Numerical Approximation of Gradient Flows for Curve Networks and Surface Clusters

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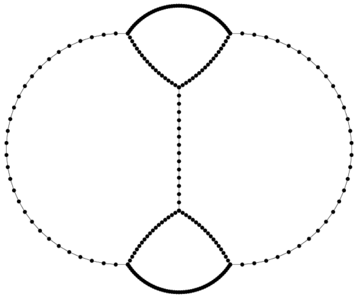
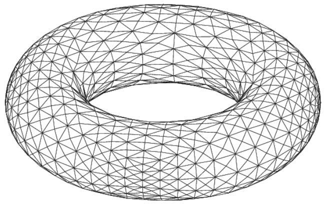
Introduction

In this talk, we consider *Geometric Evolution Equations* for

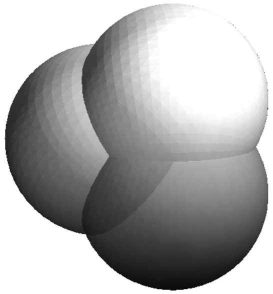
- Curves



- Curve Networks



- Surfaces



- Surface Clusters

In each case, the flow will be given as a *Gradient Flow*, of second or fourth order, of the (weighted) surface area functional, e.g.

$$E(\Gamma) = |\Gamma| := \int_\Gamma 1 \, \mathrm{d}s \,.$$

Planar Curvature Flows

Evolving simple (embedded - no intersections) planar closed curve $\Gamma(t)$.

Let $\vec{x}(\rho,t)$, $\quad \rho \in I := \mathbb{R}/\mathbb{Z}$ (periodic $[0,1]$), be a parameterization of $\Gamma(t)$.

Arclength $\quad s \quad \Rightarrow \quad$ unit tangent $\quad \vec{\tau} = \vec{x}_s = \dfrac{\vec{x}_\rho}{|\vec{x}_\rho|}$.

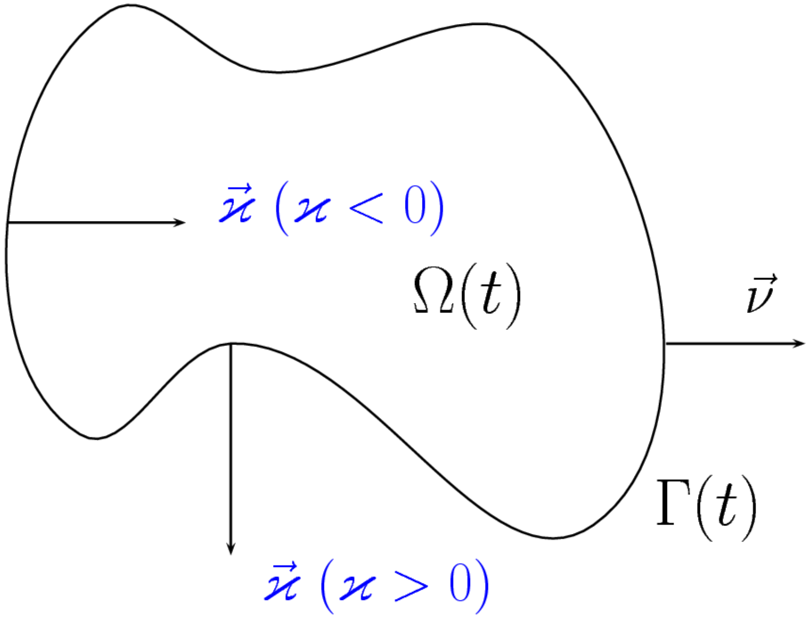
Curvature vector $\quad \vec{\varkappa} = \vec{\tau}_s = \vec{x}_{ss} = \dfrac{1}{|\vec{x}_\rho|}\left(\dfrac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho$.

Unit normal $\quad \vec{\nu} \quad \Rightarrow \quad \vec{\varkappa} \equiv \varkappa\,\vec{\nu}$, $\quad$ where $\varkappa$ is the curvature.

Let $\Omega(t)$ be the region bounded by $\Gamma(t)$.

If $\vec{\nu}$ is the outward normal, then $\varkappa$ is negative if $\Omega(t)$ is locally convex.

Planar Curvature Flows



For the evolution of $\Gamma(t)$, it suffices to prescribe its normal velocity

$$\mathcal{V} \equiv \vec{x}_t \,.\, \vec{\nu}.$$

Note that tangential velocities just change the parameterization $\vec{x}(\rho, t)$.

Mean curvature flow: $\mathcal{V} = \varkappa$ (MC)

Surface diffusion: $\mathcal{V} = -\varkappa_{ss}$ (SD)

Energy Decrease

These evolution equations have important applications in e.g. Materials Science, and they have the following properties.

$$\frac{d}{dt}|\Gamma(t)| = -\int_{\Gamma(t)} \mathcal{V}\,\varkappa\,\mathrm{d}s = \begin{cases} -\displaystyle\int_{\Gamma(t)} \varkappa^2\,\mathrm{d}s \equiv -\|\mathcal{V}\|^2_{L^2(\Gamma(t))} & \leq 0 \qquad \text{(MC)}, \\ -\displaystyle\int_{\Gamma(t)} (\varkappa_s)^2\,\mathrm{d}s \equiv -\|\mathcal{V}\|^2_{H^{-1}(\Gamma(t))} & \leq 0 \qquad \text{(SD)}. \end{cases}$$

(MC) $\mathcal{V} = \varkappa$ is the L^2 gradient flow for the energy $|\Gamma(t)|$.
(Curve-Shortening Flow).

(SD) $\mathcal{V} = -\varkappa_{ss}$ is the H^{-1} gradient flow for the energy $|\Gamma(t)|$.

$$\frac{d}{dt}|\Omega(t)| = \int_{\Gamma(t)} \mathcal{V}\,\mathrm{d}s = \begin{cases} \displaystyle\int_{\Gamma(t)} \varkappa\,\mathrm{d}s & = -2\pi \qquad \text{(MC)}, \\ -\displaystyle\int_{\Gamma(t)} \varkappa_{ss}\,\mathrm{d}s & = 0 \qquad \text{(SD)}. \end{cases}$$

Existing Parametric Approximations

There exist parametric finite element approximations of (MC) and (SD). They are based on the following formulations.

- Dziuk (91) (MC) $\mathcal{V} \equiv \vec{x}_t \,.\, \vec{\nu} = \varkappa$

  $\vec{x}_t = \varkappa\,\vec{\nu}$ and $\vec{\varkappa} = \varkappa\,\vec{\nu} \equiv \vec{x}_{ss}$ $\Rightarrow$ $\vec{x}_t = \vec{x}_{ss} \equiv \dfrac{1}{|\vec{x}_\rho|}\left(\dfrac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho .$

- Dziuk, Kluwert, Schätzle (02) (SD)

  $\vec{x}_t = -\varkappa_{ss}\,\vec{\nu} \equiv -\vec{\varkappa}_{ss} - \frac{3}{2}\,(|\vec{\varkappa}|^2\,\vec{x}_s)_s + \frac{1}{2}\,|\vec{\varkappa}|^2\,\vec{\varkappa}, \qquad \vec{\varkappa} = \vec{x}_{ss}\,.$

- Bänsch, Morin, Nochetto (05) (SD)

  $\vec{x}_t = \vec{\mathcal{V}} = \mathcal{V}\,\vec{\nu}, \qquad \mathcal{V} = -\varkappa_{ss}, \qquad \varkappa = \vec{\varkappa}\,.\,\vec{\nu}, \qquad \vec{\varkappa} = \vec{x}_{ss}\,.$

All these approaches have in common that they evolve the parameterization $\vec{x}$ only in the *normal* direction.

This can lead to **coalesence** of mesh points in practice,
and the need for remeshing.

Parametric Finite Element Approximation for (MC)

Dziuk (91): a semi-implicit approximation using piecewise linear elements.

Let $I \equiv \mathbb{R}/\mathbb{Z} = \cup_{j=1}^{N} J_j$, $N \geq 3$, partitioned into intervals $J_j = [q_{j-1}, q_j]$.

$$\underline{V}^h := \{\vec{\chi} \in C(I, \mathbb{R}^2) : \vec{\chi}|_{J_j} \text{ is linear } \forall\, j = 1 \to N\} =: [V^h]^2 \subset H^1(I, \mathbb{R}^2)$$

$0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ partitioning of $[0, T]$, $\tau_m := t_{m+1} - t_m$, $m = 0 \to M - 1$.

$\vec{X}^m \in \underline{V}^h$ approximating $\vec{x}(\cdot, t_m) \Rightarrow$ a polygonal approximation, Γ^m, to $\Gamma(t_m)$.

Find $\vec{X}^{m+1} \in \underline{V}^h$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \vec{\eta} \rangle_m^h = -\langle \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_m \equiv -\int_I \frac{\vec{X}_\rho^{m+1}}{|\vec{X}_\rho^m|} \cdot \frac{\vec{\eta}_\rho}{|\vec{X}_\rho^m|} |\vec{X}_\rho^m| \,\mathrm{d}\rho \qquad \forall\, \vec{\eta} \in \underline{V}^h;$$

where

$$\langle f, g \rangle_m := \int_{\Gamma^m} f \,.\, g \,\mathrm{d}s = \int_I f \,.\, g\, |\vec{X}_\rho^m| \,\mathrm{d}\rho$$

with $\langle \cdot, \cdot \rangle_m^h$ the mass lumped inner product.

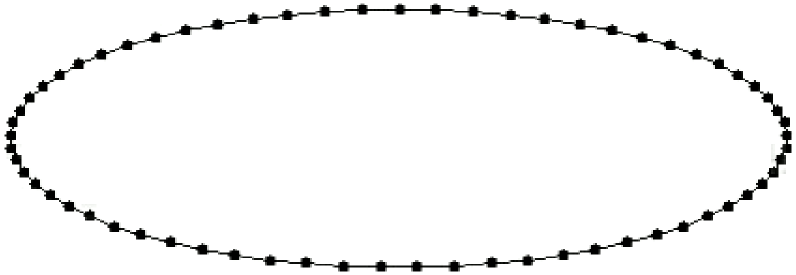
Linear tridiagonal periodic system for $\{\vec{X}^{m+1}(q_j)\}_{j=0}^{N}$.

Error analysis for semidiscrete approximation (continuous in time) Dziuk (94).

Coalesence

(SD) $\vec{x}(\cdot, 0)$ is a $3:1$ ellipse, $N = 64$, $\tau = 10^{-6}$, $T = 3 \times 10^{-4}$.

Recall (SD) $\Rightarrow$ $|\Gamma(t)|$ decreases, whilst $|\Omega(t)| = |\Omega(0)|$.



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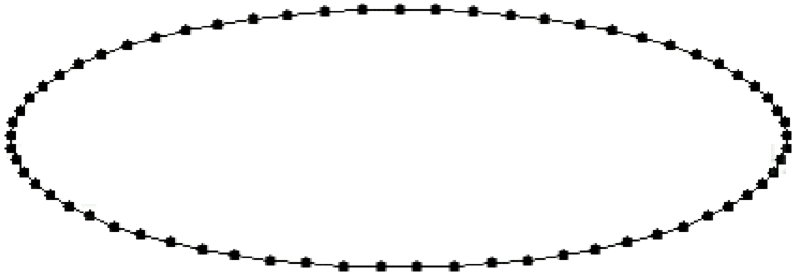

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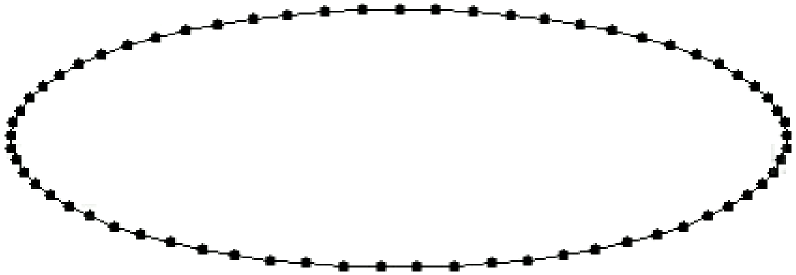


BGN

New Formulation

We use the following formulation of (MC) and (SD).

$$\vec{x}_t \,.\, \vec{\nu} = \begin{cases} \varkappa \\ -\varkappa_{ss} \end{cases}, \qquad \varkappa\,\vec{\nu} = \vec{x}_{ss}\,.$$

(MC) $\Rightarrow$ $(\vec{x}_t \,.\, \vec{\nu})\,\vec{\nu} = \vec{x}_{ss}\,.$

As the tangential component $\vec{x}_t \,.\, \vec{\tau}$ of $\vec{x}_t$ is not prescribed, $\exists$ a whole family of solutions $\vec{x}$; even though the evolution of Γ is uniquely determined.

Find $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h \times V^h$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\,\vec{\nu}^m \rangle_m^h - \begin{cases} \langle \kappa^{m+1}, \chi \rangle_m^h \\ \langle \kappa_s^{m+1}, \chi_s \rangle_m \end{cases} = 0 \qquad \forall\ \chi \in V^h,$$

$$\langle \kappa^{m+1}\,\vec{\nu}^m, \vec{\eta} \rangle_m^h + \langle \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h.$$

Linear system for $\{\vec{X}^{m+1}(q_j), \kappa^{m+1}(q_j)\}_{j=0}^N$.

(MC) $\Rightarrow$ Find $\vec{X}^{m+1} \in \underline{V}^h$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m} \,.\, \vec{\omega}^m, \vec{\eta} \,.\, \vec{\omega}^m \rangle_m^h + \langle \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h.$$

Existence and Stability

A very mild assumption on $\vec{X}^m$ needed. Let $\vec{\nu}^m_{j-\frac{1}{2}} := -\frac{[\vec{X}^m(q_j)-\vec{X}^m(q_{j-1})]^\perp}{|\vec{X}^m(q_j)-\vec{X}^m(q_{j-1})|}$.

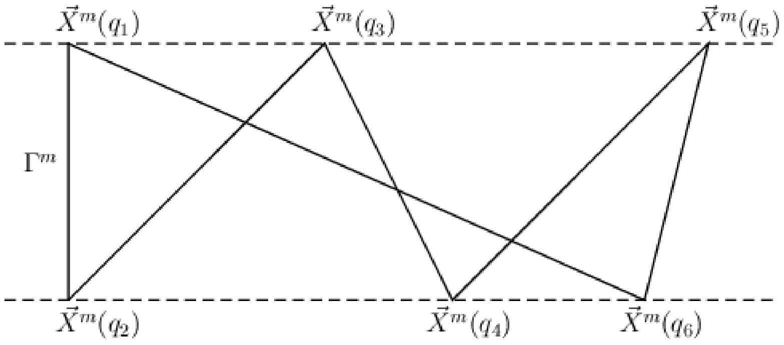
($\mathcal{A}$) Let $|\vec{X}^m_\rho| > 0$ for a.a. $\rho \in I$. For $j = 1 \to N$, introduce the vertex normal (average of neighbouring edge unit normals weighted by edge length)

$$\vec{\omega}^m_j := \frac{|\vec{X}^m(q_j)-\vec{X}^m(q_{j-1})|\,\vec{\nu}^m_{j-\frac{1}{2}}+|\vec{X}^m(q_{j+1})-\vec{X}^m(q_j)|\,\vec{\nu}^m_{j+\frac{1}{2}}}{|\vec{X}^m(q_j)-\vec{X}^m(q_{j-1})|+|\vec{X}^m(q_{j+1})-\vec{X}^m(q_j)|}$$

$$= -\tfrac{1}{\alpha}\,[\vec{X}^m(q_{j+1}) - \vec{X}^m(q_{j-1})]^\perp\,,$$ where $\cdot^\perp$ is clockwise rotation by $\frac{\pi}{2}$.

Then we further assume that $\dim \operatorname{span}\{\vec{\omega}^m_j\}_{j=1}^N = 2$.

Always true if no intersections.



A pathological example, ($\mathcal{A}$) violated.

THEOREM

Let ($\mathcal{A}$) hold. Then $\exists!$ solution $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h \times V^h$. Moreover, for $k = 1 \to M$ it holds that

$$|\Gamma^k| + \sum_{m=0}^{k-1} \tau_m \left\{\begin{array}{l} \langle \kappa^{m+1}, \kappa^{m+1} \rangle^h_m \\ \langle \kappa^{m+1}_s, \kappa^{m+1}_s \rangle_m \end{array}\right. \leq |\Gamma^0|\,.$$

Planar Equidistribution of Mesh Points

Continuous in time semidiscrete scheme:

$$\langle \vec{X}_t, \chi\,\vec{\nu}^h\rangle^h - \begin{cases} \langle \kappa, \chi\rangle^h \\ \langle \kappa_s, \chi_s\rangle \end{cases} = 0 \qquad \forall\,\chi \in V^h,$$

$$\langle \kappa\,\vec{\nu}^h, \vec{\eta}\rangle^h + \langle \vec{X}_s, \vec{\eta}_s\rangle = 0 \qquad \forall\,\vec{\eta} \in \underline{V}^h.$$

(a) Area conservation for (SD): $0 = \langle \vec{X}_t, \vec{\nu}^h\rangle^h = \int_{\Gamma^h} \vec{X}_t\,.\,\vec{\nu}^h\,\mathrm{d}s = \frac{d}{dt}|\Omega^h(t)|\,.$

(b) Equidistribution of mesh points for $\vec{X}(t)$,
where $\vec{X}(t)$ is not locally parallel, for any $t > 0$;

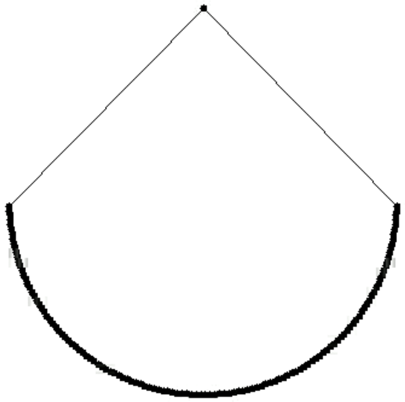
i.e. for $j = 1 \to N$ and for any $t > 0$

either $\quad |\vec{X}(q_j,t) - \vec{X}(q_{j-1},t)| = |\vec{X}(q_{j+1},t) - \vec{X}(q_j,t)|$

or $\quad [\vec{X}(q_j,t) - \vec{X}(q_{j-1},t)] \parallel [\vec{X}(q_{j+1},t) - \vec{X}(q_j,t)]\,.$

Planar Tangential Motion

Although (b) cannot be shown for the fully discrete scheme, (eventual) equidistribution is observed in practice.



(SD)

$N = 128$, $\tau = 10^{-7}$, $T = 2 \times 10^{-5}$.

There do exist schemes with tangential motion.
Note Dziuk scheme for (MC) based on

$$\vec{x}_t = \vec{x}_{ss} \equiv \frac{1}{|\vec{x}_\rho|}\left(\frac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho \equiv \frac{1}{|\vec{x}_\rho|^2}\left(\vec{x}_{\rho\rho} - [\,\vec{x}_{\rho\rho}\,.\,\vec{\tau}\,]\,\vec{\tau}\right) \quad \text{where} \quad \vec{\tau} := \frac{\vec{x}_\rho}{|\vec{x}_\rho|}$$

has no tangential motion, whereas the reparameterization $|\vec{x}_\rho|^2\,\vec{x}_t = \vec{x}_{\rho\rho}$ does.
Error analysis for a semidiscrete approximation Deckelnick, Dziuk (95).

