Fourth-Order PDEs for Image Restoration

Carola-Bibiane Schönlieb[1]

[1] Department of Applied Mathematics and Theoretical Physics (DAMTP)
University of Cambridge (UK)

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Outline

1. PDE Methods in Image Processing
   - The Variational/PDE Approach
   - The Total Variation in Imaging

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2. Image Restoration
   - The Task
   - Fourth Order Methods for Image Restoration
   - Cahn-Hilliard and TV-H^{-1} Inpainting
   - Numerical Computation of the Inpainted Image
3. Applications
   - Restoration of Ancient Frescoes
   - Road Reconstruction

This is joint work with

- Andrea Bertozzi (UCLA, University of Los Angeles California)
- Martin Burger (Institut für Numerische und Angewandte Mathematik, Westfälische Wilhelms Universität Münster)
- Shun-Yin Henry Chu (former: RICAM, University of Linz)
- Massimo Fornasier (RICAM, University of Linz)
- Lin He (former: RICAM, University of Linz)
- Peter Markowich (DAMTP, University of Cambridge)

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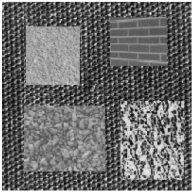
Imaging Tasks[1]

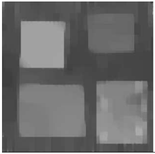
Image Denoising

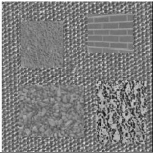
$\Downarrow$

Cartoon/Texture Decomposition



$\Downarrow$

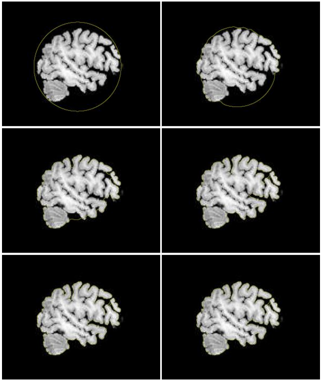




[1] Examples from L. Vese, S. Osher (2004)

Imaging Tasks

Image Segmentation[a]



[a]Example from C. Guyader, L. Vese (2008)

Image Inpainting[a]

⇓

[a]Example from Bertalmio et al (1998)

Imaging Tasks as Inverse Problems

Given: Observed image g, which is corrupted, e.g., by noise or blur.

$$\Downarrow$$

Wanted: Original image u.

In mathematical terms this problem can be written as

$$Tu = g,$$

where T models the process through which the image u went before observation.

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If T has an unbounded inverse $\Rightarrow$ this problem is **ill-posed** $\Rightarrow$ modify the problem by adding an a-priori information on u, usually in terms of a **regularizing term**, the total variation of u, for instance.

Variational approach

Let $\Omega \subset \mathbb{R}^2$ be a "nice" domain, B_1, B_2 two Banach spaces over Ω and $g \in B_1$ be the given image. A general variational approach in image processing can be then written as

$$\mathcal{J}(u) = R(u) + \frac{1}{2\lambda} \|Tu - g\|_{B_1}^2 \to \min_{u \in B_2},$$

where $\lambda > 0$ is the tuning parameter of the problem and $T \in \mathcal{L}(B_1)$ is a bounded linear operator. The minimizer u of the problem is the original version of the given image g.

- $R : B_2 \to \mathbb{R}$ denotes the **regularizing term** which smoothes the image u and represents some kind of a priori information about the minimizer u.
- $\|Tu - g\|_{B_1}^2$ is the so called **fidelity term** of the approach which forces the minimizer u to stay close to the given image g (dependent on the size of λ).

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Now for $B_1 = L^2(\Omega)$ we also have ...

PDE approach

...the corresponding Euler-Lagrange equation

$$-\lambda \nabla R(u) + T^*(g - Tu) = 0, \quad \text{in } \Omega,$$

...the corresponding steepest descent equation for $u(., t = 0) = g$ is the given image

$$u_t = -\lambda \nabla R(u) + T^*(g - Tu), \quad \text{in } \Omega.$$

...in other situations we encounter equations that do not come from variational principles, such as Cahn-Hilliard- and TV-H^{-1} inpainting. Then the image processing approach is directly given by an (evolutionary) PDE.

The right regularizing term . . .

The regularizing term should . . .

- . . . delete noise (for image denoising/deblurring), i.e., eliminate small oscillatory data,
- . . . preserve (model) important image features, like edges. . .

