Modeling, Analysis and Numerics in Electrohydrodynamics

Markus Schmuck

University Tübingen

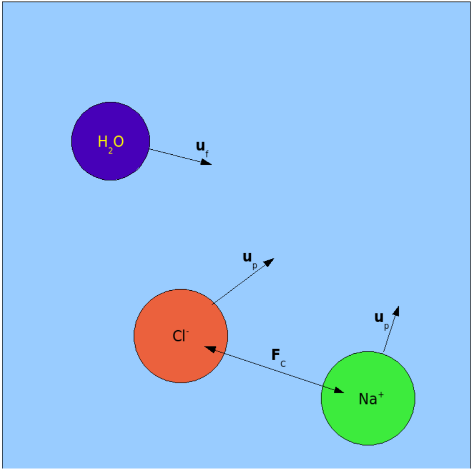
Warwick, 2009

Outline

1. Continuous Model and Analytical Results
   - Analytical Results in $3D$
   - Approximation Goal

2. Finite Element Discretization
   - Van Roosbroeck part ($\mathbf{u} = 0$)
   - Results ($\mathbf{u} = 0$)
   - The whole EHD system ($\mathbf{u} \neq 0$)
   - Results ($\mathbf{u} \neq 0$)

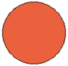
Electrokinetics describes

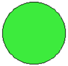


$\mathbf{u}_f$ fluid velocity (continuum hypothesis)

$\mathbf{u}_p$ particle velocity

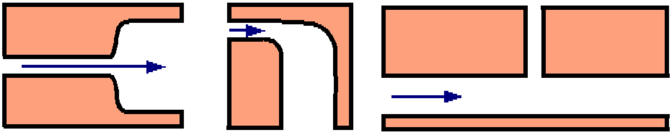
$\mathbf{F}_c$ Coulomb force


negative charged species


positive charged species

Questions in Electrokinetics: (Modeling, Numerics)

- What happens at the solid/electrolyte interphase?
- Behavior of species in special geometries?

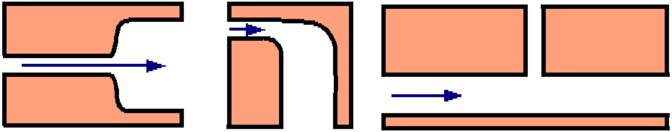


Goal: Construction/Optimazation of innovative devices:

- separation devices, micro-/nano- mixers
- supercapacitors (electric vehicles)
- desalination devices

Questions in Electrokinetics: (Modeling, Numerics)

- What happens at the solid/electrolyte interphase?
- Behavior of species in special geometries?

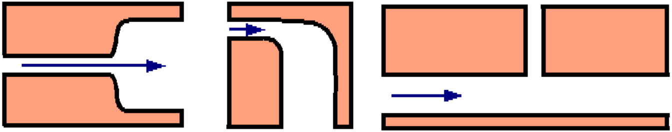


Goal: Construction/Optimazation of innovative devices:

- separation devices, micro-/nano- mixers
- supercapacitors (electric vehicles)
- desalination devices

Questions in Electrokinetics: (Modeling, Numerics)

- What happens at the solid/electrolyte interphase?
- Behavior of species in special geometries?

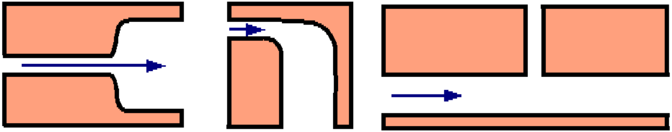


Goal: Construction/Optimazation of innovative devices:

- separation devices, micro-/nano- mixers
- supercapacitors (electric vehicles)
- desalination devices

Questions in Electrokinetics: (Modeling, Numerics)

- What happens at the solid/electrolyte interphase?
- Behavior of species in special geometries?