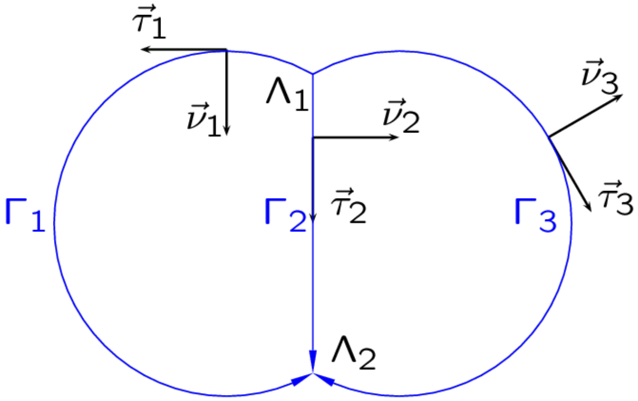
Strain (89) for (MC) uses formulation that intrinsically maintains $|\vec{x}_\rho(\rho, t)| = |\Gamma(t)|$. Extended by Hou, Lowengrub, Shelley (94) to e.g. $|\vec{x}_\rho(\rho, t)| = \alpha(\rho)\,|\Gamma(t)|$.

Mikula, Ševčovič (01) impose explicitly non-local tangential motion for the equidistribution of mesh points for (MC). See (05) for (SD).

Curve Networks

Geometric evolution equations for curve networks, e.g. three curves $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3)$ meeting at two triple junction points Λ_1 and Λ_2.

On Γ_i: $$\mathcal{V}_i := (\vec{x}_i)_t \,.\, \vec{\nu}_i = \begin{cases} \varkappa_i & \text{(MC)} \\ -(\varkappa_i)_{ss} & \text{(SD)} \end{cases}$$



At Λ_j: (a) $\vec{x}_1 = \vec{x}_2 = \vec{x}_3$ (attachment)

(b) $\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = \vec{0}$ (Young's Law)

+ for (SD): (c) $\varkappa_1 + \varkappa_2 + \varkappa_3 = 0$ (chem. pot. cont.)

(d) $(\varkappa_1)_s = (\varkappa_2)_s = (\varkappa_3)_s$ (flux balance)

(MC) :
- Evolution of phase boundaries in Materials Science, e.g. grain growth. See e.g. Bronsard, Reitich (93).
- Existing direct approaches
  Bronsard, Wetton (95), Thaddey (99), Neubauer (02).

(SD) :
- Formulation due to Garcke, Novick-Cohen (00).
- Models e.g. the evolution of soap bubbles.
- No previous work on direct numerical approximation.

Energy Decrease

As before, these evolution equations arise as gradient flows for the total surface area. Let $E(\Gamma) = |\Gamma| := \sum_{i=1}^{3} \int_{\Gamma_i} 1 \, \mathrm{d}s$. Then

$$\frac{d}{dt}|\Gamma| = -\sum_{i=1}^{3} \int_{\Gamma_i} \mathcal{V}_i \, \varkappa_i \, \mathrm{d}s = \begin{cases} -\sum_{i=1}^{3} \int_{\Gamma_i} \varkappa_i^2 \, \mathrm{d}s \equiv -\|\mathcal{V}\|^2_{L^2(\Gamma)} & \leq 0 \qquad \text{(MC)}, \\ -\sum_{i=1}^{3} \int_{\Gamma_i} ([\varkappa_i]_s)^2 \, \mathrm{d}s \equiv -\|\mathcal{V}\|^2_{H^{-1}(\Gamma)} & \leq 0 \qquad \text{(SD)}. \end{cases}$$

(MC) $\mathcal{V}_i = \varkappa_i$, $i = 1 \to 3$, and (a,b) is the L^2 gradient flow for $|\Gamma|$.

(SD) $\mathcal{V}_i = -(\varkappa_i)_{ss}$, $i = 1 \to 3$, and (a–d) is the H^{-1} gradient flow.

For (SD), the enclosed areas are conserved. E.g. for the area $a_{12}(t)$ of the phase enclosed by Γ_1 and Γ_2, it holds that

$$\frac{d}{dt} a_{12}(t) = \int_{\Gamma_2} \mathcal{V}_2 \, \mathrm{d}s - \int_{\Gamma_1} \mathcal{V}_1 \, \mathrm{d}s = -\int_{\Gamma_2} (\varkappa_2)_{ss} \, \mathrm{d}s + \int_{\Gamma_1} (\varkappa_1)_{ss} \, \mathrm{d}s = 0,$$

Variational Formulation of (SD) for Curve Networks

Let $I := [0,1]$ and set

$$\underline{V} := \{\vec{\chi} \equiv (\vec{\chi}_1, \vec{\chi}_2, \vec{\chi}_3) \in [C(I,\mathbb{R}^2)]^3 : \vec{\chi}_1 = \vec{\chi}_2 = \vec{\chi}_3 \text{ on } \partial I\}$$

and $W := \{\chi \equiv (\chi_1, \chi_2, \chi_3) \in [C(I,\mathbb{R})]^3 : \chi_1 + \chi_2 + \chi_3 = 0 \quad \text{on } \partial I\}$.

Using the following formulation of (SD)

$$\text{(i)} \quad (\vec{x}_i)_t \,.\, \vec{\nu}_i = -(\varkappa_i)_{ss}, \qquad \text{(ii)} \quad \varkappa_i \, \vec{\nu}_i = (\vec{x}_i)_{ss}\,,$$

and multiplying (ii) with $\vec{\eta} \in \underline{V} \cap [H^1(I,\mathbb{R}^2)]^3$ gives for the right hand side

$$\begin{aligned} \sum_{i=1}^{3} \int_{\Gamma_i} (\vec{x}_i)_{ss} \,.\, \vec{\eta}_i \, \mathrm{d}s &= -\sum_{i=1}^{3} \int_{\Gamma_i} (\vec{x}_i)_s \,.\, (\vec{\eta}_i)_s \, \mathrm{d}s + \sum_{i=1}^{3} \int_{\partial\Gamma_i} (\vec{x}_i)_s \,.\, \vec{\eta}_i \, \mathrm{d}s \\ &= -\sum_{i=1}^{3} \int_{\Gamma_i} (\vec{x}_i)_s \,.\, (\vec{\eta}_i)_s \, \mathrm{d}s + \sum_{j=1}^{2} (-1)^j \, \vec{\eta}(\Lambda_j) \,.\, \sum_{i=1}^{3} (\vec{\tau}_i|_{\Lambda_j})\,. \end{aligned}$$

$\Rightarrow$ Weakly enforce $\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = \vec{0}$ at Λ_j, while $\vec{x}_1 = \vec{x}_2 = \vec{x}_3$ holds via $\vec{x} \in \underline{V}$.

Similarly, testing (i) with $\chi \in W \quad \Rightarrow \quad$ weak enforcing of $(\varkappa_1)_s = (\varkappa_2)_s = (\varkappa_3)_s$.

Parametric Finite Element Approximation

Introduce piecewise linears $\underline{V}^h \subset \underline{V}$, $W^h \subset W$ and $W^h_{\mathcal{M}} \subset [C(I,\mathbb{R})]^3$.

Find $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h \times \begin{cases} W^h_{\mathcal{M}} \\ W^h \end{cases}$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\, \vec{\nu}^m \rangle^h_m - \begin{cases} \langle \kappa^{m+1}, \chi \rangle^h_m \\ \langle \kappa^{m+1}_s, \chi_s \rangle_m \end{cases} = 0 \qquad \forall\ \chi \in \begin{cases} W^h_{\mathcal{M}} \\ W^h \end{cases} ,$$

$$\langle \kappa^{m+1}\, \vec{\nu}^m, \vec{\eta} \rangle^h_m + \langle \vec{X}^{m+1}_s, \vec{\eta}_s \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h;$$

where

$$\langle f, g \rangle_m := \int_{\Gamma^m} f \,.\, g \; \mathrm{d}s := \sum_{i=1}^{3} \int_{\Gamma^m_i} f_i \,.\, g_i \; \mathrm{d}s$$

with $\langle \cdot, \cdot \rangle^h_m$ the mass lumped inner product.

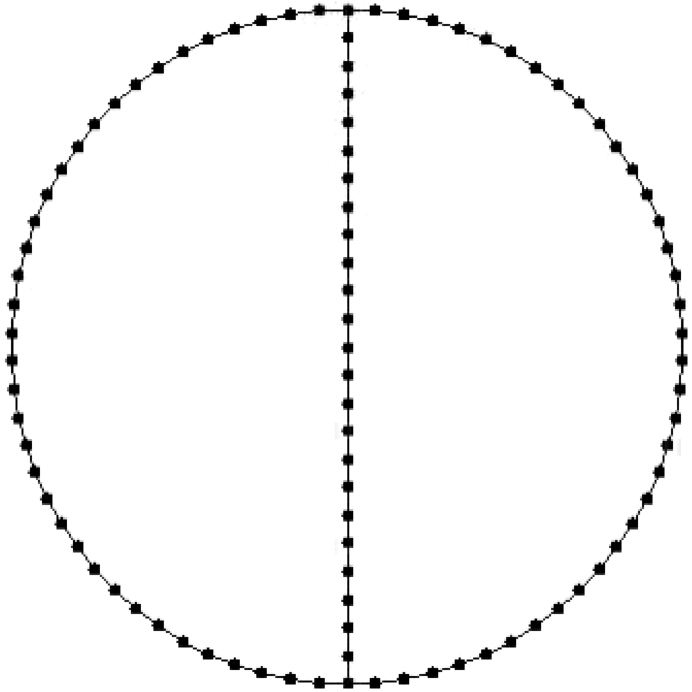
Linear system for $\{\vec{X}^{m+1}, \kappa^{m+1}\}$.

Existence, uniqueness and stability. Area conservation and equidistribution of mesh points (separate on each Γ^m_i) as before.

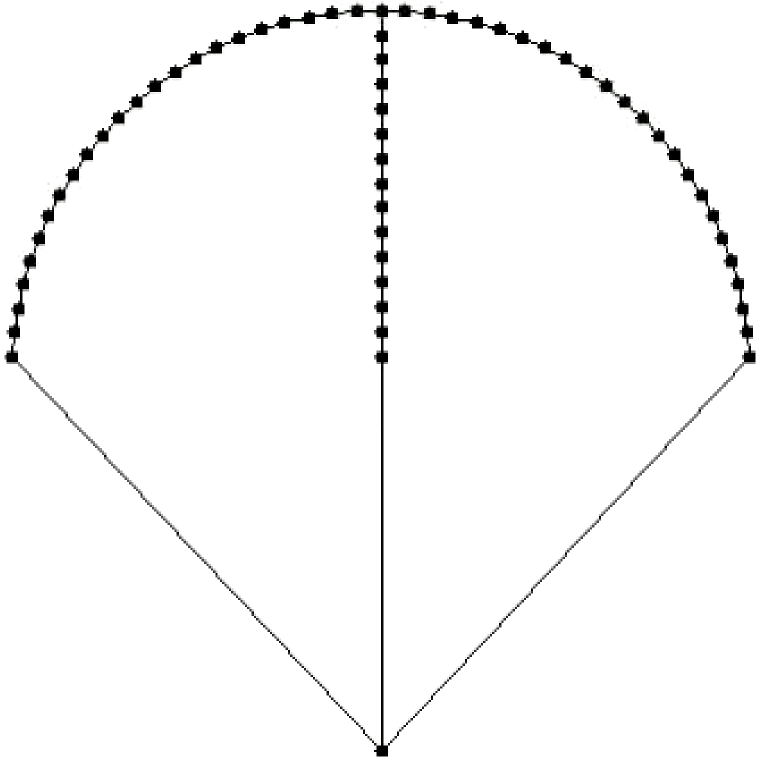
Tangential Movement

Standard double bubble

$N = 100$, $\tau = 10^{-3}$, $T = 0.2$.



$N = 64$, $\tau = 10^{-3}$, $T = 0.2$.

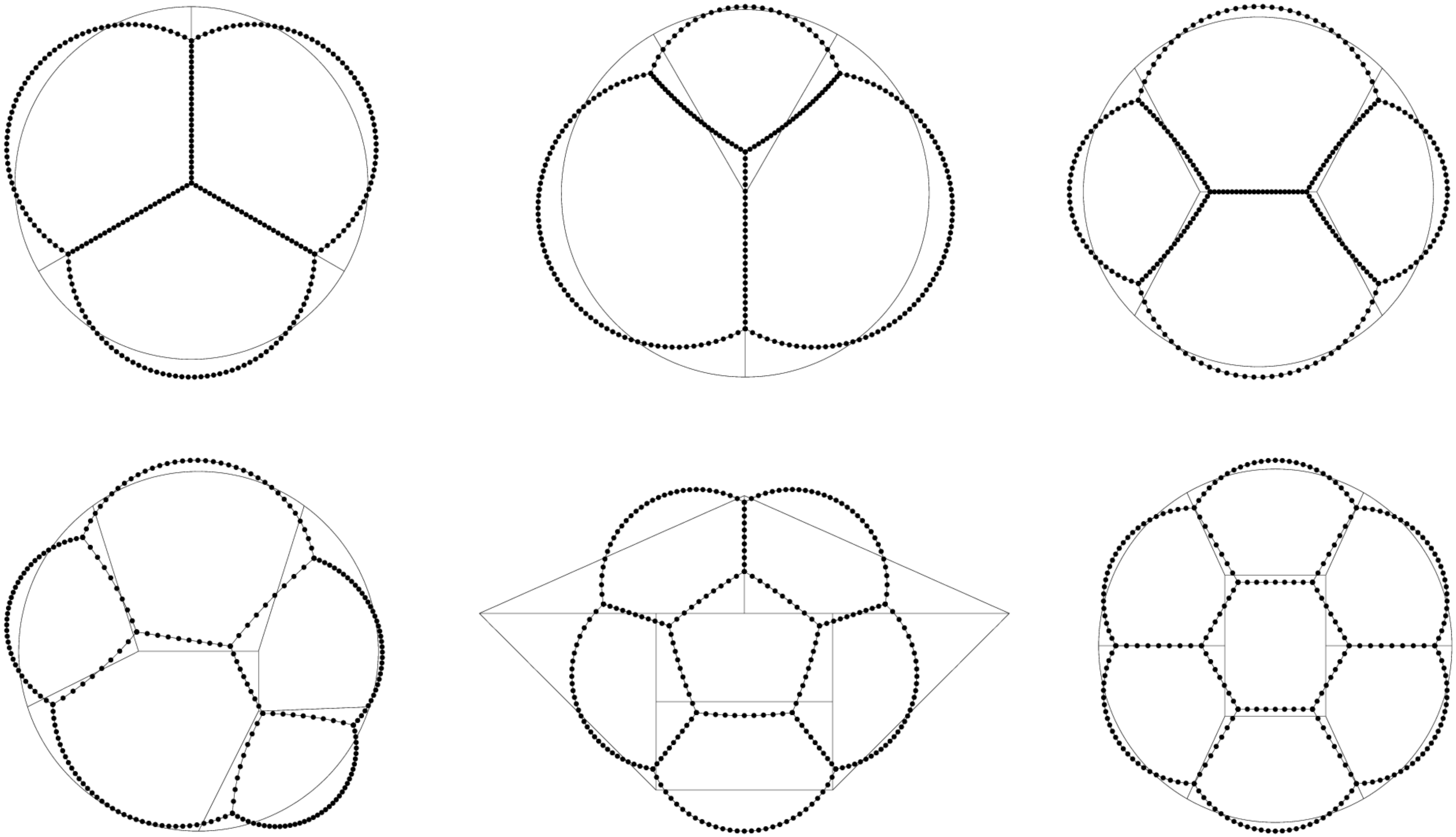


Surface diffusion:

Areas maintained, whilst the sum of the curve lengths decreases.

Curve Networks

Surface diffusion on more complicated networks



Curves in $\mathbb{R}^d$, $d \geq 3$, and Curves on Manifolds

Our scheme also easily generalises to curves in $\mathbb{R}^d$, $d \geq 3$, and curves on manifolds.

E.g. let $\mathcal{M} = \{\vec{z} \in \mathbb{R}^3 : F(\vec{z}) = 0\}$ be a 2-dimensional manifold $\subset \mathbb{R}^3$. For a closed curve Γ on $\mathcal{M}$ with a given parameterization $\vec{x}(\rho, t) \in \mathcal{M}$, mean curvature flow on the $\mathcal{M}$, also called geodesic curvature flow, is :

$$\vec{x}_t \,.\, \vec{\nu}_{\mathcal{M}} = \varkappa_{\mathcal{M}}, \qquad \varkappa_{\mathcal{M}}\, \vec{\nu}_{\mathcal{M}} + \varkappa_F\, \vec{\nu}_F = \vec{x}_{ss}\,,$$

where $\vec{\nu}_F$ is the outward unit normal of $\mathcal{M}$ and $\vec{\nu}_{\mathcal{M}} := \vec{x}_s \times \vec{\nu}_F(\vec{x})$.

Find $\{\vec{X}^{m+1}, \kappa_{\mathcal{M}}^{m+1}, \kappa_F^{m+1}\} \in \underline{V}^h \times V^h \times V^h$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\, \vec{\omega}_{\mathcal{M}}^m \rangle_m^h - \langle \kappa_{\mathcal{M}}^{m+1}, \chi \rangle_m^h = 0 \qquad \forall\ \chi \in V^h\,,$$

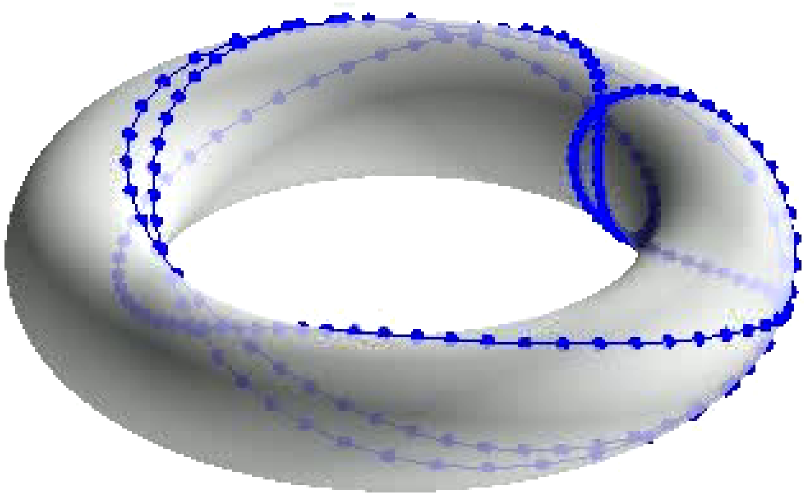
$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\, \vec{\omega}_F^m \rangle_m^h = 0 \qquad \forall\ \chi \in V^h\,,$$

$$\langle \kappa_{\mathcal{M}}^{m+1}\, \vec{\omega}_{\mathcal{M}}^m, \vec{\eta} \rangle_m^h + \langle \kappa_F^{m+1}\, \vec{\omega}_F^m, \vec{\eta} \rangle_m^h + \langle \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h\,.$$

Linear system. Existence, uniqueness, stability and equidistribution property.

Numerical Result

Geodesic curvature flow on a torus of radii $R = 4$ and $r = 1$.



$|\Gamma(0)| = 86.3,\qquad |\Gamma(T)| = 39.3.\qquad (N = 200,\ \tau = 2\times 10^{-3},\ T = 250)$

Geometric Evolution Equations of Surfaces in $\mathbb{R}^3$

Let $\{\Gamma(t)\}_t$ be a smooth family of compact orientable hypersurfaces without boundary in $\mathbb{R}^3$. Let $\Omega(t)$ be the region bounded by $\Gamma(t)$.