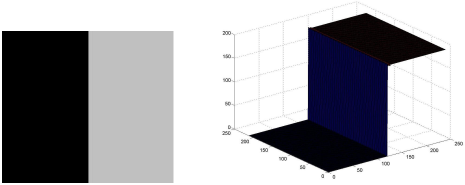

Figure: The edges in an image represent the most important information about the image contents

. . . a regularizing term which fulfills these needs is the **Total Variation**.

Definitions

For $u \in L^1_{loc}(\Omega)$

$$V(u,\Omega) := \sup\left\{\int_\Omega u\nabla\cdot\varphi\,dx : \varphi \in \left[C^1_c(\Omega)\right]^2, \|\varphi\|_\infty \leq 1\right\}$$

is the variation of u. Further

$$u \in BV(\Omega) \text{ (the space of bounded variation functions)}$$

$$\Leftrightarrow$$

$$V(u,\Omega) < \infty.$$

In such a case,

$$|Du|(\Omega) = V(u,\Omega),$$

where $|Du|(\Omega)$ is the **total variation** of the finite Radon measure Du, the derivative of u in the sense of distributions.

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For a function $u \in C^1(\Omega)$, the total variation of u is equivalent to

$$|\mathbf{Du}|(\mathbf{\Omega}) = \int_{\mathbf{\Omega}} |\nabla\mathbf{u}|\,\mathbf{dx}.$$

So what is the total variation?

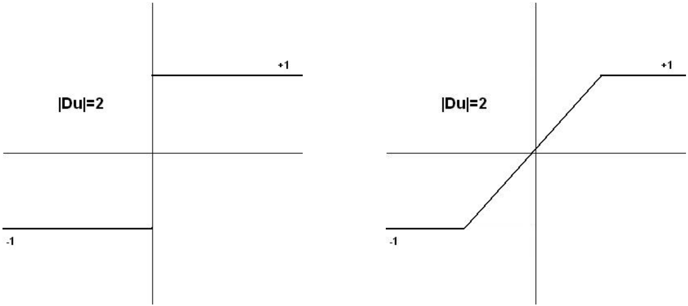
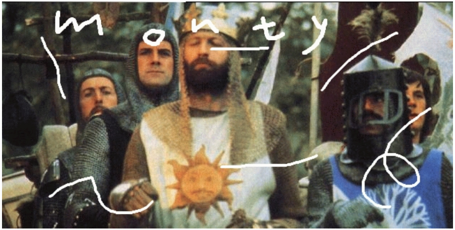


Figure: The total variation measures the size of the jump.

Outline

The task



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1

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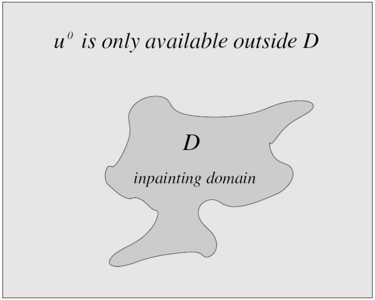

[a]

[a]source: Chan and Shen 2005

Inpainting = Image interpolation
Reconstruct the ideal image u on the missing domain D based on the data of u available outside D.

The variational approach

$$\mathcal{J}(u) = R(u) + \frac{1}{2\lambda}\left\|\chi_{\Omega\setminus D}(u-g)\right\|_{B_1}^2 \to \min_{u\in B_2},$$

where

$$\chi_{\Omega\setminus D}(x) = \begin{cases} 1 & \Omega\setminus D \\ 0 & D, \end{cases}$$

is the characteristic function of $\Omega\setminus D$.

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- $R(u)$: fills in the image content into the missing domain D, e.g., by diffusion and/or transport.
- The fidelity term only has impact on the minimizer u outside of the inpainting domain due to the characteristic function $\chi_{\Omega\setminus D}$.

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- TV-H^{-1} inpainting (M. Burger, L. He, C.-B.S.)