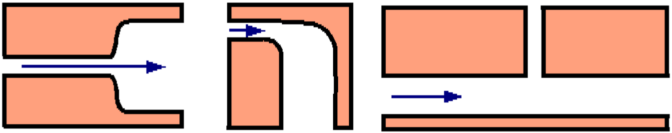


Goal: Construction/Optimazation of innovative devices:

- separation devices, micro-/nano- mixers
- supercapacitors (electric vehicles)
- desalination devices

Questions in Electrokinetics: (Modeling, Numerics)

- What happens at the solid/electrolyte interphase?
- Behavior of species in special geometries?



Goal: Construction/Optimazation of innovative devices:

- separation devices, micro-/nano- mixers
- supercapacitors (electric vehicles)
- desalination devices

Basic Continuous Model

Navier-Stokes equations:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} - \Delta \mathbf{u} + \nabla p = -(n^+ - n^-)\nabla\psi \quad \text{in } \Omega_T$$
$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_T$$

Nernst-Planck-Poisson equations:

$$\partial_t n^\pm \mp \operatorname{div}\left(n^\pm \nabla\psi\right) - \Delta n^\pm + (\mathbf{u} \cdot \nabla)\,n^\pm = 0 \quad \text{in } \Omega_T$$

$$-\Delta\psi = n^+ - n^- \quad \text{in } \Omega_T$$

Basic Continuous Model

Navier-Stokes equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \Delta\mathbf{u} + \nabla p = -(n^+ - n^-)\nabla\psi \quad \text{in } \Omega_T$$
$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_T$$

Nernst-Planck-Poisson equations:

$$\partial_t n^\pm \mp \operatorname{div}\left(n^\pm\nabla\psi\right) - \Delta n^\pm + (\mathbf{u}\cdot\nabla)\,n^\pm = 0 \quad \text{in } \Omega_T$$

$$-\Delta\psi = n^+ - n^- \quad \text{in } \Omega_T$$

Basic Continuous Model

Navier-Stokes equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \Delta\mathbf{u} + \nabla p = -(n^+ - n^-)\nabla\psi \quad \text{in } \Omega_T$$
$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega_T$$

Nernst-Planck-Poisson equations:

$$\partial_t n^\pm \mp \operatorname{div}\left(n^\pm \nabla\psi\right) - \Delta n^\pm + (\mathbf{u}\cdot\nabla)\, n^\pm = 0 \quad \text{in } \Omega_T$$

$$-\Delta\psi = n^+ - n^- \quad \text{in } \Omega_T$$

The role of the energy and the entropy:

Energy in

- **Continous Setting**: not required for (global) existence
- **Discrete Setting**: uniform bounds to study long-time asymptotics

Entropy in

- **Continuous Setting**: characterizes long-time asymptotics
- **Discrete Setting**: characterizes long-time asymptotics

The role of the energy and the entropy:

Energy in

- **Continous Setting**: not required for (global) existence
- **Discrete Setting**: uniform bounds to study long-time asymptotics

Entropy in

- **Continuous Setting**: characterizes long-time asymptotics
- **Discrete Setting**: characterizes long-time asymptotics

The role of the energy and the entropy:

Energy in

- **Continous Setting**: not required for (global) existence
- **Discrete Setting**: uniform bounds to study long-time asymptotics

Entropy in

- **Continuous Setting**: characterizes long-time asymptotics
- **Discrete Setting**: characterizes long-time asymptotics

The role of the energy and the entropy:

Energy in

- **Continous Setting**: not required for (global) existence
- **Discrete Setting**: uniform bounds to study long-time asymptotics

Entropy in

- **Continuous Setting**: characterizes long-time asymptotics
- **Discrete Setting**: characterizes long-time asymptotics

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :

Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in 3D (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :
Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :

Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Results in $3D$ (Schmuck [07])

- Existence of weak solutions: Schauder's fixed point thm
- Existence of local strong solutions
- Characterizations of weak solutions:
  (also for the pure van Roosbroeck equations)
  - $L^\infty(\Omega_T)$-bound (Moser iteration)
  - (Lyapunov) entropy-law
  - Energy law
  - Non-negativity of $n^\pm$

Based on :