Let $\vec{x}(t) : \mathcal{M} \to \Gamma(t)$ be a parameterization of $\Gamma(t)$, where $\mathcal{M} \subset \mathbb{R}^3$ is a suitable reference manifold.

Let $\vec{\nu}(t) : \Gamma(t) \to \mathbb{R}^3$ be a unit normal to $\Gamma(t)$.

For the evolution of Γ it suffices to prescribe the normal velocity $\mathcal{V} \equiv \vec{x}_t \,.\, \vec{\nu}$,

e.g. (MC) $\mathcal{V} = \varkappa\,,$ or (SD) $\mathcal{V} = -\Delta_s\,\varkappa\,.$

Here $\varkappa$ is the "mean" curvature (sum of principal curvatures) of Γ, so that

$$\vec{\varkappa} := \varkappa\,\vec{\nu} = \Delta_s\,\vec{x} \qquad (\vec{x}_{ss} \quad \text{for curves}),$$

and $\Delta_s \equiv \nabla_s \,.\, \nabla_s$ the Laplace–Beltrami operator (surface Laplacian) on $\Gamma(t)$.

Note that $\varkappa$ is positive if Γ is curved in the direction of $\vec{\nu}$.

(MC), (SD) $\Rightarrow \dfrac{d}{dt}|\Gamma(t)| \leq 0,$ (SD) $\Rightarrow \dfrac{d}{dt}|\Omega(t)| = 0.$

Finite Elements on Surfaces

Following Dziuk (91), we now introduce piecewise linear finite elements on surfaces.

Let $\Gamma \subset \mathbb{R}^d$ be a given smooth hypersurface.

Let Γ^h be a *polyhedral surface* approximating Γ,
i.e. a union of non-degenerate $(d-1)$-dimensional simplices with no hanging vertices and with vertices on Γ.

This defines a $(d-1)$-dimensional triangulation $\mathcal{T}^h$ in $\mathbb{R}^d$:

$$\Gamma^h = \bigcup_{\sigma \in \mathcal{T}^h} \overline{\sigma}$$

On this discrete surface, we define finite element spaces by

$$V^h(\Gamma^h) := \{\chi \in C(\Gamma^h, \mathbb{R}) : \chi|_\sigma \text{ is linear affine on each } \sigma \in \mathcal{T}^h\}$$

and

$$\underline{V}^h(\Gamma^h) := [V^h(\Gamma^h)]^d .$$

Parametric Finite Element Approximation

Let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partitioning of $[0,T]$, $\tau_m := t_{m+1} - t_m$, $m = 0 \to M-1$.

For $m = 0$, choose a polyhedral surface Γ^0 approximating the initial $\Gamma(0)$.

For $m \geq 0$, let $\vec{X}^m \in \underline{V}^h(\Gamma^m)$ be the identity function on Γ^m.

$(\mathcal{P}^h)$ Find $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h(\Gamma^m) \times V^h(\Gamma^m)$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\, \vec{\nu}^m \rangle_m^h - \begin{cases} \langle \kappa^{m+1}, \chi \rangle_m^h \\ \langle \nabla_s\, \kappa^{m+1}, \nabla_s\, \chi \rangle_m \end{cases} = 0 \qquad \forall\ \chi \in V^h(\Gamma^m),$$

$$\langle \kappa^{m+1}\, \vec{\nu}^m, \vec{\eta} \rangle_m^h + \langle \nabla_s\, \vec{X}^{m+1}, \nabla_s\, \vec{\eta} \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^m);$$

and set $\Gamma^{m+1} := \vec{X}^{m+1}(\Gamma^m)$.

Linear system for $\{\vec{X}^{m+1}, \kappa^{m+1}\}$.

Here

$$\langle f, g \rangle_m := \int_{\Gamma^m} f \,.\, g \;\mathrm{d}s$$

with $\langle \cdot, \cdot \rangle_m^h$ the mass lumped inner product.

Properties of the Scheme

1. Existence, Uniqueness

   Under very mild assumptions on Γ^m, there exists a unique solution $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h(\Gamma^m) \times V^h(\Gamma^m)$ to $(\mathcal{P}^h)$.

2. Stability

   For all $k = 1 \to M$ it holds that

$$|\Gamma^k| + \sum_{m=0}^{k-1} \tau_m \begin{cases} \langle \kappa^{m+1}, \kappa^{m+1} \rangle_m^h \\ \langle \nabla_s \, \kappa^{m+1}, \nabla_s \, \kappa^{m+1} \rangle_m \end{cases} \leq |\Gamma^0| \, .$$

3. Volume conservation

   For (SD) for a continuous in time semidiscrete scheme.

4. Good distribution of mesh points

   for a continuous in time semidiscrete scheme.

- For $d = 2$, equidistribution for $\Gamma^h(t)$, where $\Gamma^h(t)$ not locally parallel.
- For $d = 3$, $\Gamma^h(t)$ are “conformal polyhedral surfaces”.

Both vertex normal definitions are in the same direction;
i.e. weighted average of normals of neighbouring faces $\parallel \Delta_s^h \, \vec{X}$.

Recall Existing Parametric Approximations for $d = 3$

There exist parametric finite element approximations of (MC) and (SD). They are based on the following formulations.

- Dziuk (91) (MC)

  $\vec{x}_t = \vec{\varkappa} = \varkappa\,\vec{\nu} \equiv \Delta_s\,\vec{x}\,.$

- Bänsch, Morin, Nochetto (05) (SD)

  $\vec{x}_t = \vec{\mathcal{V}} = \mathcal{V}\,\vec{\nu}, \qquad \mathcal{V} = -\Delta_s\,\varkappa, \qquad \varkappa = \vec{\varkappa}\,.\,\vec{\nu}, \qquad \vec{\varkappa} = \Delta_s\,\vec{x}\,.$

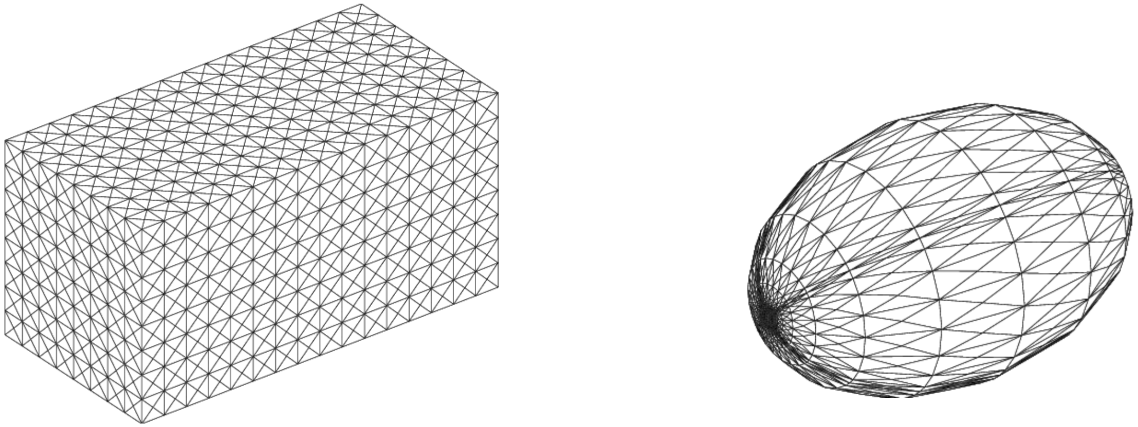
The two approaches have in common that they evolve the parameterization $\vec{x}$ and its corresponding approximation $\vec{X}$ only in the *normal* direction.

This can lead to **mesh distortions** in practice.

Can be overcome with a posteriori applied heuristic mesh smoothing steps, see e.g. Brakke (92). However, effect on normal velocity not clear.

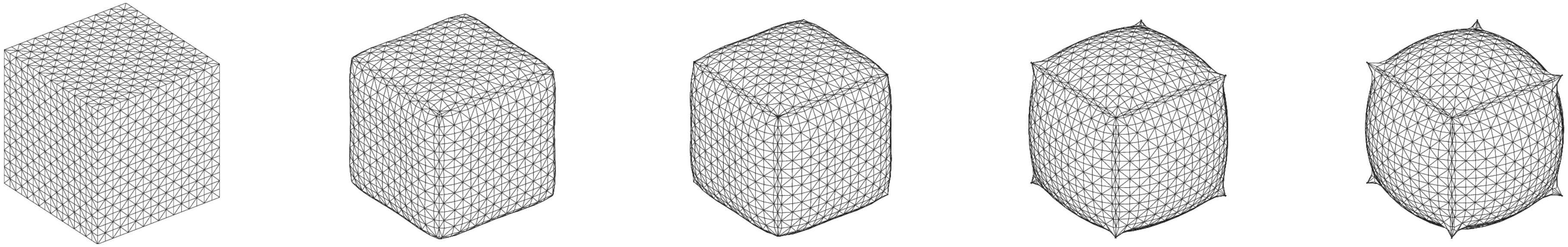
Mesh Distortions

(MC) – Dziuk :



($K = 1282$ vertices, $J = 2560$ triangles, $\tau = 10^{-3}$, $T = 0.14$)

(SD) – BMN :

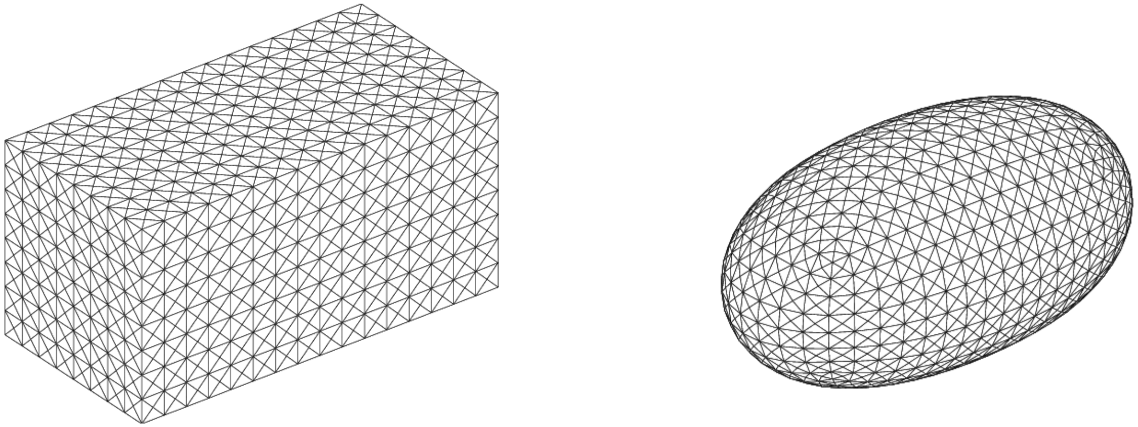


($K = 1538$, $J = 3072$, $\tau = 10^{-4}$, $T = 1.6 \times 10^{-3}$)

Finite element toolbox ALBERTA   Schmidt, Siebert (2005).

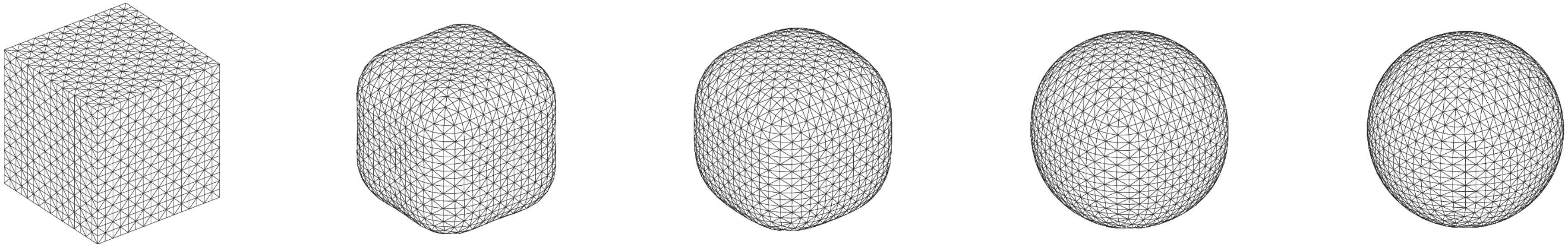
Tangential Distribution of Mesh Points

(MC) − BGN :



$(K = 1282 \text{ vertices}, J = 2560 \text{ triangles}, \tau = 10^{-3}, T = 0.14)$

(SD) − BGN :



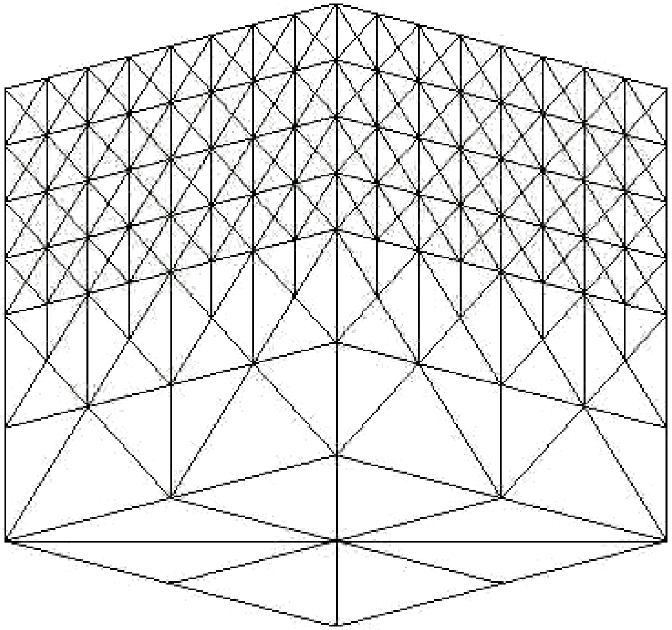
$(K = 1538, J = 3072, \tau = 10^{-4}, T = 1.6 \times 10^{-3})$

Finite element toolbox ALBERTA   Schmidt, Siebert (2005).

Tangential Distribution of Mesh Points

In general good meshes are obtained for $d = 3$:

(SD)



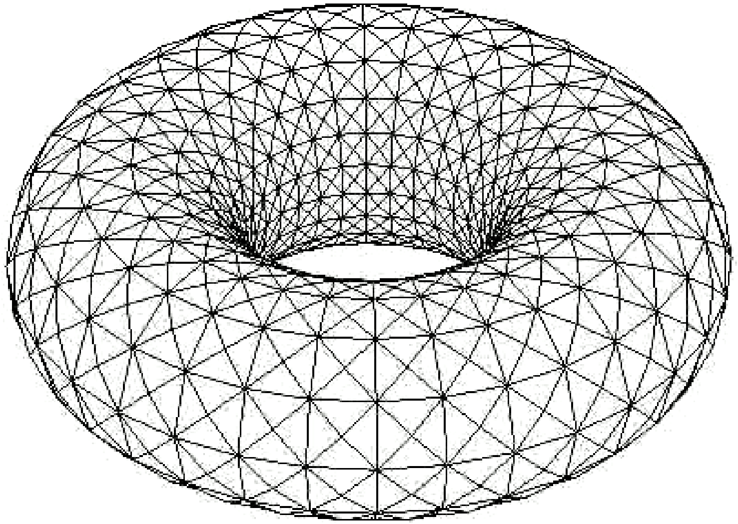
($K = 442$, $J = 880$, $\tau = 10^{-3}$, $T = 1$)

No heuristical redistribution necessary.

Note: Ideas for curves from Strain (89), Hou, Lowengrub, Shelley (94) and Mikula, Ševčovič (01) **do not** generalize to surfaces in $\mathbb{R}^3$.

Numerical Results

(MC) - torus with radii $R = 2$ and $r = 1$.

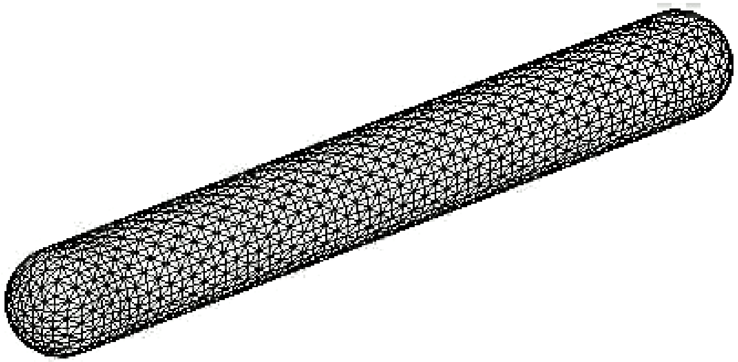


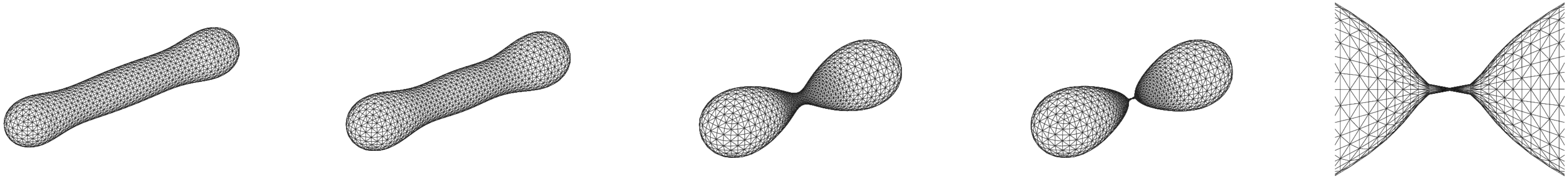
$(K = 1024,\ J = 2048,\ \tau = 10^{-3},\ T = 0.538)$

Numerical Results

(SD) leading to pinch-off

Rounded cylinder 8 × 1 × 1.





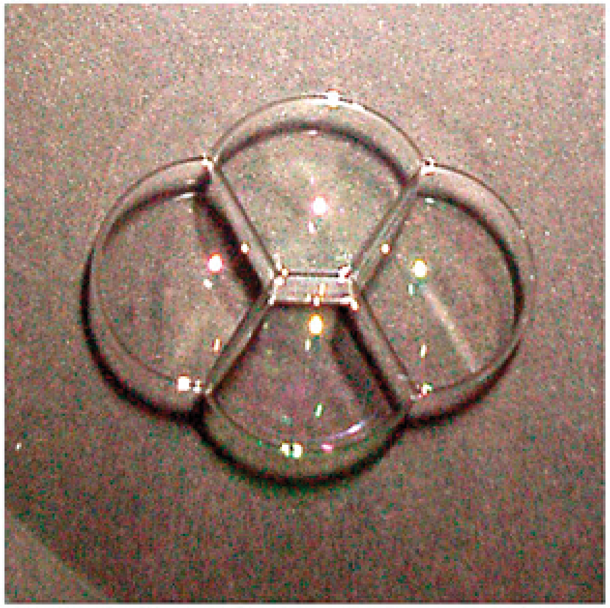
$\vec{X}(t)$ for $t = 0.05,\ 0.1,\ 0.24,\ 0.2431$. Volume loss = -0.03%.

$(K = 2178,\ J = 4352,\ \tau = 10^{-4})$

Extension of Curve Networks to $\mathbb{R}^3$: Surface Clusters

Generalisation of curve networks in the plane with triple junction points (where 3 curves meet) to

Surface clusters in $\mathbb{R}^3$ with triple junction lines (where 3 surfaces meet) and quadruple junction points (where 4 TJLs meet).