Second order approaches

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- Penalizes length of edges $\Longrightarrow$ cannot connect contours across very large distances −
- Can result in corners of the level lines across the inpainting domain −

Higher order approaches

$$\min_u \int_\Omega (a + b\nabla \cdot (\frac{\nabla u}{|\nabla u|}))|\nabla u| \, dx$$
$$\Longleftrightarrow$$
$$\min_{\Gamma_\lambda} \int_{-\infty}^{\infty} (a \, \text{length}(\Gamma_\lambda) + b \, \text{curvature}(\Gamma_\lambda)) \, d\lambda.$$

[2] Chan, Kang, Shen 2002

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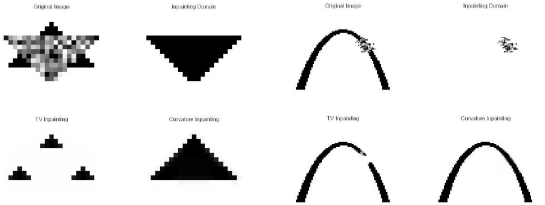


[2]

[2] Chan, Kang, Shen 2002

Higher order approaches

Challenges

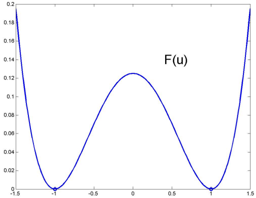
- Higher order equations are very new and little is known about them.
- Often they do not possess a maximum principle or comparison principle.
- For the proof of well-posedness of higher order inpainting models variational methods are often not applicable.
- Need of simple but effective models.
- Need of stable and fast numerical solvers.

The Cahn-Hilliard equation

We consider the Cahn-Hilliard equation

$$\begin{aligned} u_t &= \Delta(-\epsilon^2 \Delta u + F'(u)) && \Omega \times (0,\infty) \\ u(x,t=0) &= u_0 && \Omega \times \{t=0\}, \end{aligned}$$

$\Omega \in \mathbb{R}^n$ bounded with Neumann boundary conditions. F is a so called double well potential with $F'(u) = \frac{1}{2}(u^3 - u)$ and $\epsilon > 0$.



Phase separation and coarsening

The Cahn-Hilliard equation models phase separation and subsequent phase coarsening of binary alloys.

Figure: $\epsilon = 0.1$

Cahn-Hilliard Inpainting Model

Following the approach of Bertozzi, Esedoglu and Gillette (2006) inpainting with Cahn-Hilliard is given by the evolution equation

$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u),$$

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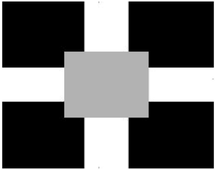
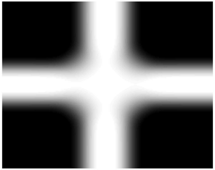
where g is a given binary image.

Setting:

- Analytically: this is **not a gradient flow** anymore!
- Numerically: we have to solve a nonlinear, **fourth order** evolution equation.

Inpainting examples $\lambda = 10^{-5}$





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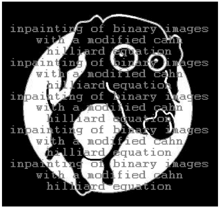
[3] u(1200) with $\epsilon = 0.1$
[4] u(2400) with $\epsilon = 0.01$

Inpainting examples $\lambda = 10^{-9}$



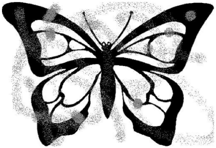
 [5]

 [6]

[5] u(200) with $\epsilon = 0.8$

[6] u(500) with $\epsilon = 0.01$

Inpainting examples $\lambda = 10^{-9}$



7

8

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[7] u(800) with $\epsilon = 0.8$

[8] u(1600) with $\epsilon = 0.01$

Analytic Challenges

Results for the stationary solution are difficult because of the missing energy for the equation.

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$$u_t = \Delta(-\epsilon \Delta u + \frac{1}{\epsilon} F'(u)) + \frac{1}{\lambda} \chi_{\Omega \setminus D}(g - u)$$

gradient flow in H^{-1} of the energy

$$\mathcal{J}^1(u) = \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u)\, dx$$

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Results for the stationary solution are difficult because of the missing energy for the equation.