Gajewski/Gröger [86]; Bihler/Nadziedja [94]; Temam (book) [01]

Approximation Goal

Discretization by affine finite elements:

Recover results from the continuous setting to the discrete world

Part II

Outline

Energies and Characterizations not immediate:

Problems		**Tools**
$\operatorname{div} \mathbf{u}_h \neq 0$	$\Rightarrow$	skew symmetricity of $\left(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\right)$
No Moser-iteration	$\Rightarrow$	1) **M-matrix** $\Rightarrow$ d.m.p 2) entropy provider S_ϵ
$\log$ not linear	$\Rightarrow$	1) **perturbed entropy**, 2) entropy provider S_ϵ

Approach: **1**) energy based,
2) entropy based

2) **bases on :** Grün/Rumpf [00]; Barrett/Nürnberg [04];

Energies and Characterizations not immediate:

Problems		**Tools**
$\operatorname{div} \mathbf{u}_h \neq 0$	$\Rightarrow$	skew symmetricity of $\left(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\right)$
No Moser-iteration	$\Rightarrow$	**1**) **M-matrix** $\Rightarrow$ d.m.p 2) entropy provider S_ϵ
$\log$ not linear	$\Rightarrow$	1) **perturbed entropy**, 2) entropy provider S_ϵ

Approach: **1**) energy based,
2) entropy based

2) **bases on :** Grün/Rumpf [00]; Barrett/Nürnberg [04];

Energies and Characterizations not immediate:

Problems | **Tools**

- $\operatorname{div} \mathbf{u}_h \neq 0$ $\Rightarrow$ skew symmetricity of $\big(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\big)$
- No Moser-iteration $\Rightarrow$ **1**) **M-matrix** $\Rightarrow$ d.m.p
  2) entropy provider S_ϵ
- $\log$ not linear $\Rightarrow$ **1**) **perturbed entropy**,
  2) entropy provider S_ϵ

Approach: **1**) energy based,
2) entropy based

2) **bases on :** Grün/Rumpf [00]; Barrett/Nürnberg [04];

Energies and Characterizations not immediate:

Problems		**Tools**
$\operatorname{div} \mathbf{u}_h \neq 0$	$\Rightarrow$	skew symmetricity of $\big(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\big)$
No Moser-iteration	$\Rightarrow$	**1**) **M-matrix** $\Rightarrow$ d.m.p 2) entropy provider S_ϵ
$\log$ not linear	$\Rightarrow$	**1**) **perturbed entropy**, 2) entropy provider S_ϵ

Approach: **1**) energy based,
2) entropy based

2) **bases on :** Grün/Rumpf [00]; Barrett/Nürnberg [04];

Energies and Characterizations not immediate:

Problems		**Tools**
$\operatorname{div} \mathbf{u}_h \neq 0$	$\Rightarrow$	skew symmetricity of $\left(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\right)$
No Moser-iteration	$\Rightarrow$	**1**) **M-matrix** $\Rightarrow$ d.m.p 2) entropy provider S_ϵ
$\log$ not linear	$\Rightarrow$	**1**) **perturbed entropy**, 2) entropy provider S_ϵ

Approach: **1**) energy based, 2) entropy based

2) **bases on :** Grün/Rumpf [00]; Barrett/Nürnberg [04];

Two Discretization Strategies

Schemes:

$$(\mathrm{d}_t(\mathcal{N}^{\pm})^j, \Phi)_h + (\nabla(\mathcal{N}^{\pm})^j, \nabla\Phi) \pm (R_\epsilon[(\mathcal{N}^{\pm})^j]\nabla\Psi^j, \nabla\Phi) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = (P^j - N^j, \Phi)_h,$$

where

$$R_\epsilon[X] = \begin{cases} X & \text{energy based} \\ S_\epsilon(X) & \text{entropy based} \end{cases}$$

Remark: $S_\epsilon(\cdot)$ is a perturbation of the identity!