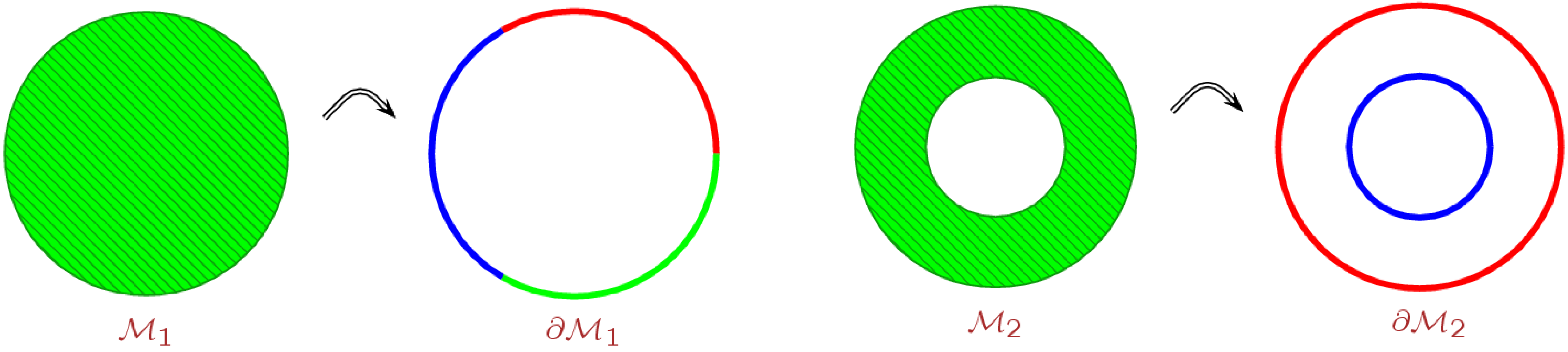
Once again the conditions on these triple junction lines will be naturally imposed in our formulation. Conditions on QJs implied by TJL conditions.

See Garcke, Novick-Cohen (00) and Bronsard, Garcke, Stoth (98).

Surface Clusters

Geometric evolution equations for surface clusters. Let $\Gamma := (\Gamma_1, \dots, \Gamma_{I_S})$ be a cluster of surface patches meeting at I_T triple junction lines $\mathcal{T}_1, \dots, \mathcal{T}_{I_T}$.

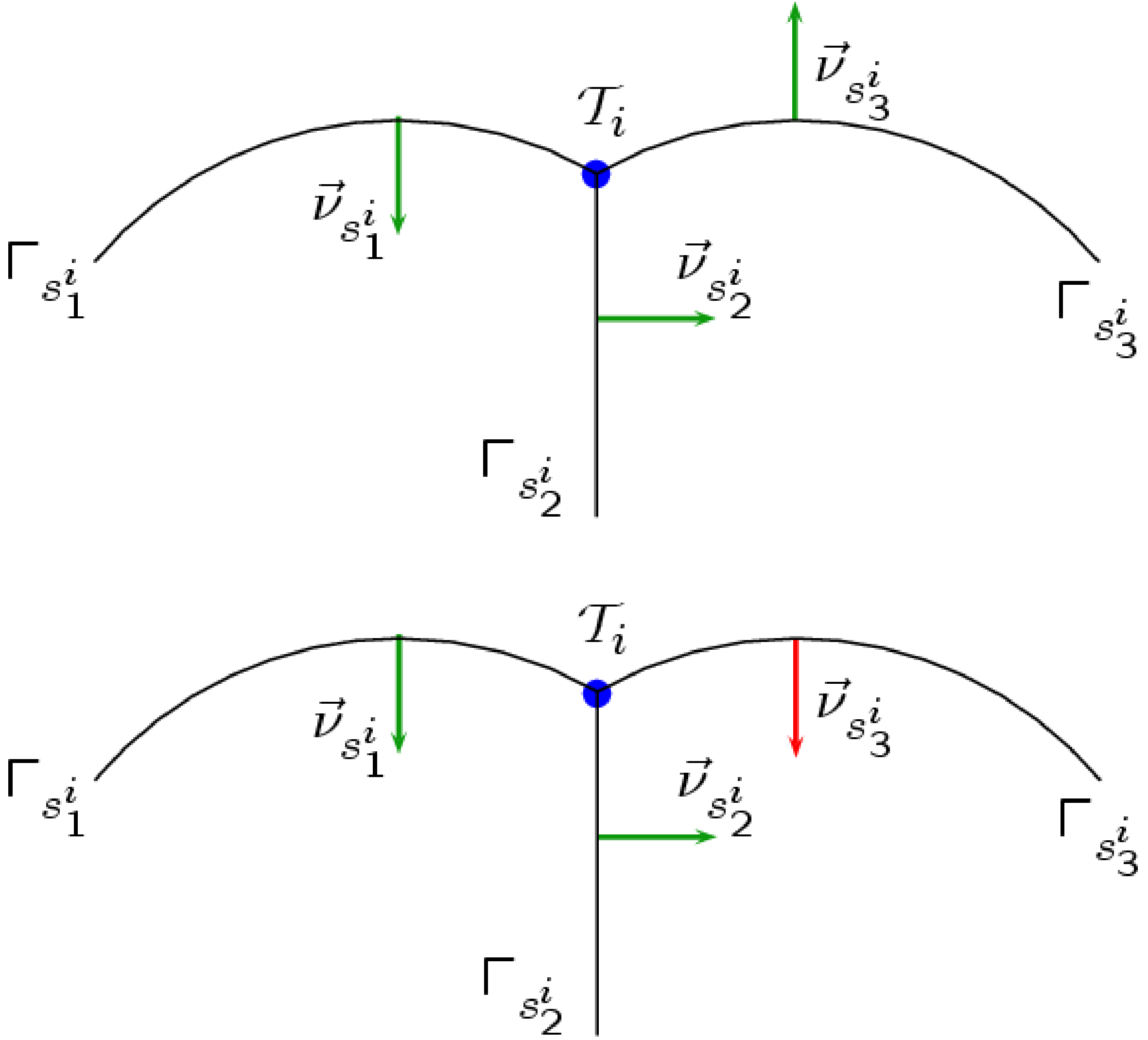
Let $\mathcal{M} := (\mathcal{M}_1, \dots, \mathcal{M}_{I_S})$ be a collection of reference manifolds with boundaries $\partial\mathcal{M}_i$ partitioned into $\partial\mathcal{M}_i = \bigcup_{j=1}^{I_{P_i}} \partial_j \mathcal{M}_i$, $i = 1 \to I_S$.



Then the triple junction lines $\mathcal{T}_i$, $i = 1 \to I_T$, are given by sets $\{(s^i_j, p^i_j, o^i_j) : j = 1 \to 3\} \in \{(s,p,o) : s \in \{1, \dots, I_S\}, p \in \{1, \dots, I_{P_s}\}, o \in \{-1, 1\}\}^3$.

Here $o^i_j \in \{-1, 1\}$ denotes the choice of local normal for the surface $\Gamma_{s^i_j}$ meeting at triple junction $\mathcal{T}_i$, in order align the orientation of the three normals.

Local Orientation of Normals at Triple Junction Lines



Sketch of the local orientation of $(\Gamma_{s_1^i}, \Gamma_{s_2^i}, \Gamma_{s_3^i})$ at the triple junction line $\mathcal{T}_i$ (**blue**). Depicted is a plane that is perpendicular to $\mathcal{T}_i$. In the example above, $o^i := (o_1^i, o_2^i, o_3^i)$ can be chosen as $o^i = \pm(1,1,1)$. However, in the example below we require $o^i = \pm(1,1,-1)$.

Geometric Evolution Equations

Let $\vec{x}(t) := (\vec{x}_1, \ldots, \vec{x}_{I_S})(t) : \mathcal{M} \to \Gamma(t)$ be a parameterization of $\Gamma(t)$.

On Γ_i: $$\mathcal{V}_i := (\vec{x}_i)_t \,.\, \vec{\nu}_i = \begin{cases} \varkappa_i & \text{(MC)} \\ -\Delta_s \, \varkappa_i & \text{(SD)} \end{cases}, \qquad i = 1 \to I_S\,.$$

In addition, the following triple junction line conditions have to hold for $i = 1 \to I_T$.

(a) $$\mathcal{T}_i(t) := \vec{x}_{s_1^i}(\partial_{p_1^i} \mathcal{M}_{s_1^i}) = \vec{x}_{s_2^i}(\partial_{p_2^i} \mathcal{M}_{s_2^i}) = \vec{x}_{s_3^i}(\partial_{p_3^i} \mathcal{M}_{s_3^i})\,, \qquad \text{(attachment)}$$

(b) $$\sum_{j=1}^{3} o_j^i \, \vec{\nu}_{s_j^i} = \vec{0} \quad \text{on } \mathcal{T}_i, \qquad \text{(force balance)}$$

(c) $$o_1^i \, \vec{\mu}_{s_1^i} \,.\, \nabla_s \, \varkappa_{s_1^i} = o_2^i \, \vec{\mu}_{s_2^i} \,.\, \nabla_s \, \varkappa_{s_2^i} = o_3^i \, \vec{\mu}_{s_3^i} \,.\, \nabla_s \, \varkappa_{s_3^i} \quad \text{on } \mathcal{T}_i, \qquad \text{(flux balance)}$$

(d) $$\sum_{j=1}^{3} o_j^i \, \varkappa_{s_j^i} = 0 \quad \text{on } \mathcal{T}_i, \qquad \text{(chem. pot. cont.)}$$

Here $\vec{\mu}_i$ denotes the intrinsic unit normal to $\partial\Gamma_i$, that lies within the tangent plane of Γ_i.

$$E(\Gamma) = |\Gamma| := \sum_{i=1}^{I_S} \int_{\Gamma_i} 1 \, \mathrm{d}s \qquad \Rightarrow \quad \frac{d}{dt}|\Gamma(t)| \leq 0 \qquad \text{(MC) \& (a,b), (SD) \& (a–d)}$$

Weak Formulation

Given the triple junction lines defined by $\{(s^i_j, p^i_j, o^i_j) : j = 1 \to 3\}$, set

$$\underline{V} := \{\vec\chi \equiv (\vec\chi_1, \ldots, \vec\chi_{I_S}) \in \overset{I_S}{\underset{i=1}{\times}} H^1(\mathcal{M}_i, \mathbb{R}^3) : \vec\chi_{s^i_1} = \vec\chi_{s^i_2} = \vec\chi_{s^i_3} \ \text{ on } \mathcal{T}_i,\ i = 1 \to I_T\}$$

$$\text{and } W := \{\chi \equiv (\chi_1, \ldots, \chi_{I_S}) \in \underbrace{\overset{I_S}{\underset{i=1}{\times}} H^1(\mathcal{M}_i, \mathbb{R})}_{=:W_0} : \sum_{j=1}^{3} o^i_j\, \chi_{s^i_j} = 0 \ \text{ on } \mathcal{T}_i,\ i = 1 \to I_T\}\,.$$

Find $\vec x \in \underline{V}$ and $\varkappa \in \begin{cases} W_0 \\ W \end{cases}$ such that

$$\langle \vec x_t, \eta\, \vec\nu \rangle - \begin{cases} \langle \varkappa, \eta \rangle \\ \langle \nabla_s\, \varkappa, \nabla_s\, \eta \rangle \end{cases} = 0 \qquad \forall\ \eta \in \begin{cases} W_0 \\ W \end{cases} \quad ,$$

$$\langle \varkappa\, \vec\nu, \vec\chi \rangle + \langle \nabla_s\, \vec x, \nabla_s\, \vec\chi \rangle = 0 \qquad \forall\ \vec\chi \in \underline{V}\,,$$

where

$$\langle \eta, \chi \rangle := \sum_{i=1}^{I_S} \int_{\Gamma_i} \eta_i \,.\, \chi_i \; \mathrm{d}s\,.$$

(a,d) enforced via trial spaces, (c,d) enforced weakly via test spaces.

Parametric Finite Element Approximation

Introduce piecewise linears $\underline{V}^h \subset \underline{V}$, $W^h \subset W$ and $W_0^h \subset W_0$.

Find $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}^h \times \begin{cases} W_0^h \\ W^h \end{cases}$ such that

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\, \vec{\nu}^m \right\rangle_m^h - \begin{cases} \langle \kappa^{m+1}, \chi \rangle_m^h \\ \langle \nabla_s\, \kappa^{m+1}, \nabla_s\, \chi \rangle_m \end{cases} = 0 \qquad \forall\ \chi \in \begin{cases} W_0^h \\ W^h \end{cases},$$

$$\langle \kappa^{m+1}\, \vec{\nu}^m, \vec{\eta} \rangle_m^h + \langle \nabla_s\, \vec{X}^{m+1}, \nabla_s\, \vec{\eta} \rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h;$$

where

$$\langle f, g \rangle_m := \int_{\Gamma^m} f \,.\, g \, \mathrm{d}s := \sum_{i=1}^{I_S} \int_{\Gamma_i^m} f_i \,.\, g_i \, \mathrm{d}s$$

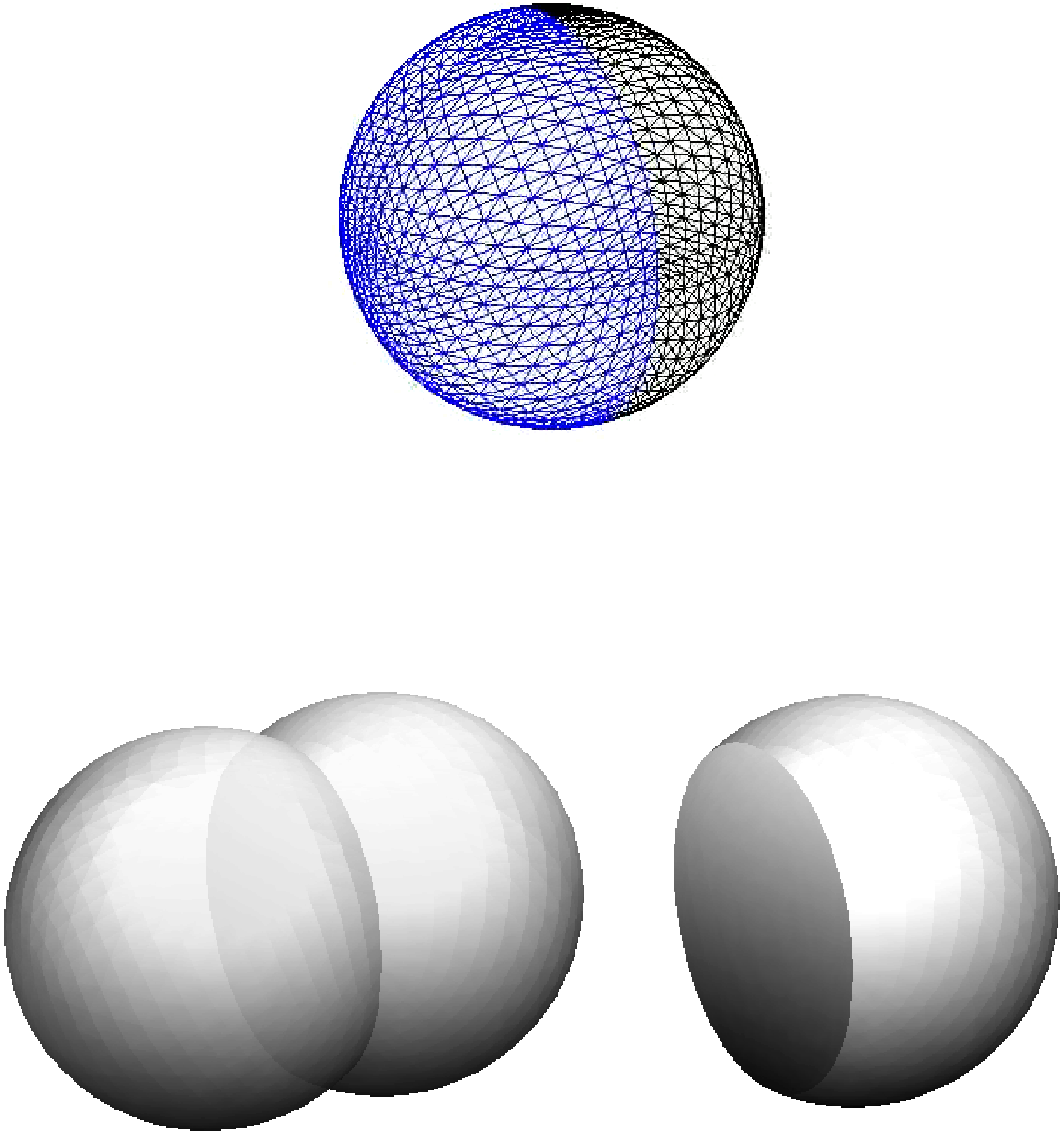
with $\langle \cdot, \cdot \rangle_m^h$ the mass lumped inner product.

Linear system for $\{\vec{X}^{m+1}, \kappa^{m+1}\}$.

Existence, uniqueness and stability. Volume conservation and good mesh properties as before.