$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u)$$

gradient flow in L^2 of the energy

$$\mathcal{J}^2(u) = \frac{1}{2\lambda}\int_\Omega \chi_{\Omega\setminus D}(g-u)^2\,dx.$$

Main existing results

Bertozzi, Esedoglu, Gillette 06 (BEG 06):

- Global existence for the evolution equation: For initial data $u_0 \in L^2(\Omega)$ and $\lambda \leq O(\epsilon^3)$:
  there $\exists^1$ $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2_N(\Omega))$

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  there $\exists^1$ $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2_N(\Omega))$
- The stationary solution of the limiting case $\lambda \to 0$ solves

$$
\begin{aligned}
\Delta(\epsilon\Delta u - \frac{1}{\epsilon}F'(u)) &= 0 \quad \text{in } D\\
u &= g \quad \text{on } \partial D\\
\nabla u &= \nabla g \quad \text{on } \partial D,
\end{aligned}
$$

for g regular enough ($g \in C^2$).

New results

Theorem (Existence of a stationary solution)

The stationary equation

$$\Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u) = 0 \quad \text{in } \Omega,$$

has a unique weak solution in $H^1(\Omega)$ *if* $\lambda \leq O(\epsilon^3)$ *and* $|D| < \frac{2}{C}$ *for a constant* $C > 0$.

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has a unique weak solution in $H^1(\Omega)$ if $\lambda \le O(\epsilon^3)$ and $|D| < \frac{2}{C}$ for a constant $C > 0$.

Main ideas of the proof:

- Formulation of a fixed point equation;
- Existence of a fixed point with Schauder;
- Fixed point = stationary solution.

$\epsilon \to 0$

The sequence of Cahn-Hilliard functionals

$$CH(u) = \int_\Omega (\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u))\, dx$$

Γ-converges in the topology $L^1(\Omega)$ to

$$TV(u) = \begin{cases} C_0 \, |Du| \, (\Omega) & \text{if } |u(x)| = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

as $\epsilon \to 0$, where $C_0 = 2 \int_{-1}^{1} \sqrt{F(s)}\, ds$.

TV-H^{-1} inpainting[9]

Motivated by this Γ-convergence result we propose the following higher-order inpainting method: the inpainted image u of $g \in L^2(\Omega)$, shall evolve via

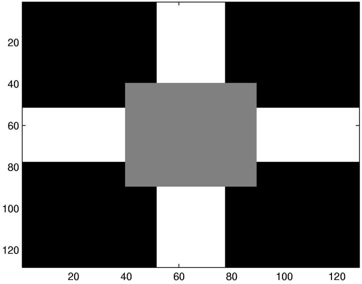
$$u_t = \Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u), \quad p \in \partial TV(u),$$

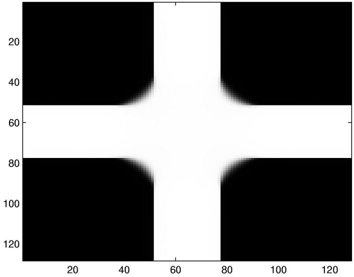
where $\partial TV(u)$ denotes the subdifferential of

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

[9] $TV - H^{-1}$ denoising: S. Osher, A. Sole and L. Vese (2002); L. Lieu and L. Vese (2005)

Inpainting examples

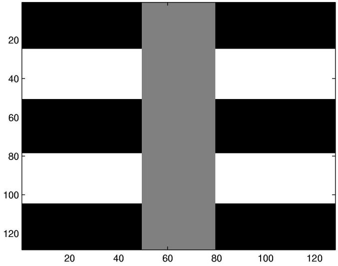


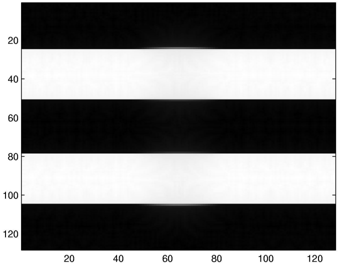


[10]

[10] u(10000) with $\lambda = 2 \cdot 10^{-5}$.

Inpainting examples





[11]

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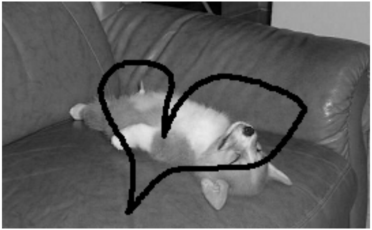
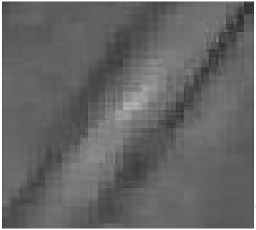
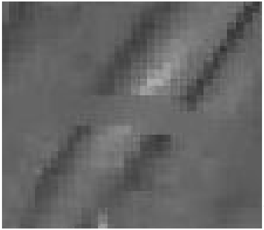


[12]

[12] u(1000) with $\lambda = 10^{-3}$.