Two Discretization Strategies

Schemes:

$$(\mathrm{d}_t(\mathcal{N}^{\pm})^j, \Phi)_h + (\nabla(\mathcal{N}^{\pm})^j, \nabla\Phi) \pm \left(R_\epsilon[(\mathcal{N}^{\pm})^j]\nabla\Psi^j, \nabla\Phi\right) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = (P^j - N^j, \Phi)_h,$$

where

$$R_\epsilon[X] = \begin{cases} X & \text{energy based} \\ S_\epsilon(X) & \text{entropy based} \end{cases}$$

Remark: $S_\epsilon(\cdot)$ is a perturbation of the identity!

Two Discretization Strategies

Schemes:

$$(\mathrm{d}_t(\mathcal{N}^{\pm})^j, \Phi)_h + (\nabla(\mathcal{N}^{\pm})^j, \nabla\Phi) \pm \left(R_\epsilon[(\mathcal{N}^{\pm})^j]\nabla\Psi^j, \nabla\Phi\right) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = (P^j - N^j, \Phi)_h,$$

where

$$R_\epsilon[X] = \begin{cases} X & \text{energy based} \\ S_\epsilon(X) & \text{entropy based} \end{cases}$$

Remark: $S_\epsilon(\cdot)$ is a perturbation of the identity!

Two Discretization Strategies

Schemes:

$$(\mathrm{d}_t(\mathcal{N}^{\pm})^j, \Phi)_h + (\nabla(\mathcal{N}^{\pm})^j, \nabla\Phi) \pm \left(R_\epsilon[(\mathcal{N}^{\pm})^j]\nabla\Psi^j, \nabla\Phi\right) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = (P^j - N^j, \Phi)_h,$$

where

$$R_\epsilon[X] = \begin{cases} X & \text{energy based} \\ S_\epsilon(X) & \text{entropy based} \end{cases}$$

Remark: $S_\epsilon(\cdot)$ is a perturbation of the identity!

Two Discretization Strategies

Schemes:

$$(\mathrm{d}_t(\mathcal{N}^{\pm})^j, \Phi)_h + (\nabla(\mathcal{N}^{\pm})^j, \nabla\Phi) \pm \left(R_\epsilon[(\mathcal{N}^{\pm})^j]\nabla\Psi^j, \nabla\Phi\right) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = (P^j - N^j, \Phi)_h,$$

where

$$R_\epsilon[X] = \begin{cases} X & \text{energy based} \\ S_\epsilon(X) & \text{entropy based} \end{cases}$$

Remark: $S_\epsilon(\cdot)$ is a perturbation of the identity!

Explanations to the Schemes:

Fully implicit Scheme:

$\Rightarrow$ Necessary for energy and entropy properties

Consequence:

$\Rightarrow$ In practice a linearization required

Tool:

$\Rightarrow$ Banach's fixed point theorem (constructive proof for the energy based strategy)

$\Rightarrow$ fully practical algorithm

Explanations to the Schemes:

Fully implicit Scheme:

$\Rightarrow$ Necessary for energy and entropy properties

Consequence:

$\Rightarrow$ In practice a linearization required

Tool:

$\Rightarrow$ Banach's fixed point theorem (constructive proof for the energy based strategy)

$\Rightarrow$ fully practical algorithm

Explanations to the Schemes:

Fully implicit Scheme:

$\Rightarrow$ Necessary for energy and entropy properties

Consequence:

$\Rightarrow$ In practice a linearization required

Tool:

$\Rightarrow$ Banach's fixed point theorem (constructive proof for the energy based strategy)

$\Rightarrow$ fully practical algorithm

Explanations to the Schemes:

Fully implicit Scheme:

$\Rightarrow$ Necessary for energy and entropy properties

Consequence:

$\Rightarrow$ In practice a linearization required

Tool:

$\Rightarrow$ Banach's fixed point theorem (constructive proof for the energy based strategy)

$\Rightarrow$ fully practical algorithm

Energy based approach $\Rightarrow$ perturbed Entropy

Problems:

