovery a posteriori error estimates)

$$\left( \max_{t \in [0, T]} \|e(t)\|^2 + 2 \int_0^T \|\rho(t)\|^2 \, dt \right)^{1/2} \leq \|e(0)\| C \left( N \sum_{n=1}^N (\beta_n \tau_n + \theta_n \tau_n + \gamma_n \tau_n) + 8 \int_0^T \|\epsilon\|^2 \right)^{1/2}.$$
Recovery estimates
Fully discrete estimators

- elliptic (gradient recovery error) estimator $\varepsilon_n := \| \nabla U^n - G^n[U^n] \|$,
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Recovery estimates
Fully discrete estimators

- elliptic (gradient recovery error) estimator \( \varepsilon_n := \| \nabla U^n - G^n[U^n] \| \),
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- time error indicator \( \theta_n := C \tau_n \| I^n U^{n-1} - U^n \|_a \)
- data approximation error indicator \( \beta_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} \| I^n f^n - f \| \).
Recovery estimates

Fully discrete estimators

- elliptic (gradient recovery error) estimator \( \varepsilon_n := \| \nabla U^n - G^n[U^n] \| \),
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Recovery estimates

Fully discrete estimators

- elliptic (gradient recovery error) estimator \( \varepsilon_n := \| \nabla U^n - G^n[U^n] \| \),
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- data approximation error indicator \( \beta_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} \| I^n f^n - f \| \).

Lemma (gradient recovery a posteriori error estimates)

\[
\left( \max_{t \in [0,T]} \| e(t) \|^2 + 2 \int_0^T \| \rho(t) \|^2_a \, dt \right)^{1/2} \\
\leq \| e(0) \| C \left( \sum_{n=1}^{N} (\beta_n \tau_n + \theta_n \tau_n + \gamma_n \tau_n) + 8 \int_0^T \| \varepsilon \|^2_a \right)^{1/2}.
\]
Convergence tests for ZZ-type estimators

[Lakkis and Pryer, 2008]

$P^1$ elements $\tau = h$

---

**EOC**

- EOC[$|e|$]$_{L^2(0, t_m; H^1)}$
- EOC[$\tau(\Sigma \epsilon_n^2 + \epsilon^2_{n-1})^{1/2}$]
- EOC[$\tau \Sigma_n \theta_n$]
- Effectivity Index

- $||e||_{L^2(0, t_m; H^1(\Omega))}$
- $\tau(\Sigma \epsilon_n^2 + \epsilon^2_{n-1})^{1/2}$
- $\tau \Sigma_n \theta_n$
- Combined Estimator
Convergence tests for ZZ-type estimators

[Lakkis and Pryer, 2008]

\( P^1 \) elements \( \tau = h^2 \)

\[ \text{EOC}[||e||_{L^2(0, t_m; H^1)}] \text{EOC}[\tau(\Sigma \epsilon^2_n + \epsilon^2_{n-1})^{1/2}] \text{EOC}[\tau \Sigma_n \theta_n] \text{Effectivity Index} \]

\[ ||e||_{L^2(0, t_m; H^1)} \tau(\Sigma \epsilon^2_n + \epsilon^2_{n-1})^{1/2} \tau \Sigma_n \theta_n \text{Combined Estimator} \]
Convergence tests for ZZ-type estimators

\[ P^2 \text{ elements } \tau = h^3 \]
ZZ Adaptive Mesh Refinement

[Lakkis and Pryer, 2008]
### 'Easy" (regular) problem

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### Space-error dominated problem

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Recovery adaptive method

Time-error dominated problem (implicit strategy)

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Time-error dominated problem (explicit strategy)

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Discontinuous Galerkin (DG) bilinear form discretizing $\Delta$ is given by

$$B(w, v) := \sum_{K \in \mathcal{T}} \int_K \nabla w \nabla v + \int_{\Gamma} (\theta \{\nabla v\} [w] - \{\nabla w\} [v] + \sigma [w] [v]) \, ds$$

where $[\phi] := \phi^+_S n^+ + \phi^- n^-$ is the jump of the scalar $\phi$, and $\{\phi\} := (\phi^+ + \phi^-)/2$ is the average of field $\phi$, across the triangulations sides; $\theta, \sigma$ are method, penalty parameters, respectively.

DG space with respect to triangulation $\mathcal{T}$, with no hanging nodes restriction:

$$V_h = \{ v \in L^2(\Omega) : v|_K \in \mathbb{P}^p \}$$

where $p$ is a fixed polynomial degree.
Definition (the DG elliptic reconstruction)

Let \( U_{DG}(t) \) be the (semidiscrete) DG solution at time \( t \in [0, T] \), define **elliptic reconstruction** \( w(t) \in H^1_0(\Omega) \), of \( U_{DG}(t) \) solves elliptic problem

\[
A w(t) = g(t) \quad \forall \ t \in [0, T],
\]

where

\[
g(t) := AU_{DG}(t) + f - \Pi f,
\]

and \( A : \mathcal{V}_h \to \mathcal{V}_h \) is the discrete DG-operator defined by for \( V \in \mathcal{V}_h \) by

\[
\langle AV, \Phi \rangle = B(V, \Phi) \quad \forall \ \Phi \in \mathcal{V}_h.
\]
Here \( w \in H^1_0(\Omega) \) and \( u \in H^1_0(\Omega) \) \( \Rightarrow \rho \in H^1_0(\Omega) \). Define **discontinuous part**:

\[
e_d := U_d := \epsilon_d,
\]

and its **continuous part**:

\[
e_c := e - e_d = e - U_d = \rho + \epsilon_c.
\]

**Lemma (basic nonconforming energy estimate)**

\[
\frac{1}{2} \frac{d}{dt} \| e_c \|^2 + \| \rho \|^2_a = B(\epsilon_c, \rho) + \langle \partial_t U_d, e_c \rangle + l_{n-1} \langle A^{n-1} U^{n-1} - A^n U^n, e_c \rangle + \langle (I^n U^{n-1} - U^{n-1}) / \tau_n + f^n - f, e_c \rangle.
\]

- **elliptic**
- **nonconforming**
- **time & mesh-change**
- **data & mesh-change**
Closing remarks and outlook

Conclusions

- Elliptic reconstruction unifies known a posteriori analysis (with Makridakis, Nochetto).
- Leads to new optimal-order estimates (with Demlow, Makridakis).
- Rigorous justification the use of recovery techniques (with Pryer).
- Nonconforming methods (with Georgoulis).

Current developments (new directions?)

- Wave equation (with Georgoulis & Makridakis).
- Semilinear equations, e.g., Allen–Cahn (with Georgoulis & Makridakis), with applications to stochastic Monte-Carlo simulations (with Katsoulakis, Kossioris & Romito).
- Quasilinear equations, e.g., MCF of function graphs.

Taxpayer’s support

O Lakkis (Sussex)

Marie Curie/HYKE (GR/EU), Nuffield Foundation (UK), EPSRC (UK), Hausdorff Institute Bonn (DE).


A posteriori error estimates by recovered gradients in parabolic finite element equations.
Technical report, University of Texas, Austin.
Preprint (submitted to Math. Comp.).

Elliptic reconstruction and a posteriori error estimates for parabolic problems.

Pointwise a posteriori error estimates for elliptic problems on highly graded meshes.