4th order versus 2nd order method


[13]


[14]

[13] TV-H^{-1} u(1000) with $\lambda = 10^{-3}$.

[14] TV-L^2 u(5000) with $\lambda = 10^{-3}$.

Inpainting examples



[15]

[15] u(1000) with $\lambda = 10^{-3}$.

TV-H^{-1} inpainting

Applying the same strategy that we used for the existence of a stationary solution in the Cahn-Hilliard case, we can prove:

Theorem

Let $g \in L^2(\Omega)$. The stationary equation

$$\Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u) = 0, \quad p \in \partial TV(u)$$

admits a solution $u \in BV(\Omega)$.

TV-H^{-1} inpainting

Applying the same strategy that we used for the existence of a stationary solution in the Cahn-Hilliard case, we can prove:

Theorem

Let $g \in L^2(\Omega)$. The stationary equation

$$\Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u) = 0, \quad p \in \partial TV(u)$$

admits a solution $u \in BV(\Omega)$.

Reference: M. Burger, L. He, C.-B. Schönlieb, *Cahn-Hilliard inpainting and a generalization for grayvalue images*, submitted 25 p

Open Problems

- Asymtotic behaviour of fourth-order inpainting approaches; convergence of the evolution equation to a stationary state.

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- TV-H^{-1} Inpainting: Boundary conditions on ∂D as $\lambda \to \infty$ (similar to results by Bertozzi, Esedoglu, and Gillette for Cahn-Hilliard inpainting).

Open Problems

- Asymtotic behaviour of fourth-order inpainting approaches; convergence of the evolution equation to a stationary state.
- TV-H^{-1} Inpainting: Boundary conditions on ∂D as $\lambda \to \infty$ (similar to results by Bertozzi, Esedoglu, and Gillette for Cahn-Hilliard inpainting).
- Continuation of the gradient into the inpainting domain also for other higher-order inpainting approaches!

Convexity Splitting (1)

A minimizer u of an energy $\mathcal{J}(u)$ is formally computed as a stationary solution of

$$\begin{aligned} u_t &= -\nabla \mathcal{J}(u) \\ u(0) &= u_0. \end{aligned} \quad (1)$$

Under certain assumptions on $\mathcal{J}$ this is called a gradient system.

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If $\mathcal{J}(u)$ is convex then only a single equilibrium for the gradient system exists.

If $\mathcal{J}(u)$ is **not convex** multiple minimizers may exist and the gradient flow can expand $u(t)$. An **explicit iterative algorithm**, i.e. $u_{k+1} = u_k - \Delta t \nabla \mathcal{J}(u_k)$ in this case may require extremely small time steps, depending of course on E. For the higher order equations $\mathcal{J}(u_k)$ contains second order derivatives resulting in a **restriction of $\Delta \mathrm{t}$ up to order $(\Delta \mathrm{x})^4$**.

Convexity splitting (2)

The idea of convexity splitting is to derive a semi-impicit iterative scheme that is unconditionally stable.

Eyre (1998): Let

$$\mathcal{J}(u) = \mathcal{J}_c(u) - \mathcal{J}_e(u)$$

where $\mathcal{J}_c, \mathcal{J}_e$ are strictly convex. Under certain assumptions on the functionals, the numerical scheme

$$u_{k+1} = u_k - \Delta t(\nabla \mathcal{J}_c(u_{k+1}) - \nabla \mathcal{J}_e(u_k))$$

is gradient stable for every initial condition $u_0 \in \mathbb{R}$ and all $\Delta t > 0$, and possesses a unique solution for each iteration step.

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Although **our inpainting models** do not obey a variational principle (the are **not gradient flows!**), we can apply the convexity splitting method in a **modified** form . . .