- $\ln \mathcal{N}^{\pm}$ not allowed test fct. $\Rightarrow$ linear interpolation $\mathcal{I}_h[\ln(\mathcal{N}^{\pm} + \delta)]$
- $\left(\mathcal{N}^{\pm}\nabla\Psi, \nabla\mathcal{I}_h[\ln(\mathcal{N}^{\pm}+\delta)]\right) \neq \left(\nabla\Psi, \nabla\mathcal{N}^{\pm}\right)$
  "$=$" is desired, since $\left(\nabla\Psi, \nabla(\mathcal{N}^+ - \mathcal{N}^-)\right) = \|\mathcal{N}^+ - \mathcal{N}^-\|^2$

  Idea: Use the interpolation error

  $$\mathcal{I}_h[\ln(\mathcal{N}^{\pm}+\delta)] = \left(\mathcal{I}_h[\ln(\mathcal{N}^{\pm}+\delta)] - \ln(\mathcal{N}^{\pm}+\delta)\right) + \ln(\mathcal{N}^{\pm}+\delta)$$

  $\Longrightarrow$ perturbation term depending on δ, more regular initial data and the mesh coupling $k < Ch^2$

Energy based approach $\Rightarrow$ perturbed Entropy

Problems:

- $\ln \mathcal{N}^{\pm}$ not allowed test fct. $\Rightarrow$ linear interpolation $\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm} + \delta)\big]$
- $\Big(\mathcal{N}^{\pm}\nabla\Psi, \nabla\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm} + \delta)\big]\Big) \neq \Big(\nabla\Psi, \nabla\mathcal{N}^{\pm}\Big)$
  "$=$" is desired, since $\Big(\nabla\Psi, \nabla(\mathcal{N}^{+} - \mathcal{N}^{-})\Big) = \|\mathcal{N}^{+} - \mathcal{N}^{-}\|^2$

Idea: Use the interpolation error

$$\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm}+\delta)\big] = \Big(\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm}+\delta)\big] - \ln(\mathcal{N}^{\pm}+\delta)\Big) + \ln(\mathcal{N}^{\pm}+\delta)$$

$\Longrightarrow$ perturbation term depending on δ, more regular initial data and the mesh coupling $k < Ch^2$

Energy based approach $\Rightarrow$ perturbed Entropy

Problems:

- $\ln \mathcal{N}^{\pm}$ not allowed test fct. $\Rightarrow$ linear interpolation $\mathcal{I}_h\left[\ln(\mathcal{N}^{\pm} + \delta)\right]$
- $\left(\mathcal{N}^{\pm} \nabla \Psi, \nabla \mathcal{I}_h\left[\ln(\mathcal{N}^{\pm} + \delta)\right]\right) \neq \left(\nabla \Psi, \nabla \mathcal{N}^{\pm}\right)$

  "$=$" is desired, since $\left(\nabla \Psi, \nabla(\mathcal{N}^{+} - \mathcal{N}^{-})\right) = \|\mathcal{N}^{+} - \mathcal{N}^{-}\|^2$

  Idea: Use the interpolation error

  $$\mathcal{I}_h\left[\ln(\mathcal{N}^{\pm}+\delta)\right] = \left(\mathcal{I}_h\left[\ln(\mathcal{N}^{\pm}+\delta)\right] - \ln(\mathcal{N}^{\pm}+\delta)\right) + \ln(\mathcal{N}^{\pm}+\delta)$$

$\Longrightarrow$ perturbation term depending on δ, more regular initial data and the mesh coupling $k < Ch^2$

Energy based approach $\Rightarrow$ perturbed Entropy

Problems:

- $\ln \mathcal{N}^{\pm}$ not allowed test fct. $\Rightarrow$ linear interpolation $\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm} + \delta)\big]$
- $\Big(\mathcal{N}^{\pm}\nabla\Psi, \nabla\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm} + \delta)\big]\Big) \neq \Big(\nabla\Psi, \nabla\mathcal{N}^{\pm}\Big)$