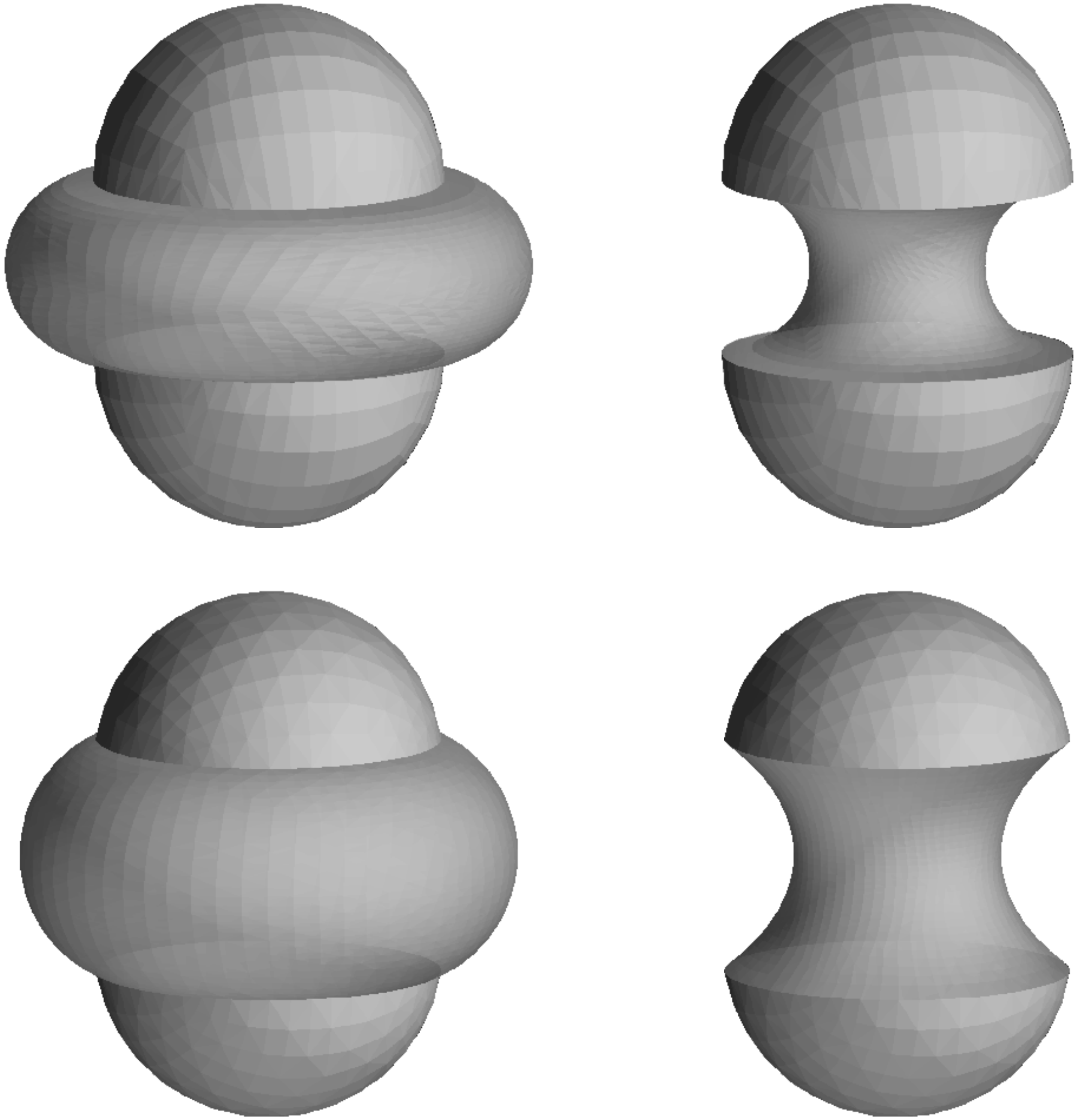
Numerical Results

(SD)



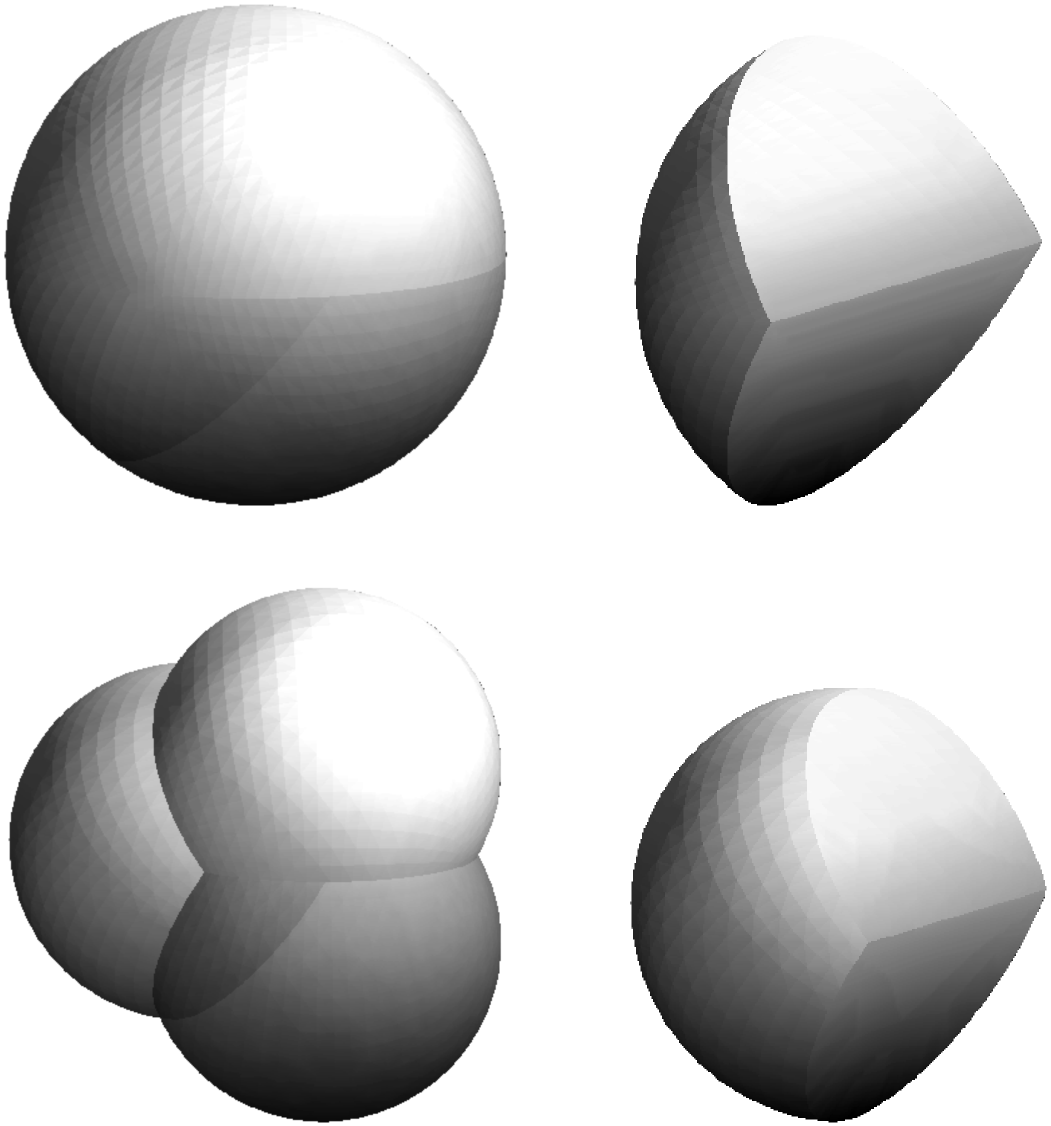
($K = 3267$, $J = 6240$, $\tau = 10^{-3}$ and $T = 1$)

Numerical Results



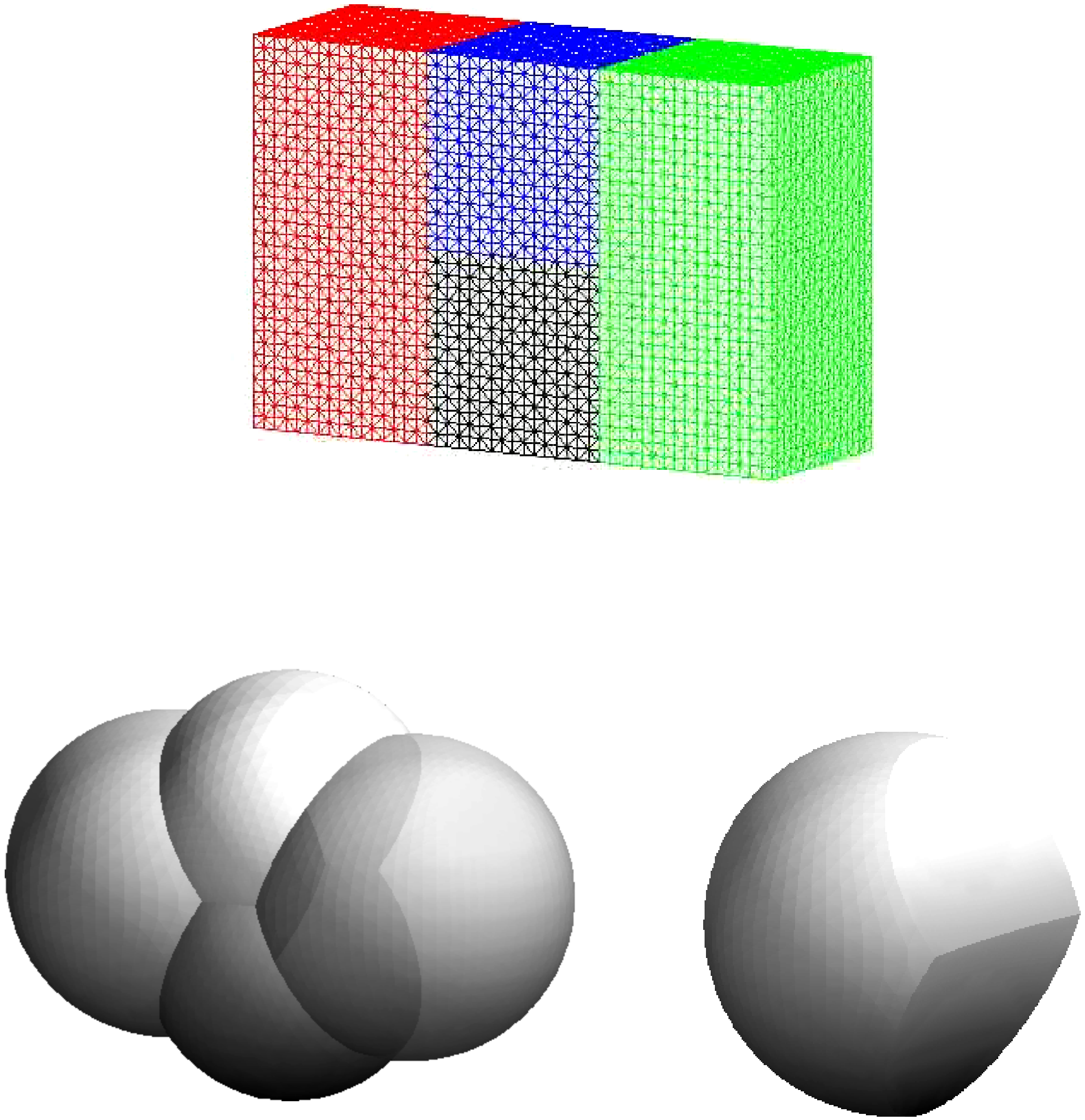
(SD) Two volumes (dumbbell and torus) with 4 surfaces.
Plot of Γ^0 and Γ^M. On the right, Γ_i^m, $i = 1, 2, 4$, $m = 0, M$.
($K = 4802$, $J = 9216$, $\tau = 10^{-3}$ and $T = 1$)

Numerical Results



(SD) Triple bubble. Γ^0 is partitioned unit ball into 3 equal segments.
Plot of Γ^0 and Γ^M. On the right, Γ_i^m, $i = 1, 4, 5$, $m = 0, M$.
($K = 6534$, $J = 12288$, $\tau = 10^{-4}$ and $T = 1$)

Numerical Results



$(K = 7305,\ J = 13824,\ \tau = 10^{-4}$ and $T = 0.5)$

Anisotropic Surface Energy

All of the presented schemes naturally generalize to an anisotropic surface energy

$$|\Gamma|_\gamma := \int_\Gamma \gamma(\vec\nu)\,\mathrm{d}s$$

where $\gamma : \mathbb{R}^d \setminus \{\vec 0\} \to \mathbb{R}_{>0}$ is a given anisotropy function,
which we will assume is absolutely homogeneous of degree one, i.e.

$$\gamma(\lambda\,\vec p) = |\lambda|\,\gamma(\vec p) \quad \forall\ \vec p \in \mathbb{R}^d \setminus \{\vec 0\},\ \forall\ \lambda \in \mathbb{R} \setminus \{0\}\,.$$

In order to obtain unconditionally stable approximations, we restrict ourselves to the following class of anisotropies:

$$\gamma(\vec p) = \left(\sum_{\ell=1}^{L} [\gamma_\ell(\vec p)]^r\right)^{\frac{1}{r}}, \qquad \gamma_\ell(\vec p) := \sqrt{\vec p\,.\,G_\ell\,\vec p}, \qquad r \in [1,\infty)\,,$$

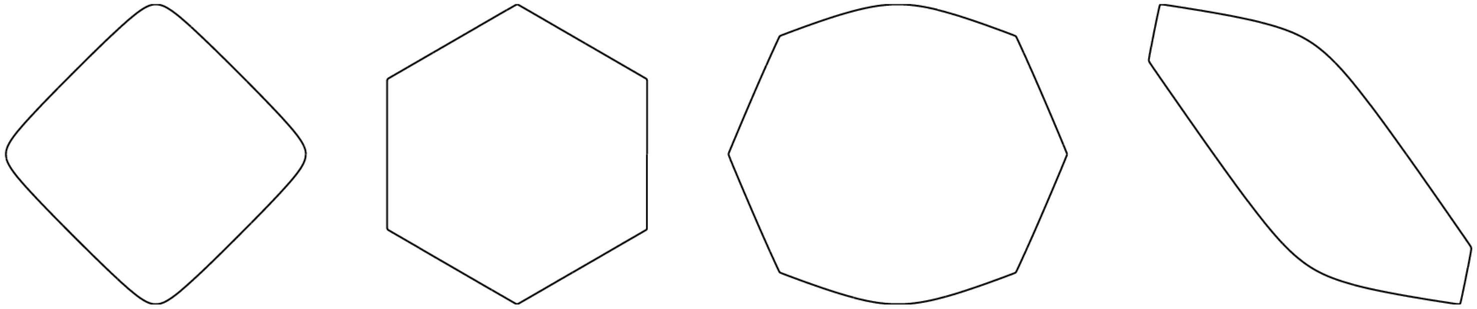
where $G_\ell \in \mathbb{R}^{d\times d}$, $\ell = 1 \to L$, are symmetric and positive definite.

Example Anisotropies, $d = 2$

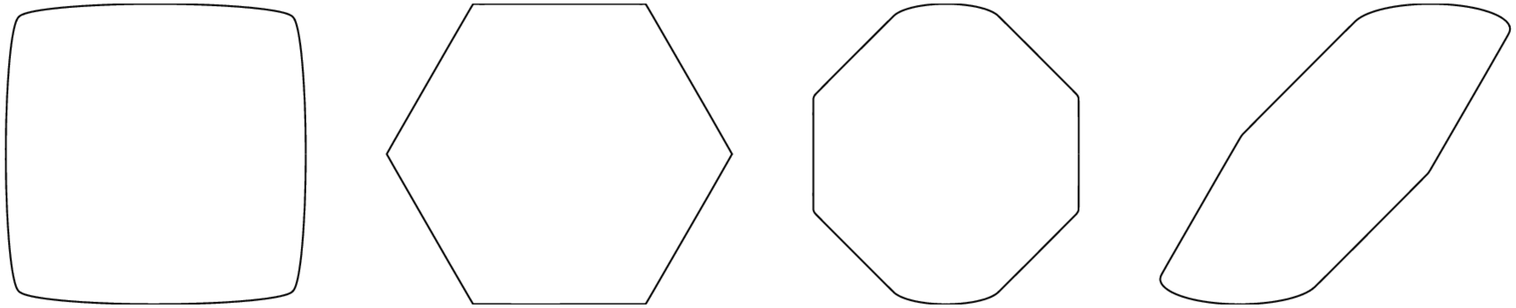
Frank diagram: $\mathcal{F} := \{\vec{p} \in \mathbb{R}^d : \gamma(\vec{p}) \leq 1\}$

Wulff shape: $\mathcal{W} := \{\vec{q} \in \mathbb{R}^d : \gamma^*(\vec{q}) \leq 1\}$, where $\gamma^*(\vec{q}) = \sup_{\vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}} \frac{\vec{p} \, . \, \vec{q}}{\gamma(\vec{p})}$.

$\mathcal{F}$:



$\mathcal{W}$:

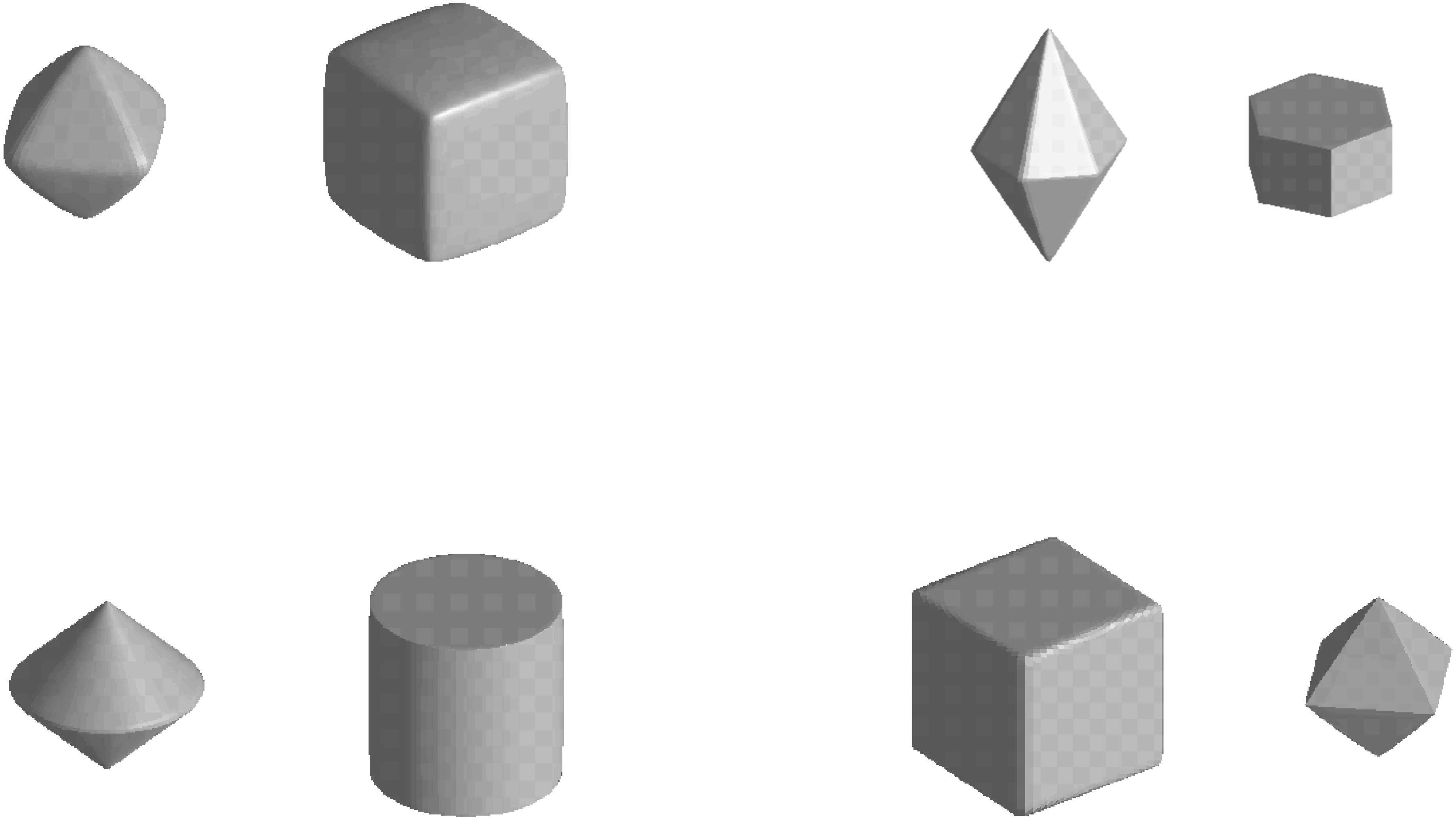


A wide class of anisotropies can be modelled by this choice.

Including regularized crystalline surface energies.

Example Anisotropies, $d = 3$

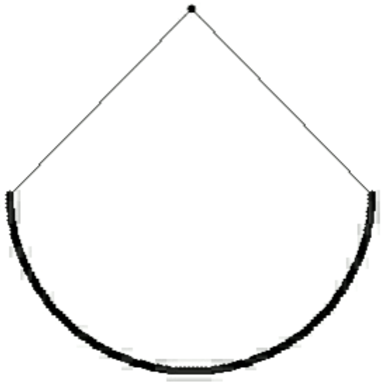
Examples of Frank diagrams $\mathcal{F}$ and associated Wulff shapes $\mathcal{W}$:



Tangential Movement

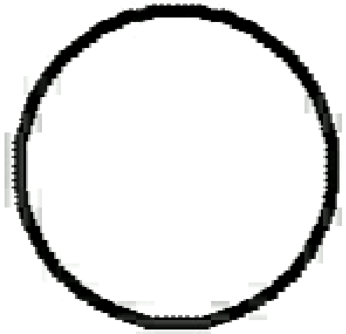
(SD$_\gamma$) $\gamma(\vec{p}) = \sqrt{p_1^2 + 0.1\, p_2^2}$ $\Rightarrow$ Wulff shape is $\sqrt{10} : 1$ ellipse

$N = 128$, $\tau = 10^{-3}$, $T = 1$.