Convexity splitting for Cahn-Hilliard inpainting (1)

$$\mathcal{J}^1(u) = \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u)\, dx,$$

$\mathcal{J}^1 = \mathcal{J}^1_c - \mathcal{J}^1_e$ with

$$\mathcal{J}^1_c(u) = \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{C_1}{2} |u|^2 \; dx,$$

and

$$\mathcal{J}^1_e(u) = \int_\Omega -\frac{1}{\epsilon} F(u) + \frac{C_1}{2} |u|^2 \; dx.$$

Convexity splitting for Cahn-Hilliard inpainting (1)

$$\mathcal{J}^2(u) = \frac{1}{2\lambda}\int_\Omega \chi_{\Omega\setminus D}(g-u)^2\,dx.$$

$\mathcal{J}^2 = \mathcal{J}^2_c - \mathcal{J}^2_e$ with

$$\mathcal{J}^2_c(u) = \int_\Omega \frac{C_2}{2}\left|u\right|^2\,dx,$$

and

$$\mathcal{J}^2_e = \frac{1}{2\lambda}\int_\Omega -\chi_{\Omega\setminus D}(g-u)^2\,dx + \int_\Omega \frac{C_2}{2}\left|u\right|^2\,dx.$$

Convexity splitting for Cahn-Hilliard inpainting (2)

The resulting time-stepping scheme is

$$\frac{u_{k+1} - u_k}{\tau} = -\nabla_{H^{-1}}(\mathcal{J}_c^1(u^{k+1}) - \mathcal{J}_e^1(u^k)) - \nabla_{L^2}(\mathcal{J}_c^2(u^{k+1}) - \mathcal{J}_e^2(u^k)),$$

where $\nabla_{H^{-1}}$ and ∇_{L^2} represent the Fréchet derivative with respect to the H^{-1} inner product and the L^2 inner product respectively. This translates to a numerical scheme of the form

$$\begin{aligned} & \frac{u_{k+1} - u_k}{\tau} + \epsilon\Delta\Delta u_{k+1} - C_1\Delta u_{k+1} + C_2 u_{k+1} \\ = \quad & \frac{1}{\epsilon}\Delta F'(u_k) - C_1\Delta u_k + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g - u_k) + C_2 u_k. \end{aligned}$$

To make sure that $\mathcal{J}_c^i, \mathcal{J}_e^i$, $i = 1, 2$, are convex the constants $C_1 > \frac{1}{\epsilon}$, $C_2 > 1/\lambda$.

Convexity splitting for TV-H^{-1} inpainting

A similar technique can be applied to TV-H^{-1} inpainting:

$$\begin{aligned}\frac{u^{k+1}-u^k}{\tau}+C_1\Delta^2 u^{k+1}+C_2 u^{k+1} &= C_1\Delta^2 u^k-\Delta(\nabla\cdot(\frac{\nabla u^k}{|\nabla u^k|})) \\ &\quad +C_2 u^k+\frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u^k),\end{aligned}$$

with constants $C_1 > \frac{1}{\epsilon}$ (where here ϵ comes from the regularization of the total variation), $C_2 > 1/\lambda$.

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Reference:

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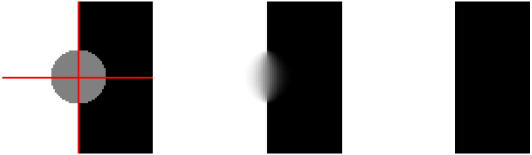
Since usually in inpainting tasks λ is chosen comparatively small, e.g., $\lambda = 10^{-3}$, the condition on C_2 damps the convergence of this method $\Rightarrow$ **converging slow**!

Other strategies for TV-H^{-1} inpainting

- TV-H^{-1} minimization via the **dual formulation of the total variation** (modeled on Chambolle's algorithm):

$$u = g + \Delta \left(\mathbb{P}^1_{\lambda K}(\Delta^{-1} g) \right),$$

  and versions of it . . .
- **Domain decomposition** for total variation approaches



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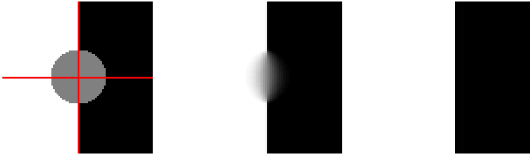

References:

- C.-B. Schönlieb, *Total variation minimization with an H^{-1} constraint*, submitted 23 p.
- M. Fornasier, C.-B. Schönlieb, *Subspace correction methods for total variation and ℓ_1- minimization*, submitted 33 p.

Outline

Digital Restoration of Frescoes

Fife Senses Project on *Mathematical Methods for Image Analysis and Processing in the Visual Arts* sponsored by the Viennese Technology Fund (WWTF).
Collaboration between

- Faculty of Mathematics, University of Vienna (Peter A. Markowich, Massimo Fornasier):
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  Physcial Restoration of the Frescoes $\Rightarrow$ Comparison with the digital result

The Neidhart Frescoes (1)

Fragments of 14th century wall frescoes found beneath the crumbling plaster of an old apartment in the heart of Vienna, depict a popular medieval cycle of songs of the 13th century minnesinger, Neidhart von Reuental.