  "$=$" is desired, since $\Big(\nabla\Psi, \nabla(\mathcal{N}^{+} - \mathcal{N}^{-})\Big) = \|\mathcal{N}^{+} - \mathcal{N}^{-}\|^2$

  Idea: Use the interpolation error

  $$\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm}+\delta)\big] = \Big(\mathcal{I}_h\big[\ln(\mathcal{N}^{\pm}+\delta)\big] - \ln(\mathcal{N}^{\pm}+\delta)\Big) + \ln(\mathcal{N}^{\pm}+\delta)$$

  $\Longrightarrow$ perturbation term depending on δ, more regular initial data and the mesh coupling $k < Ch^2$

Entropy based approach $\Rightarrow$ unperturbed Entropy

- **Idea:** Imitate the continuous situation :

$$\left(n^{\pm}\nabla\psi, \nabla \ln n^{\pm}\right) = \left(\nabla\psi, \nabla n^{\pm}\right)$$

- **Tool:** Define $S_\epsilon(\mathcal{N}^{\pm})$ such that

$$S_\epsilon(\mathcal{N}^{\pm})\nabla\mathcal{I}_h\left[\ln \mathcal{N}^{\pm}\right] = \nabla\mathcal{N}^{\pm} .$$

$$\Longrightarrow \quad \left(\nabla\Psi S_\epsilon(\mathcal{N}^{\pm}), \nabla\mathcal{I}_h\left[\ln\mathcal{N}^{\pm}\right]\right) = \left(\nabla\Psi, \nabla\mathcal{N}^{\pm}\right)$$

$\Longrightarrow$ no perturbation

Entropy based approach $\Rightarrow$ unperturbed Entropy

- **Idea:** Imitate the continuous situation :

$$\left(n^{\pm}\nabla\psi, \nabla \ln n^{\pm}\right) = \left(\nabla\psi, \nabla n^{\pm}\right)$$

- **Tool:** Define $S_{\epsilon}(\mathcal{N}^{\pm})$ such that

$$S_{\epsilon}(\mathcal{N}^{\pm})\nabla\mathcal{I}_h\left[\ln\mathcal{N}^{\pm}\right] = \nabla\mathcal{N}^{\pm} .$$

$$\Longrightarrow \quad \left(\nabla\Psi S_{\epsilon}(\mathcal{N}^{\pm}), \nabla\mathcal{I}_h\left[\ln\mathcal{N}^{\pm}\right]\right) = \left(\nabla\Psi, \nabla\mathcal{N}^{\pm}\right)$$

$\Longrightarrow$ no perturbation

Entropy based approach $\Rightarrow$ unperturbed Entropy

- **Idea:** Imitate the continuous situation :

$$\left(n^{\pm}\nabla\psi, \nabla \ln n^{\pm}\right) = \left(\nabla\psi, \nabla n^{\pm}\right)$$

- **Tool:** Define $S_\epsilon(\mathcal{N}^{\pm})$ such that

$$S_\epsilon(\mathcal{N}^{\pm})\nabla\mathcal{I}_h\left[\ln \mathcal{N}^{\pm}\right] = \nabla\mathcal{N}^{\pm} .$$

$$\Longrightarrow \quad \left(\nabla\Psi S_\epsilon(\mathcal{N}^{\pm}), \nabla\mathcal{I}_h\left[\ln \mathcal{N}^{\pm}\right]\right) = \left(\nabla\Psi, \nabla\mathcal{N}^{\pm}\right)$$

$\Longrightarrow$ no perturbation

Entropy based approach $\Rightarrow$ unperturbed Entropy

- **Idea:** Imitate the continuous situation :

$$\left(n^{\pm}\nabla\psi, \nabla \ln n^{\pm}\right) = \left(\nabla\psi, \nabla n^{\pm}\right)$$