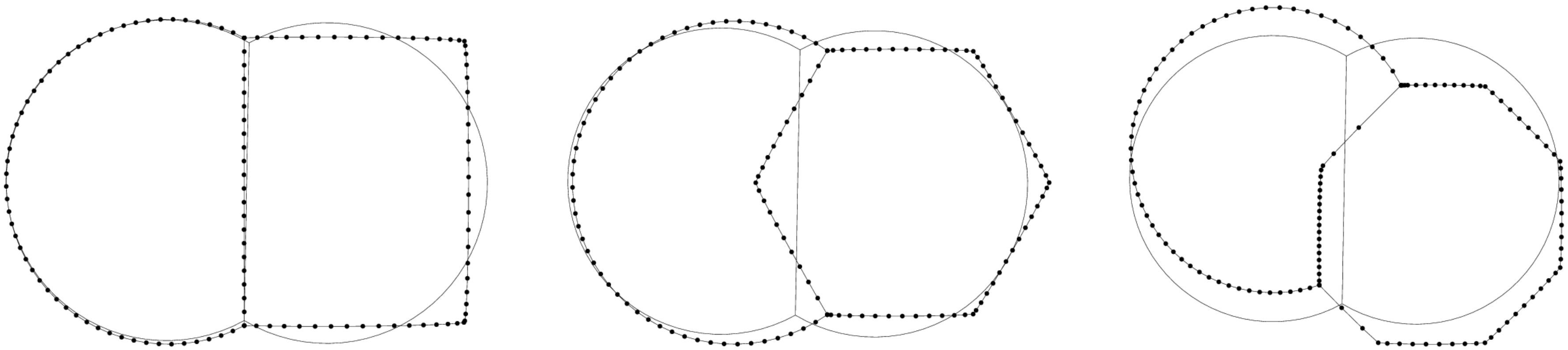
(SD$_\gamma$) $\gamma(\vec{p}) = \sqrt{p_1^2 + 0.01\, p_2^2}$ $\Rightarrow$ Wulff shape is $10 : 1$ ellipse

$N = 128$, $\tau = 10^{-3}$, $T = 1$.

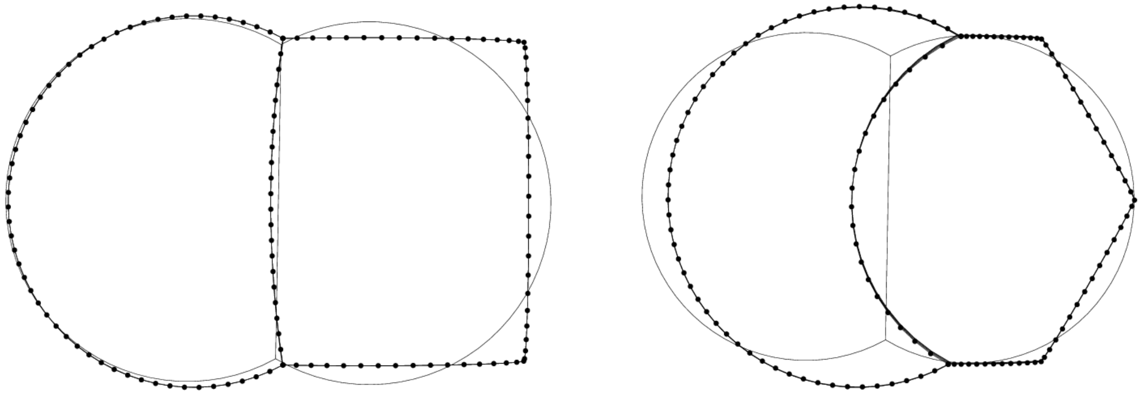


Anisotropic Planar Double Bubbles

E.g. (SD$_\gamma$) : $\gamma_1(\vec{p}) = |\vec{p}|$, $\gamma_i(\vec{p}) = \gamma_0(\vec{p})$, $i = 2 \to 3$. $\varepsilon_\ell^2 \equiv 0.001$



$\gamma_1(\vec{p}) = \gamma_2(\vec{p}) = |\vec{p}|$, $\gamma_3(\vec{p}) = \gamma_0(\vec{p})$. $\varepsilon_\ell^2 \equiv 0.001$

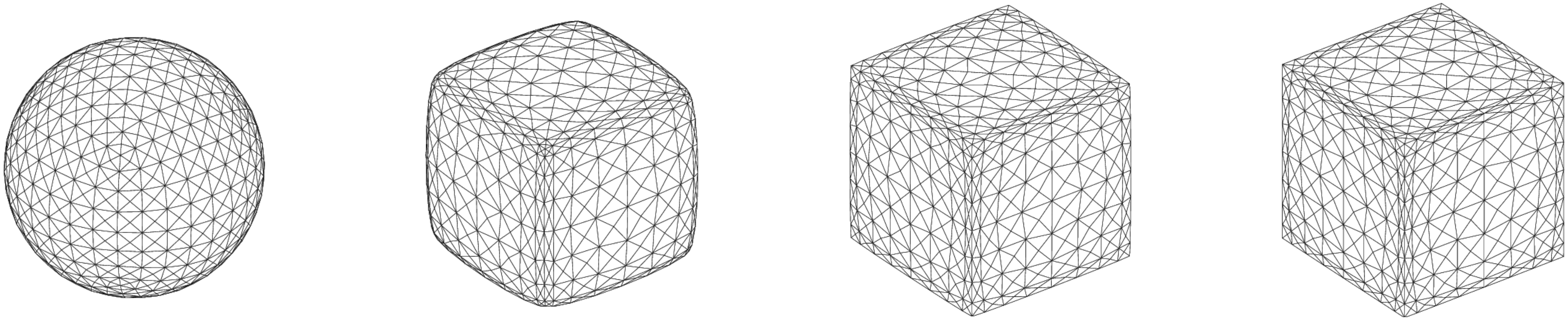


Numerical Results

(SD$_\gamma$) - Cubic Wulff Shape,

i.e. $\gamma(\vec{p}) = \sum_{i=1}^{d} \sqrt{\varepsilon^2 \, |\vec{p}|^2 + p_i^2 (1 - \varepsilon^2)}$ is a regularized l^1-norm.

($r = 1$, $L = 3$, $G_1 = \text{diag}(1, \varepsilon^2, \varepsilon^2)$, $G_2 = \text{diag}(\varepsilon^2, 1, \varepsilon^2)$, $G_3 = \text{diag}(\varepsilon^2, \varepsilon^2, 1)$)



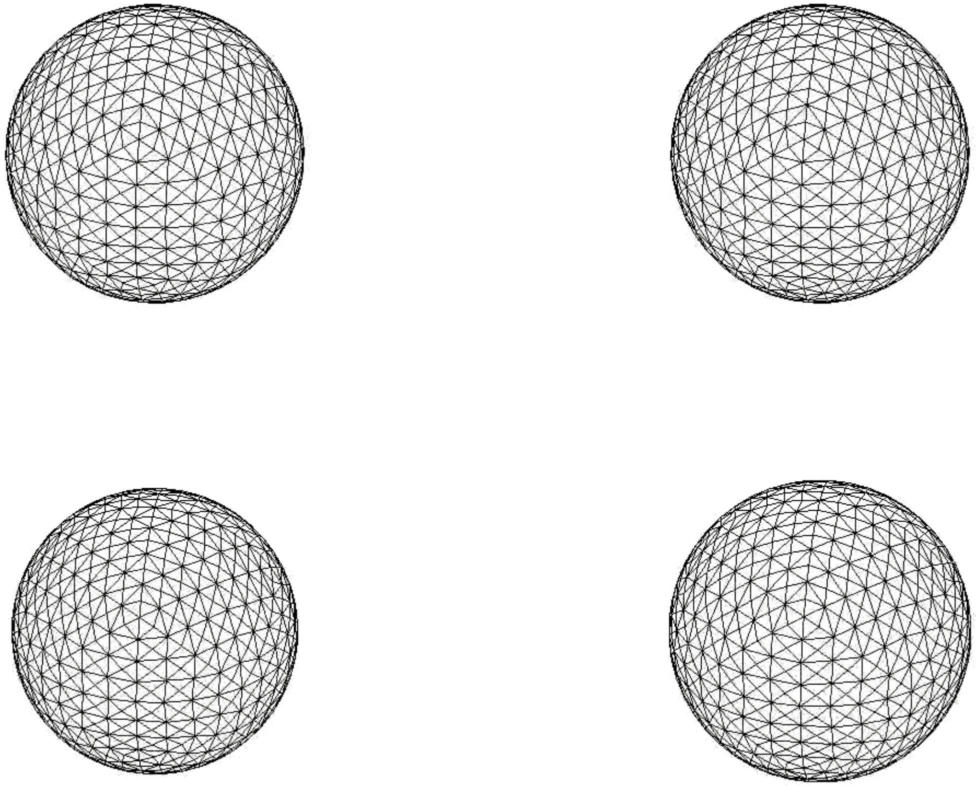
A plot of $\vec{X}(0.5)$ for $\varepsilon = 10^{-k}$, $k = 0 \rightarrow 3$.

$(K = 770,\ J = 1536,\ \tau = 10^{-3})$

Numerical Results

(SD_γ) - Cube ($r = 1$, $L = 3$), Hexagonal Prism ($r = 1$, $L = 4$),

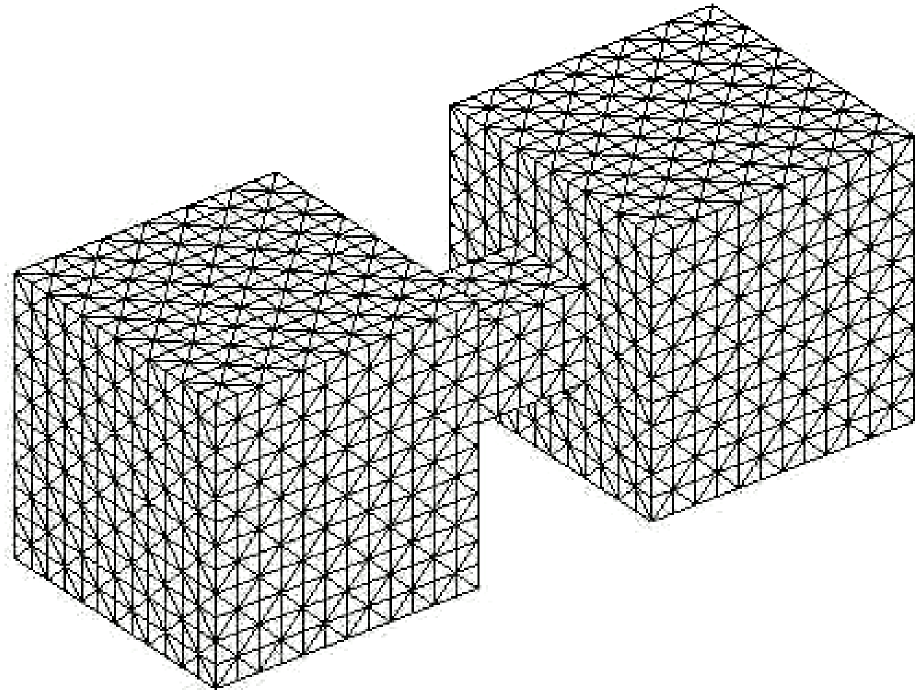
Octahedron ($r = 30$, $L = 3$), Cylinder Wulff Shape ($r = 1$, $L = 2$).



($\varepsilon = 10^{-2}$, $K = 770$, $J = 1536$, $\tau = 10^{-4}$, $T = 0.1,\ 0.25,\ 0.02,\ 0.25$)

Numerical Results

(MC$_\gamma$) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-2}$)

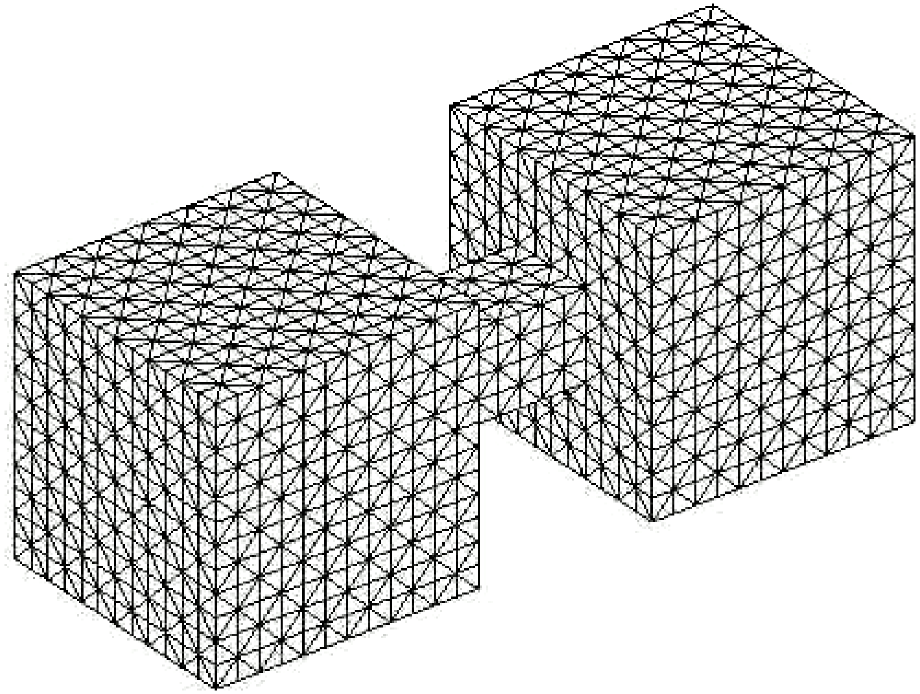


Dumbbell

($K = 1826$, $J = 3648$, $\tau = 10^{-4}$ and $T = 0.18$)

Numerical Results

(SD$_\gamma$) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-2}$)

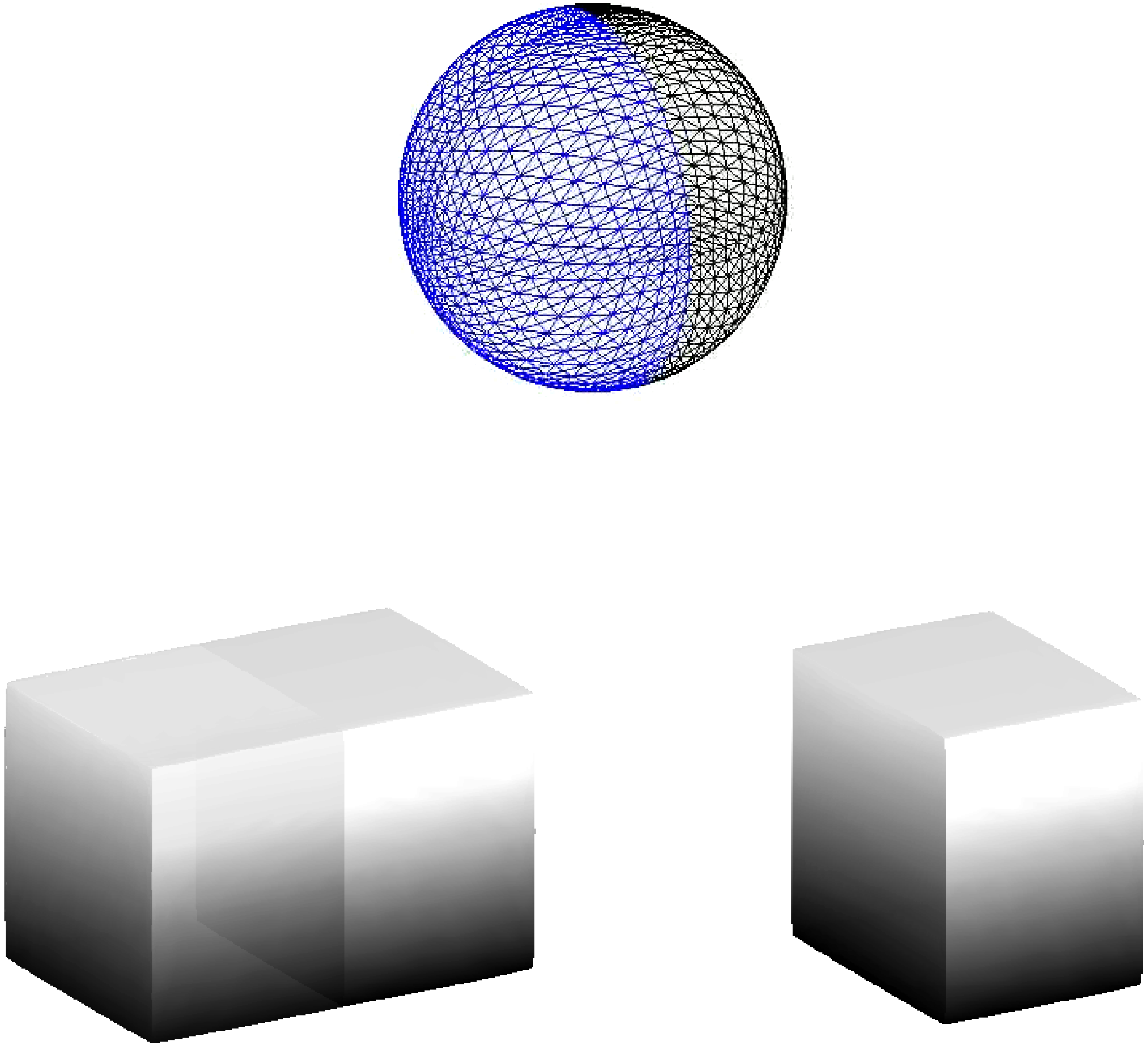


Dumbbell

($K = 1826$, $J = 3648$, $\tau = 10^{-4}$ and $T = 3.5$)

Numerical Results

(SD_γ) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-1}$)



($K = 3267$, $J = 6240$, $\tau = 10^{-3}$ and $T = 1$)

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5. ——, *On the parametric finite element approximation of evolving hypersurfaces in* $\mathbb{R}^3$, J. Comput. Phys., **227** (2008), pp. 4281–4307.

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7. ——, *Parametric approximation of three dimensional surface clusters*, 2008 (in preparation).

Anisotropic Surface Energy

$$|\Gamma|_\gamma := \int_\Gamma \gamma(\vec{\nu})\,\mathrm{d}s$$

where $\gamma : \mathbb{R}^d \setminus \{\vec{0}\} \to \mathbb{R}_{>0}$ is a given anisotropy function,
which we will assume is absolutely homogeneous of degree one, i.e.

$$\gamma(\lambda\,\vec{p}) = |\lambda|\,\gamma(\vec{p}) \quad \forall\ \vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\},\ \forall\ \lambda \in \mathbb{R} \setminus \{0\}\,.$$

Let $\Gamma(\varepsilon) := \{\vec{z} + \varepsilon\,\vec{g}(\vec{z}) : \vec{z} \in \Gamma\}$. First variation of this energy yields

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,|\Gamma(\varepsilon)|_\gamma\,|_{\varepsilon=0} = -\int_\Gamma \vec{\varkappa}_\gamma \,.\, \vec{g}\,\mathrm{d}s\,;$$

where

Weighted mean curvature vector: $\vec{\varkappa}_\gamma := \varkappa_\gamma\,\vec{\nu}$,

Weighted mean curvature: $\varkappa_\gamma := -\nabla_s \,.\, \vec{\nu}_\gamma$,

Cahn–Hoffmann vector: $\vec{\nu}_\gamma := \gamma'(\vec{\nu})$ Cahn, Hoffmann (74).