The Neidhart Frescoes (2)

The white holes in the fresco are due to the wall which covered the fresco until a few years ago. They arised when the wall was removed. In the following we want to apply digital restoration methods to these frescoes.

The Neidhart Frescoes (3)

Goal:

- Advanced mathematical inpainting tools should restore the frescoes digitally.
- The virtually restored frescoes then serve as a courtesy and template for museums artists with the challenge of the physical restoration of the frescoes.

The Neidhart Frescoes (3)

Goal:

- Advanced mathematical inpainting tools should restore the frescoes digitally.
- The virtually restored frescoes then serve as a courtesy and template for museums artists with the challenge of the physical restoration of the frescoes.

Challenges

- Due to their great age and almost 600 years of living by owners and tenants in the apartment, saturation, hue and contrast quality of the colors in the frescoes suffered.
- Digital grayvalue, i.e., color interpolation, in the damaged parts of the fresco therefore demands sophisticated algorithms taking these lacks into account.

Fresco restoration (1)

Fresco restoration (2)

 [16]

[16] Cahn-Hilliard inpainting after 200 timesteps with $\epsilon = 3$ and additional 800 timesteps with $\epsilon = 0.01$

Fresco restoration (3)

Outlook:

The next step will be to recolorize the damaged parts by using the recovered binary structure as underlying information. A possible variational approach would then be

$$\mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2\lambda}\int_{\Omega\setminus D}(u-g)^2\,dx + \frac{1}{2\mu}\int_D (u_{bin} - u_{ch}),$$

where u_{ch} is the result of the Cahn-Hilliard inpainting algorithm and u_{bin} is the binarized version of u.

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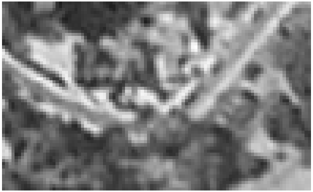
Road data from Los Angeles[17]

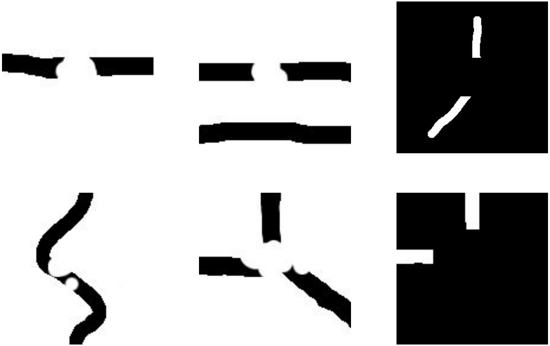
Our data consists of satellite images of roads in Los Angeles of the following kind ...



[17] Joint work with Andrea Bertozzi from UCLA

First approach: Inpainting of binary roads

Consider binarized roads[18] . . .



[18] Thanks to Shao-Ching Huang (UCLA) for the preparation of the data

First approach: Inpainting of binary roads

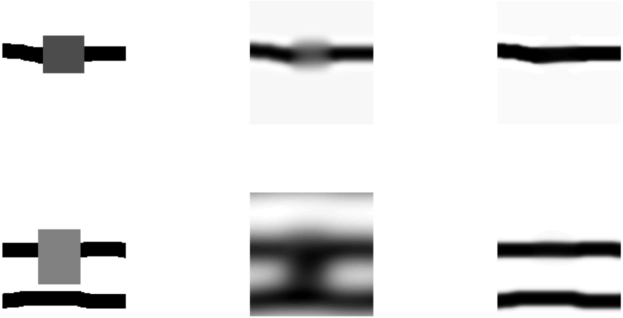


Figure: Cahn-Hilliard inpainting in two steps, namely with $\epsilon = 0.1$ and $\epsilon = 0.01$ (i.e. $\epsilon = 1.6$ and $\epsilon = 0.01$)

First approach: Inpainting of binary roads

Figure: Cahn-Hilliard inpainting in two steps, namely with $\epsilon = 0.1$ and $\epsilon = 0.01$ (i.e. $\epsilon = 1.6$ and $\epsilon = 0.01$)