- **Tool:** Define $S_\epsilon(\mathcal{N}^{\pm})$ such that

$$S_\epsilon(\mathcal{N}^{\pm})\nabla\mathcal{I}_h\left[\ln \mathcal{N}^{\pm}\right] = \nabla\mathcal{N}^{\pm}\,.$$

$$\Longrightarrow \qquad \left(\nabla\Psi S_\epsilon(\mathcal{N}^{\pm}), \nabla\mathcal{I}_h\left[\ln \mathcal{N}^{\pm}\right]\right) = \left(\nabla\Psi, \nabla\mathcal{N}^{\pm}\right)$$

$\Longrightarrow$ no perturbation

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Summary of Results (Prohl/Schmuck [07])

- Existence and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law
- **1**) perturbed **versus** 2) unperturbed entropy law
- **1**) Maximum principle **versus** 2) nothing
- **1**) Non-negativity **versus** 2) quasi-non-negativity

Topic of the remaining part:

Extend the energy based strategy to the whole electrohydrodynamic system

Energy based Scheme:

Let $\mathbf{F}_C^j := ((\mathcal{N}^+)^j - (\mathcal{N}^-)^j)\nabla\Psi^j$, and $(\mathbf{V},\ \Phi,\ Q) \in \mathbf{X}_h \times Y_h \times M_h$

$$(\mathrm{d}_t\mathbf{U}^j, \mathbf{V}) + (\nabla\mathbf{U}^j, \nabla\mathbf{V}) + \sigma(\nabla\mathrm{d}_t\mathbf{U}^j, \nabla\mathbf{V}) + ((\mathbf{U}^{j-1}\cdot\nabla)\mathbf{U}^j, \mathbf{V})$$

$$+\frac{1}{2}((\mathrm{div}\,\mathbf{U}^{j-1})\mathbf{U}^j, \mathbf{V}) - (\Pi^j, \mathrm{div}\,\mathbf{V}) = (-\mathbf{F}_C^j, \mathbf{V})$$

$$(\mathrm{div}\,\mathbf{U}^j,\ Q) = 0$$

$$(\mathrm{d}_t(\mathcal{N}^\pm)^j, \Phi)_h + (\nabla(\mathcal{N}^\pm)^j, \nabla\Phi)$$

$$\pm((\mathcal{N}^\pm)^j\nabla\Psi^j, \nabla\Phi) + (\mathbf{U}^j(\mathcal{N}^\pm)^j, \nabla\Phi) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = ((\mathcal{N}^+)^j - (\mathcal{N}^-)^j, \Phi)_h.$$

Energy based Scheme:

Let $\mathbf{F}_C^j := ((\mathcal{N}^+)^j - (\mathcal{N}^-)^j)\nabla\Psi^j$, and $(\mathbf{V}, \Phi, Q) \in \mathbf{X}_h \times Y_h \times M_h$

$$
\begin{aligned}
&(\mathrm{d}_t\mathbf{U}^j, \mathbf{V}) + (\nabla\mathbf{U}^j, \nabla\mathbf{V}) + \sigma(\nabla\mathrm{d}_t\mathbf{U}^j, \nabla\mathbf{V}) + ((\mathbf{U}^{j-1}\cdot\nabla)\mathbf{U}^j, \mathbf{V}) \\
&\quad + \frac{1}{2}((\operatorname{div}\mathbf{U}^{j-1})\mathbf{U}^j, \mathbf{V}) - (\Pi^j, \operatorname{div}\mathbf{V}) = (-\mathbf{F}_C^j, \mathbf{V}) \\
&(\operatorname{div}\mathbf{U}^j,\ Q) = 0
\end{aligned}
$$

$$
\begin{aligned}
&(\mathrm{d}_t(\mathcal{N}^\pm)^j, \Phi)_h + (\nabla(\mathcal{N}^\pm)^j, \nabla\Phi) \\
&\quad \pm((\mathcal{N}^\pm)^j\nabla\Psi^j, \nabla\Phi) + (\mathbf{U}^j(\mathcal{N}^\pm)^j, \nabla\Phi) = 0
\end{aligned}
$$

$$(\nabla\Psi^j, \nabla\Phi) = ((\mathcal{N}^+)^j - (\mathcal{N}^-)^j, \Phi)_h.$$

Energy based Scheme:

Let $\mathbf{F}_C^j := ((\mathcal{N}^+)^j - (\mathcal{N}^-)^j)\nabla\Psi^j$, and $(\mathbf{V}, \Phi, Q) \in \mathbf{X}_h \times Y_h \times M_h$

$$(\mathrm{d}_t\mathbf{U}^j, \mathbf{V}) + (\nabla\mathbf{U}^j, \nabla\mathbf{V}) + \sigma(\nabla\mathrm{d}_t\mathbf{U}^j, \nabla\mathbf{V}) + ((\mathbf{U}^{j-1}\cdot\nabla)\mathbf{U}^j, \mathbf{V})$$
$$+\frac{1}{2}((\mathrm{div}\,\mathbf{U}^{j-1})\mathbf{U}^j, \mathbf{V}) - (\Pi^j, \mathrm{div}\,\mathbf{V}) = (-\mathbf{F}_C^j, \mathbf{V})$$
$$(\mathrm{div}\,\mathbf{U}^j,\ Q) = 0$$

$$(\mathrm{d}_t(\mathcal{N}^\pm)^j, \Phi)_h + (\nabla(\mathcal{N}^\pm)^j, \nabla\Phi)$$
$$\pm((\mathcal{N}^\pm)^j\nabla\Psi^j, \nabla\Phi) + (\mathbf{U}^j(\mathcal{N}^\pm)^j, \nabla\Phi) = 0$$

$$(\nabla\Psi^j, \nabla\Phi) = \big((\mathcal{N}^+)^j - (\mathcal{N}^-)^j, \Phi\big)_h.$$

New Difficulties:

M-matrix property of the Nernst-Planck-Poisson part:

- Problem: lack of regularity for $\mathbf{U}$
  $\Longrightarrow$ Tool: σ-regularization: $\sigma\left(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V}\right)$
- Dimensional argument in the Nernst-Planck-Poisson part (requires h, k small)
  $\Longrightarrow \quad \sigma := h^\alpha \quad \text{with} \quad 0 < \alpha < \frac{6-N}{3}$

Convergence:

- $\sigma\left(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V}\right)$ complicates the application of Aubin-Lions' compactness result

New Difficulties:

M-matrix property of the Nernst-Planck-Poisson part:

- Problem: lack of regularity for $\mathbf{U}$
  $\Longrightarrow$ Tool: σ-regularization: $\sigma\big(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V}\big)$
- Dimensional argument in the Nernst-Planck-Poisson part (requires h, k small)
  $\Longrightarrow$ $\sigma := h^\alpha$ with $0 < \alpha < \frac{6-N}{3}$

Convergence:

- $\sigma\big(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V}\big)$ complicates the application of Aubin-Lions' compactness result

New Difficulties:

M-matrix property of the Nernst-Planck-Poisson part:

- Problem: lack of regularity for **U**
  $\Longrightarrow$ Tool: σ-regularization: $\sigma(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V})$
- Dimensional argument in the Nernst-Planck-Poisson part (requires h, k small)
  $\Longrightarrow$ $\sigma := h^\alpha \quad \text{with} \quad 0 < \alpha < \frac{6-\mathrm{N}}{3}$

Convergence:

- $\sigma(\nabla d_t \mathbf{U}^j, \nabla \mathbf{V})$ complicates the application of Aubin-Lions' compactness result

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Summary of Results (Prohl/Schmuck [08])

- Existence ($k < Ch^{\frac{N}{3}+\epsilon}$, $\epsilon > 0$) and convergence
- Mass conservation for $\mathcal{N}^{\pm}$
- Energy law (fully implicit scheme)
- Perturbed entropy law
- Maximum principle for $\mathcal{N}^{\pm}$
- Non-negativity for $\mathcal{N}^{\pm}$

Tool: Fully practical Fixed Point Iteration

Conclusion:

All results from the continuous setting can be recovered to the FE setting.

Thank You for Your Attention!

Conclusion:

All results from the continuous setting can be recovered to the FE setting.

Thank You for Your Attention!