In the isotropic case, $\gamma(\vec{p}) = |\vec{p}|$, we have that

$|\Gamma|_\gamma \equiv |\Gamma|$, $\vec{\nu}_\gamma \equiv \vec{\nu}$, $\varkappa_\gamma \equiv \varkappa$ and $\vec{\varkappa}_\gamma \equiv \vec{\varkappa}$.

Anisotropic Energy Decrease

This leads to the geometric evolution equations

Anisotropic mean curvature flow: $\mathcal{V} = \beta(\vec{\nu})\varkappa_\gamma$ (MC_γ),

Anisotropic surface diffusion: $\mathcal{V} = -\nabla_s\,.\,(\beta(\vec{\nu})\,\nabla_s\,\varkappa_\gamma)$ (SD_γ);

where $\beta : S^{d-1} \to \mathbb{R}_{>0}$ is an anisotropic mobility.

Similarly to the isotropic case, $\gamma(\vec{p}) = |\vec{p}|$, we have that

$$\frac{d}{dt}|\Gamma(t)|_\gamma = -\int_{\Gamma(t)} \mathcal{V}\,\varkappa_\gamma\,\mathrm{d}s = \begin{cases} -\displaystyle\int_{\Gamma(t)} \beta(\vec{\nu})\,\varkappa_\gamma^2\,\mathrm{d}s \leq 0 & (\mathrm{MC}_\gamma), \\ -\displaystyle\int_{\Gamma(t)} \beta(\vec{\nu})\,(\nabla_s\,\varkappa_\gamma)^2\,\mathrm{d}s \leq 0 & (\mathrm{SD}_\gamma). \end{cases}$$

Let $\Omega(t)$ be bounded region with boundary $\Gamma(t)$, $\vec{\nu}$ outward unit normal $\Rightarrow$

$$\frac{d}{dt}|\Omega(t)| = \int_{\Gamma(t)} \mathcal{V}\,\mathrm{d}s = -\int_{\Gamma(t)} \nabla_s\,.\,(\beta(\vec{\nu})\,\nabla_s\,\varkappa_\gamma)\,\mathrm{d}s = 0 \qquad (\mathrm{SD}_\gamma).$$

$\vec{x} : \mathcal{M} \to \mathbb{R}^d$ a parameterization of Γ, anisotropic analogue of $\varkappa\,\vec{\nu} = \Delta_s\,\vec{x}$ is

$$\varkappa_\gamma\,\vec{\nu} = -\nabla_s\,.\,(\vec{\nu}\,[\gamma'(\vec{\nu})]^T) + \nabla_s\,.\,(\gamma(\vec{\nu})\,\nabla_s\,\vec{x}) - \gamma(\vec{\nu})\,\Delta_s\,\vec{x}$$

$$\equiv \quad [\gamma'(\vec{\nu})]_s^\perp \qquad (\text{if } d = 2\text{, i.e. a curve, and } \vec{\nu} = -\vec{x}_s^\perp).$$

Existing Parametric Approaches

There exist parametric finite element approximations for (MC_γ) and (SD_γ).

- Dziuk (99) (MC_γ) $d = 2$, i.e. curves in $\mathbb{R}^2$ $\quad \vec\nu = -\vec x_s^\perp$

  $$\int_\Gamma [\beta(\vec\nu)]^{-1}\, \vec x_t \,.\, \vec\varphi \,\mathrm{d}s + \int_\Gamma \gamma'(\vec x_s^\perp) \,.\, \vec\varphi_s^\perp \,\mathrm{d}s = 0\,.$$

  Stability and error bounds for semidiscrete scheme, for an admissable class of anisotropies.

  See also Pozzi (07) for an extension to curves in $\mathbb{R}^d$, $d \geq 2$.

- Clarenz, Dziuk, Rumpf (03) (MC_γ) $\beta(\vec\nu) \equiv 1$ and $d = 3$, i.e. surfaces in $\mathbb{R}^3$

  $$\int_\Gamma \vec x_t \,.\, \vec\varphi \,\mathrm{d}s + \int_\Gamma \gamma(\vec\nu)\, \nabla_s \vec x \,.\, \nabla_s \vec\varphi \,\mathrm{d}s = \int_\Gamma [\vec\nu\, [\gamma'(\vec\nu)]^T\,] \,.\, \nabla_s \vec\varphi \,\mathrm{d}s$$

- Haußer, Voigt (06) (SD_γ) $\beta(\vec\nu) \equiv 1$ and $d = 3$

  $$\vec x_t = \vec{\mathcal{V}} = \mathcal{V}\,\vec\nu, \qquad \mathcal{V} = -\Delta_s\, \varkappa_\gamma, \qquad \varkappa_\gamma = \vec\varkappa_\gamma \,.\, \vec\nu,$$

  $$\int_\Gamma \vec\varkappa_\gamma \,.\, \vec\varphi \,\mathrm{d}s + \int_\Gamma \gamma(\vec\nu)\, \nabla_s \vec x \,.\, \nabla_s \vec\varphi \,\mathrm{d}s = \int_\Gamma [\vec\nu\, [\gamma'(\vec\nu)]^T\,] \,.\, \nabla_s \vec\varphi \,\mathrm{d}s$$

There exist **no stability results** for the corresponding fully discrete approximations. All of these approaches have zero tangential velocity.

Anisotropic Surface Energy

The desired stability property leads us to the following class of anisotropies.

$$\gamma(\vec{p}) = \sum_{\ell=1}^{L} \gamma_\ell(\vec{p}) = \sum_{\ell=1}^{L} \sqrt{\vec{p}\,.\,G_\ell\,\vec{p}} \qquad \text{in the case } d = 2,$$

where $G_\ell \in \mathbb{R}^{2\times 2}$, $\ell = 1 \to L$, are symmetric and positive definite.

Example: $\gamma(\vec{p}) = \left[\sum_{i=1}^{2} \alpha_i^2\, p_i^2\right]^{\frac{1}{2}}$ and the regularized l^1-norm

$$\gamma(\vec{p}) = \sum_{i=1}^{2} \sqrt{\varepsilon^2\,|\vec{p}|^2 + p_i^2}\,, \qquad \varepsilon > 0\,.$$

In the latter case

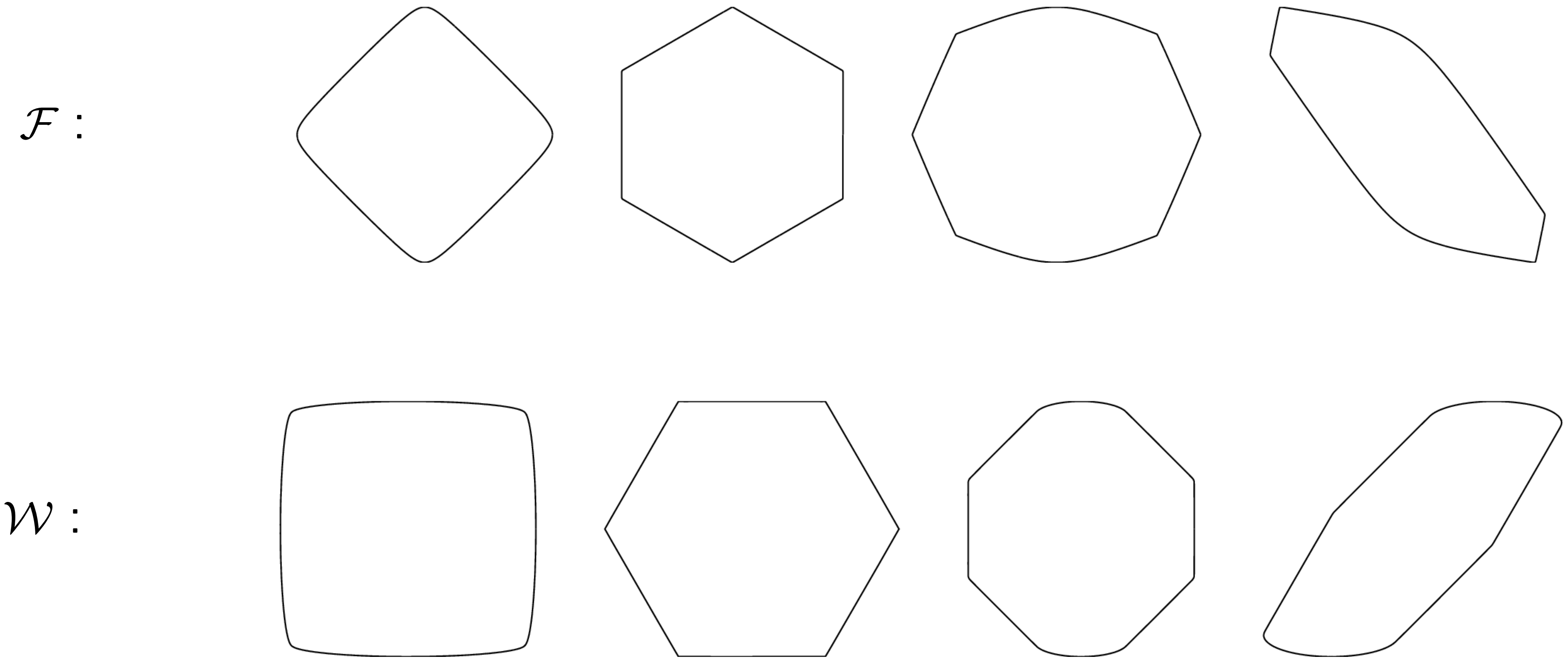
$$G_1 = \begin{pmatrix} 1+\varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \qquad \text{and} \qquad G_2 = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1+\varepsilon^2 \end{pmatrix}.$$

Note that $\gamma'(\vec{p}) = \sum_{\ell=1}^{L} [\gamma_\ell(\vec{p})]^{-1}\, G_\ell\, \vec{p}\,.$

Visualization of Anisotropy

Frank diagram: $\mathcal{F} := \{\vec{p} \in \mathbb{R}^d : \gamma(\vec{p}) \leq 1\}$

Wulff shape: $\mathcal{W} := \{\vec{q} \in \mathbb{R}^d : \gamma^*(\vec{q}) \leq 1\}$, where $\gamma^*(\vec{q}) = \sup_{\vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}} \frac{\vec{p} \, . \, \vec{q}}{\gamma(\vec{p})}$.



A wide class of anisotropies can be modelled by this choice.

Including regularized crystalline surface energies.

Parametric Finite Element Approximation

Recall

$$\int_\Gamma \vec{x}_t \,.\, \vec{\nu}\,\varphi \,\mathrm{d}s = \begin{cases} \displaystyle\int_\Gamma \beta(\vec{\nu})\,\varkappa_\gamma\,\varphi\,\mathrm{d}s \\ \displaystyle\int_\Gamma \beta(\vec{\nu})\,\nabla_s \varkappa_\gamma \,.\, \nabla_s\varphi\,\mathrm{d}s \end{cases}, \qquad \int_\Gamma \varkappa_\gamma\,\vec{\nu}\,.\,\vec{\varphi}\,\mathrm{d}s + \int_\Gamma \gamma'(\vec{x}_s^\perp)\,.\,\vec{\varphi}_s^\perp\,\mathrm{d}s = 0\,.$$

$(\mathcal{P}^h_\gamma)$ Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}^h \times V^h$ such that

$$\langle\frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\,\vec{\nu}^m\rangle_m^h - \begin{cases} \langle\beta(\vec{\nu}^m)\,\kappa_\gamma^{m+1}, \chi\rangle_m^h \\ \langle\beta(\vec{\nu}^m)\,\nabla_s\,\kappa_\gamma^{m+1}, \nabla_s\,\chi\rangle_m \end{cases} = 0 \qquad \forall\ \chi \in V^h,$$

$$\langle\kappa_\gamma^{m+1}\,\vec{\nu}^m, \vec{\eta}\rangle_m^h + \sum_{\ell=1}^{L}\langle[\gamma_\ell(\vec{\nu}^m)]^{-1}\,G_\ell\,[\vec{X}_s^{m+1}]^\perp, \vec{\eta}_s^\perp\rangle_m = 0 \qquad \forall\ \vec{\eta} \in \underline{V}^h,$$

where

$$\langle f, g\rangle_m := \int_{\Gamma^m} f\,.\,g\,\mathrm{d}s = \int_I f\,.\,g\,|\vec{X}_\rho^m|\,\mathrm{d}\rho\,.$$

Linear system for $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\}$.

Note that for $\gamma(\vec{p}) = |\vec{p}|$ and $\beta \equiv 1$ the scheme $(\mathcal{P}^h_\gamma)$ collapses to the isotropic scheme discussed earlier.

THEOREM

Let $(\mathcal{A})$ hold. Then $\exists!$ solution $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}^h \times V^h$. Moreover, for $k = 1 \to M$ it holds that

$$|\Gamma^k|_\gamma + \sum_{m=0}^{k-1} \tau_m \begin{cases} \langle \beta(\vec{\nu}^m)\, \kappa_\gamma^{m+1}, \kappa_\gamma^{m+1} \rangle_m^h \\ \langle \beta(\vec{\nu}^m)\, \nabla_s\, \kappa_\gamma^{m+1}, \nabla_s\, \kappa_\gamma^{m+1} \rangle_m \end{cases} \leq |\Gamma^0|_\gamma \,.$$

Continuous in time semidiscrete scheme $\{\vec{X}(t), \kappa_\gamma(t)\}$:

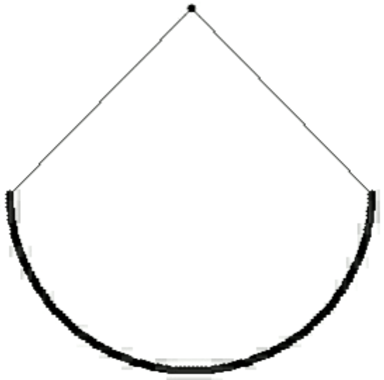
(a) Area conservation for (SD$_\gamma$): $0 = \langle \vec{X}_t, \vec{\nu}^h \rangle^h = \int_{\Gamma^h} \vec{X}_t \,.\, \vec{\nu}^h \,\mathrm{d}s = \frac{d}{dt} |\Omega^h(t)|$

(b) Equidistribution of mesh points for $\vec{X}(t)$, where $\vec{X}(t)$ not locally parallel, for any $t > 0$, with respect to a non-trivial weighting function.

Tangential Movement

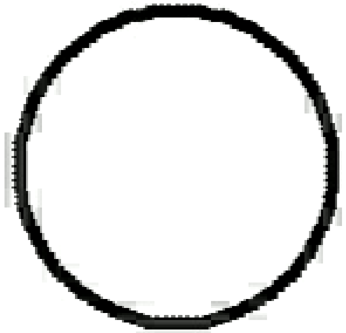
(SD$_\gamma$) $\gamma(\vec{p}) = \sqrt{p_1^2 + 0.1\,p_2^2}$ $\Rightarrow$ Wulff shape is $\sqrt{10} : 1$ ellipse

$N = 128$, $\tau = 10^{-3}$, $T = 1$.



(SD$_\gamma$) $\gamma(\vec{p}) = \sqrt{p_1^2 + 0.01\,p_2^2}$ $\Rightarrow$ Wulff shape is $10 : 1$ ellipse

$N = 128$, $\tau = 10^{-3}$, $T = 1$.

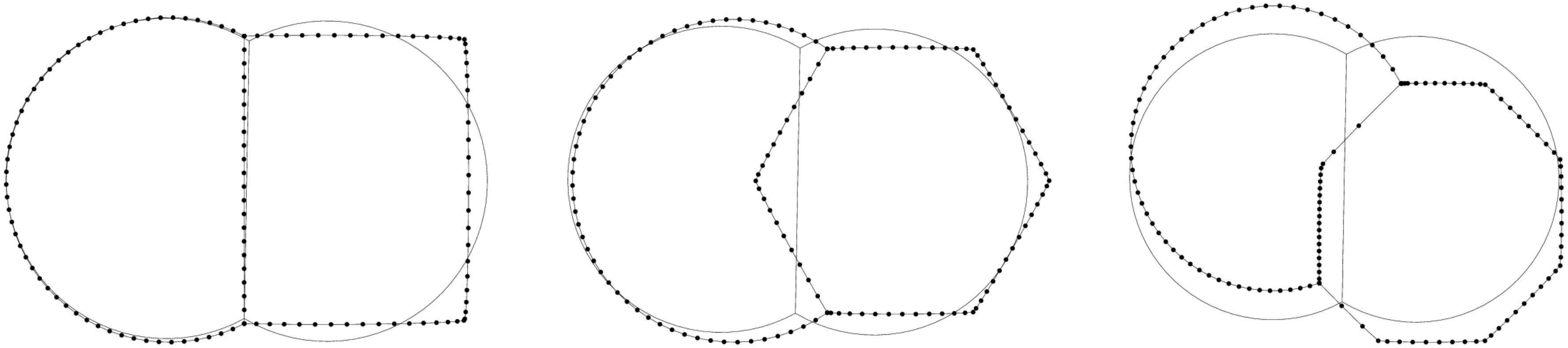


All the previous algorithms for curves (e.g networks, $\mathbb{R}^d$, geodesic flows) can be generalised to the anisotropic case.

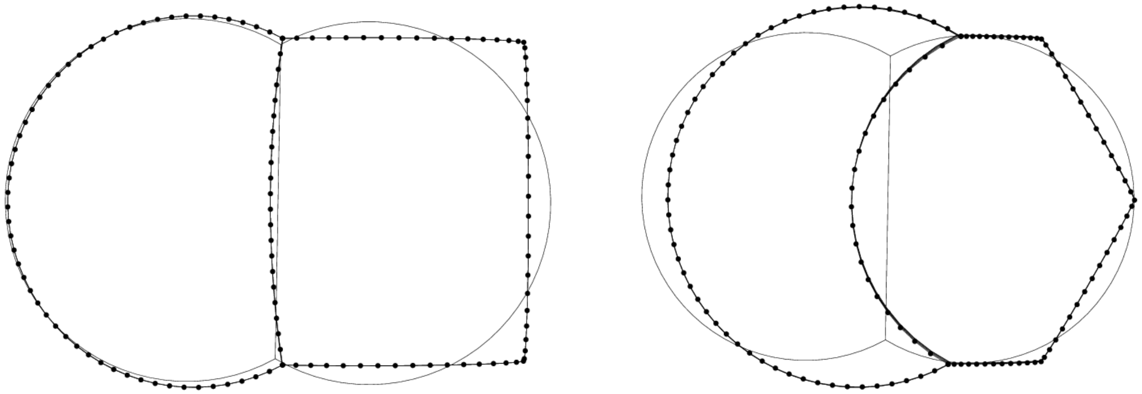
Linear system. Existence, uniqueness, stability and equidistribution.