First approach: Inpainting of binary roads

Figure: Cahn-Hilliard inpainting in two steps, namely with $\epsilon = 0.1$ and $\epsilon = 0.01$

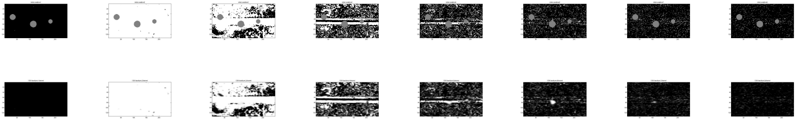
Second approach: Bitwise Cahn-Hilliard

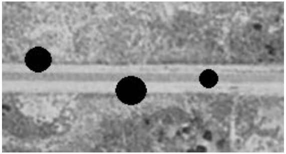
One possible generalization of Cahn-Hilliard inpainting for grayscale images is to split the grayscale image bit-wise into channels

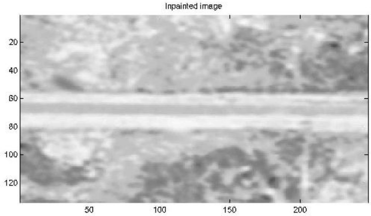
$$u(x) \rightsquigarrow \sum_{k=1}^{K} u_k(x) 2^{-(k-1)},$$

where $K > 0$ and then apply the Cahn-Hilliard inpainting approach to each binary channel u_k. In the following numerical examples $K = 8$.

Second approach: Bitwise Cahn-Hilliard







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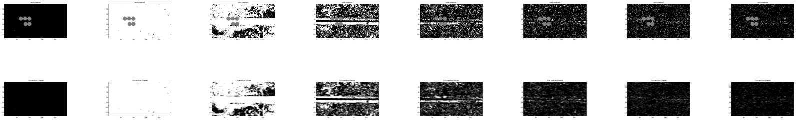

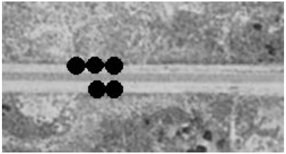

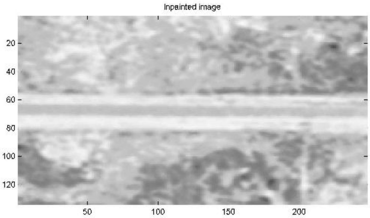

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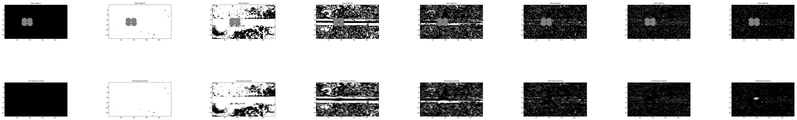

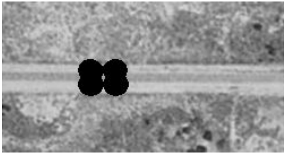

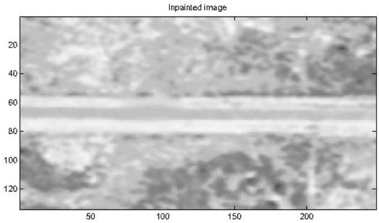

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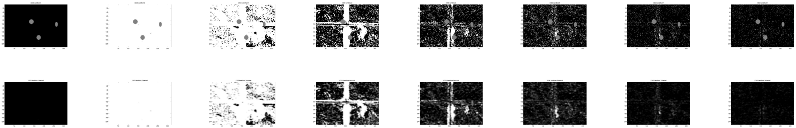

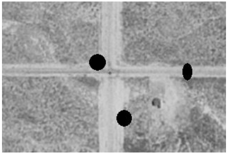

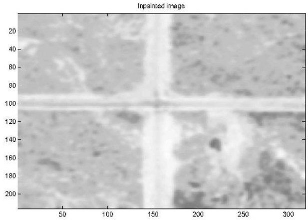

Conclusion

- PDE techniques in image processing
- . . . in particular in image inpainting
- Cahn-Hilliard- and TV-H^{-1} inpainting
- Numerical approaches (convexity splitting, dual formulation, domain decomposition)
- Restoration of frescoes and roads.

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For more details see `http://homepage.univie.ac.at/carola.schoenlieb`

or write to: `c.b.schonlieb@damtp.cam.ac.uk`