Anisotropic Double Bubbles

E.g. (SD$_\gamma$) : $\gamma_1(\vec{p}) = |\vec{p}|$, $\gamma_i(\vec{p}) = \gamma_0(\vec{p})$, $i = 2 \rightarrow 3.$ $\varepsilon_\ell^2 \equiv 0.001$



$\gamma_1(\vec{p}) = \gamma_2(\vec{p}) = |\vec{p}|$, $\gamma_3(\vec{p}) = \gamma_0(\vec{p})$. $\varepsilon_\ell^2 \equiv 0.001$



Anisotropic Surface Energies for $d = 3$

Introduce a symmetric anisotropic version of $\varkappa \vec\nu = \Delta_s \vec x$, as opposed to

$$\varkappa_\gamma \vec\nu = -\nabla_s \,.\, (\vec\nu\, [\gamma'(\vec\nu)]^T) + \nabla_s \,.\, (\gamma(\vec\nu)\, \nabla_s \vec x) - \gamma(\vec\nu)\, \Delta_s \vec x \,.$$

For

$$\gamma(\vec p) = \left(\sum_{\ell=1}^{L} [\gamma_\ell(\vec p)]^r \right)^{\frac{1}{r}}, \qquad \gamma_\ell(\vec p) := \sqrt{\vec p \,.\, G_\ell\, \vec p}, \qquad r \in [1, \infty)\,,$$

where $G_\ell \in \mathbb{R}^{3\times 3}$, $\ell = 1 \to L$, are symmetric and positive definite, we obtain the identity

$$\varkappa_\gamma \vec\nu = \sum_{\ell=1}^{L} \gamma_\ell(\vec\nu)\, \widetilde G_\ell\, \nabla_s^{\widetilde G_\ell} \,.\, \left[\left[\frac{\gamma_\ell(\vec\nu)}{\gamma(\vec\nu)} \right]^{r-1} \nabla_s^{\widetilde G_\ell}\, \vec x \right] .$$

Here

$$\widetilde G_\ell := [\det G_\ell]^{\frac{1}{2}}\, G_\ell^{-1}, \qquad \ell = 1 \to L\,,$$

and $\nabla_s^{\widetilde G_\ell}$, $\nabla_s^{\widetilde G_\ell} .$ are anisotropic surface gradient and divergence operators induced by the inner product

$$(\vec u, \vec v)_{\widetilde G_\ell} := \vec u \,.\, \widetilde G_\ell\, \vec v \qquad \forall\ \vec u,\, \vec v \in \mathbb{R}^3\,.$$

Parametric Finite Element Approximation

Based on the identity

$$\varkappa_\gamma\,\vec{\nu} = \sum_{\ell=1}^{L} \gamma_\ell(\vec{\nu})\,\widetilde{G}_\ell\,\nabla_s^{\widetilde{G}_\ell} \cdot \left[\left[\frac{\gamma_\ell(\vec{\nu})}{\gamma(\vec{\nu})}\right]^{r-1} \nabla_s^{\widetilde{G}_\ell}\,\vec{x}\right],$$

we obtain

$(\mathcal{P}^h_\gamma)$ Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}^h(\Gamma^m) \times V^h(\Gamma^m)$ such that

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi\,\vec{\nu}^m\rangle_m^h - \begin{cases} \langle \kappa_\gamma^{m+1}, \chi\rangle_m^h \\ \langle \nabla_s\,\kappa_\gamma^{m+1}, \nabla_s\,\chi\rangle_m \end{cases} = 0 \qquad \forall\ \chi \in V^h(\Gamma^m),$$

$$\langle \kappa_\gamma^{m+1}\,\vec{\nu}^m, \vec{\eta}\rangle_m^h + \sum_{\ell=1}^{L} \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{\nu}^{m+1})}{\gamma(\vec{\nu}^{m+1})}\right]^{r-1} (\nabla_s^{\widetilde{G}_\ell}\,\vec{X}^{m+1}, \nabla_s^{\widetilde{G}_\ell}\,\vec{\eta})_{\widetilde{G}_\ell}\,\gamma_\ell(\vec{\nu}^m)\,\mathrm{d}s = 0$$

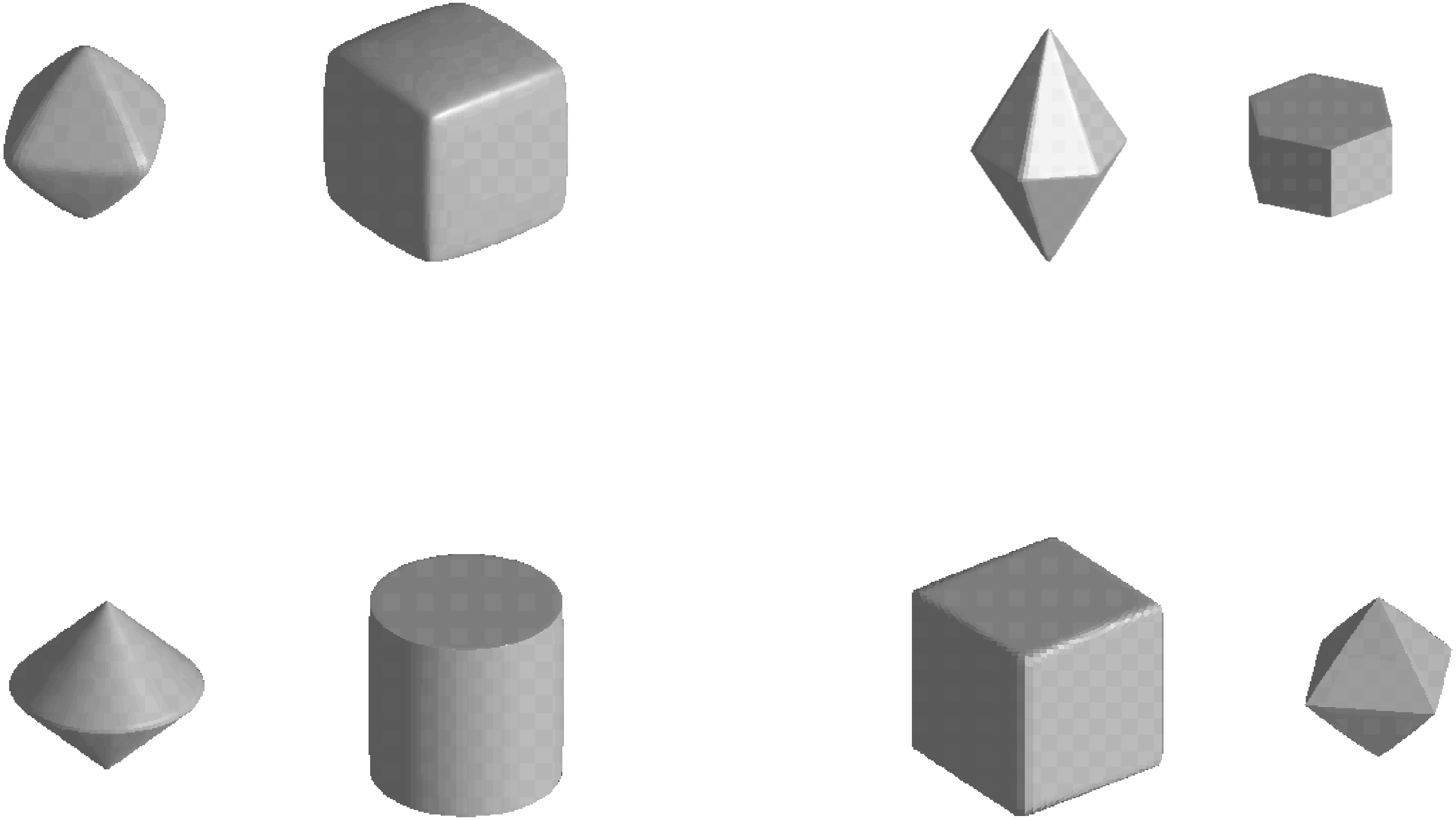
$$\forall\ \vec{\eta} \in \underline{V}^h(\Gamma^m)\,.$$

$r = 1$: Linear system. Existence, uniqueness and stability.

$r > 1$: Nonlinear system. Stability.

Wulff Shapes $d = 3$

Examples of Frank diagrams $\mathcal{F}$ and associated Wulff shapes $\mathcal{W}$:

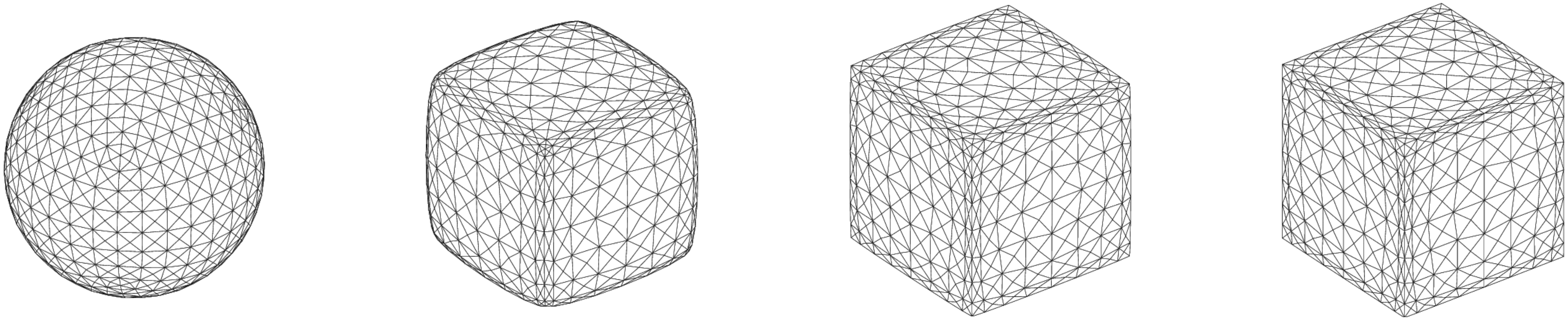


Numerical Results

(SD$_\gamma$) - Cubic Wulff Shape,

i.e. $\gamma(\vec{p}) = \sum_{i=1}^{d} \sqrt{\varepsilon^2 |\vec{p}|^2 + p_i^2(1-\varepsilon^2)}$ is a regularized l^1-norm.

$(r = 1,\ L = 3,\ G_1 = \text{diag}(1, \varepsilon^2, \varepsilon^2),\ G_2 = \text{diag}(\varepsilon^2, 1, \varepsilon^2),\ G_3 = \text{diag}(\varepsilon^2, \varepsilon^2, 1))$



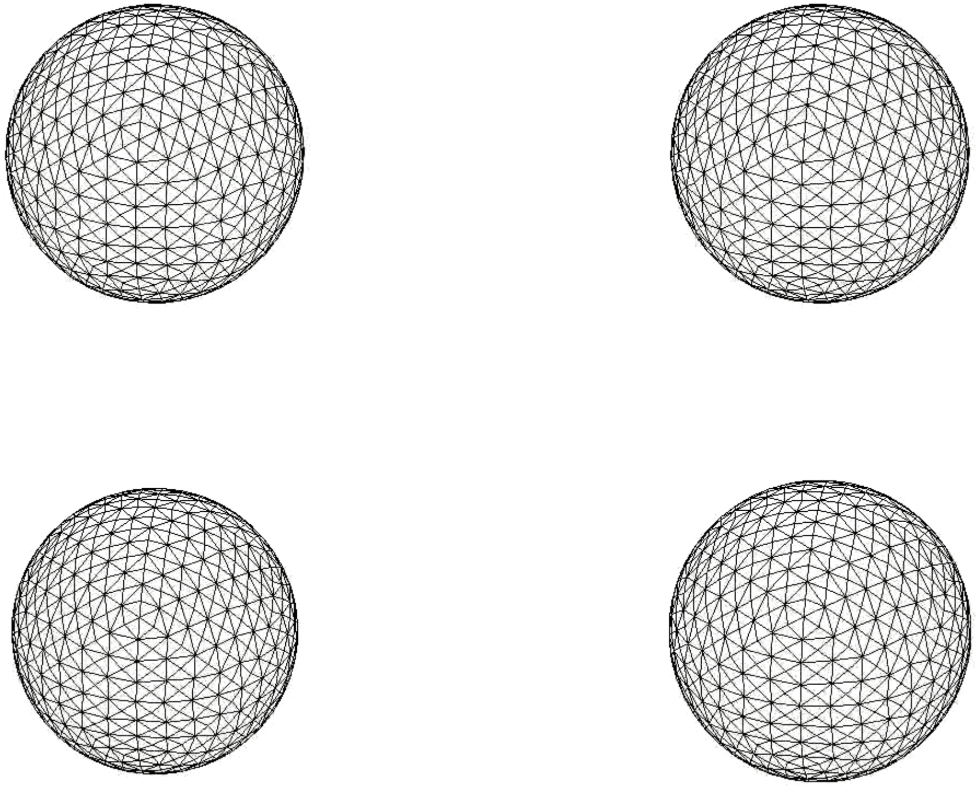
A plot of $\vec{X}(0.5)$ for $\varepsilon = 10^{-k}$, $k = 0 \to 3$.

$(K = 770,\ J = 1536,\ \tau = 10^{-3})$

Numerical Results

(SD$_\gamma$) - Cube ($r = 1$, $L = 3$), Hexagonal Prism ($r = 1$, $L = 4$),

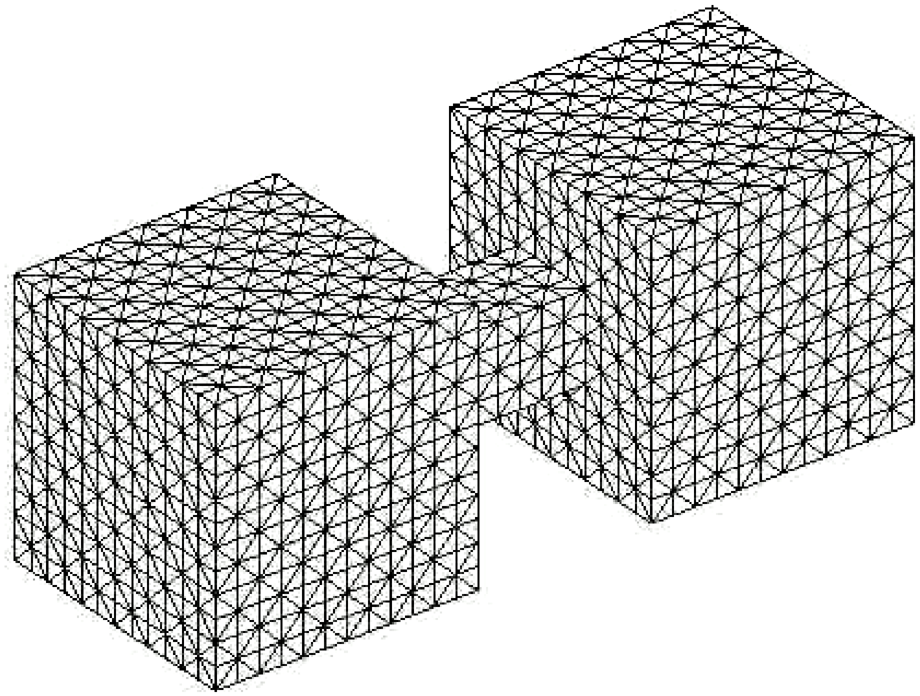
Octahedron ($r = 30$, $L = 3$), Cylinder Wulff Shape ($r = 1$, $L = 2$).



($\varepsilon = 10^{-2}$, $K = 770$, $J = 1536$, $\tau = 10^{-4}$, $T = 0.1,\ 0.25,\ 0.02,\ 0.25$)

Numerical Results

(MC_γ) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-2}$)

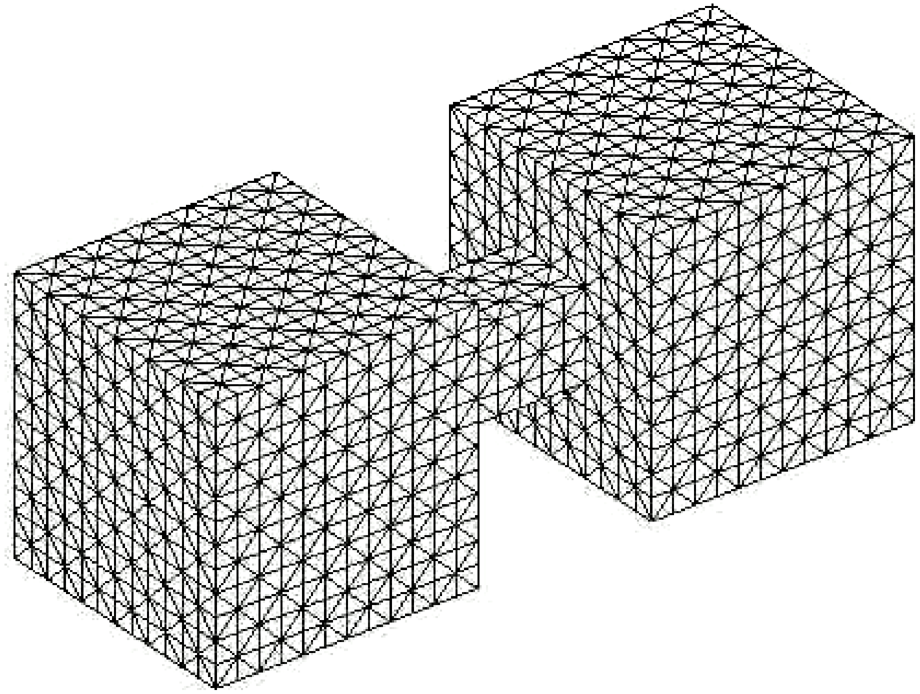


Dumbbell

($K = 1826,\ J = 3648,\ \tau = 10^{-4}$ and $T = 0.18$)

Numerical Results

(SD$_\gamma$) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-2}$)

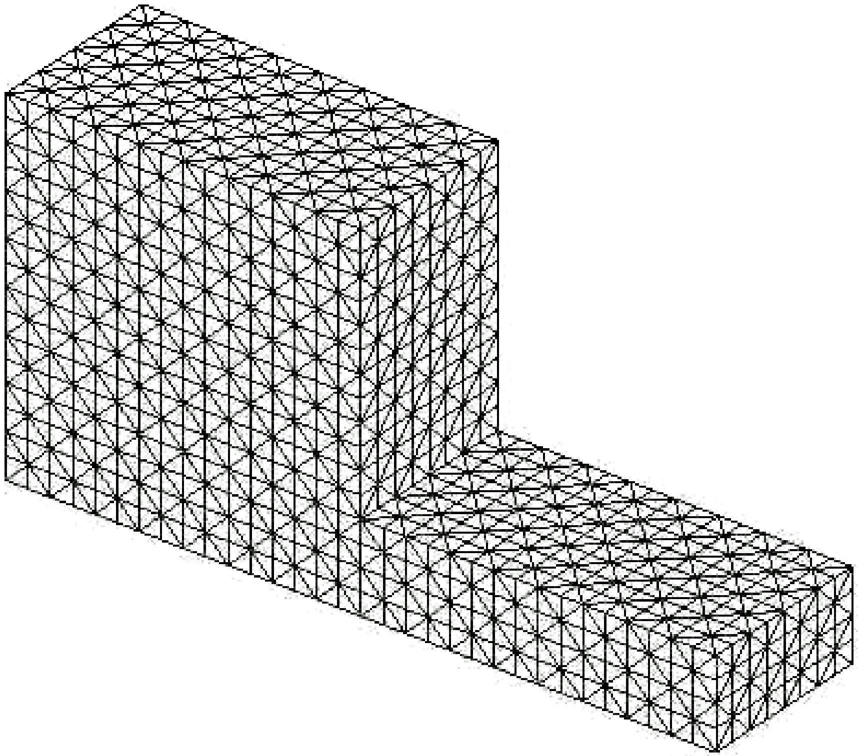


Dumbbell

($K = 1826$, $J = 3648$, $\tau = 10^{-4}$ and $T = 3.5$)

Numerical Results

(SD$_\gamma$) - Cubic Wulff Shape ($r = 1$, $L = 3$, $\varepsilon = 10^{-2}$)



Facet breaking

($K = 1410$, $J = 2816$, $\tau = 10^{-3}$, $T = 5$)

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