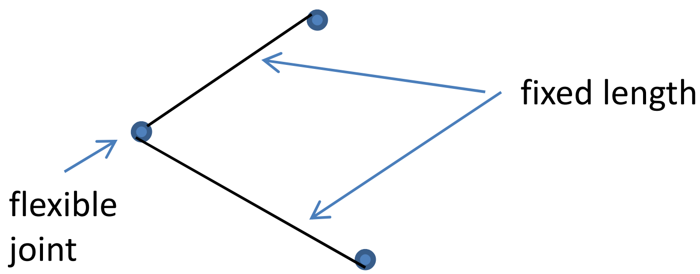
The Geometry of Euler's equation

Introduction

Part 1

Mechanical systems with constraints, symmetries



In principle can be dealt with by applying F=ma, but this can become complicated

Lagrangian mechanics:

M manifold of possible configurations – "configuration space"

q points of M (either abstract, or in specific coordinates $q=(q^1, \ldots, q^n)$)

$t \to q(t)$ trajectories in M (to be computed)

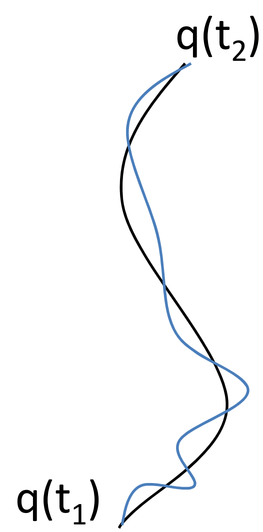
Effective algorithm (due to Lagrange) for writing the equations for q(t) without some complicated analysis of the forces:

For a curve q(t) in M (not necessarily a real motion), find the expression for the kinetic energy T and the potential energy V of the system.

Typically $T = T(q, \dot{q})$ with a quadratic expression in $\dot{q}$

and $V = V(q)$

Consider the Lagrangian $$L = L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$



Hamilton's principle:

The actual motions are extremals of the "action functional"

$$\int_{t_1}^{t_2} L(q, \dot{q})\, dt$$ in the class of trajectories with fixed endpoints $q(t_1)$, $q(t_2)$

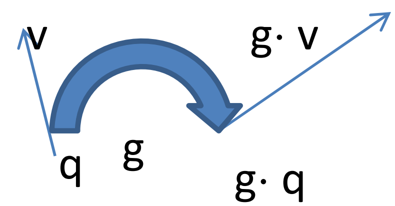
Lagrange equations: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$

Systems with symmetries

G group acting on M $q \in M \to g \cdot q \in M$

Groups of interest: q(t) a solution $\Rightarrow$ $g \cdot q(t)$ a solution

sufficient condition: L is invariant under G



$L: TM \to R$

G acts also on TM: $(q, v) \in T_x(M) \quad \to \quad g \cdot (q, v) = (g \cdot q, Dg(q) \cdot v) \in T_{gx} M$

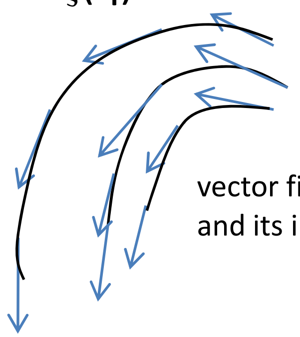
Invariant L: $L(g \cdot (q, v)) = L(q,v)$

Goal: use the symmetries to simplify the equations

For reducing the equations, we need "continuous symmetry groups" (Lie groups) and the corresponding "infinitesimal symmetries" (Lie algebras).

Infinitesimal transformation of M = vector fields ξ(q) on M. Their fluxes generate a 1-parameter family of transformtions g^t .

$$\frac{d}{dt}\,(g^t \cdot q) = \xi(g^t \cdot q)$$

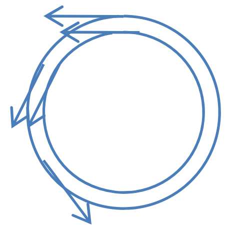


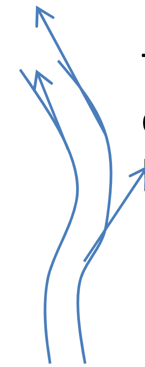
vector field ξ(q)
and its integral lines

Example:

$$g^t \cdot x = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\xi(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$





f const.
on integral
lines

Functions f on M invariant under an infinitesimal transformation $\xi(q)$

$\xi \cdot f = d/d\epsilon\, f(q+\epsilon\, \xi(q))_{\,\epsilon=0} = 0$ (derivative in the ξ direction vanishes)

Infinitesimal transformations ξ on M extend to infinitesimal transformations of TM, (still denoted by ξ)

$q \rightarrow q+\epsilon\, \xi(q)$

original transformation

$(q,v) \rightarrow (q+\epsilon\xi(q),\, v + \epsilon D\xi(q)\, v\ )$

extension

just linearize
$g\cdot(q, v) = (g\cdot q,\, Dg(q)\cdot v)$

Lagrangian **L: TM $\rightarrow$ R** invariant under an infinitesimal transformation ξ

$$\boldsymbol{\xi \cdot L = 0}$$

the derivative of L in the direction
of the extended ξ

A simple version of Noether's theorem:

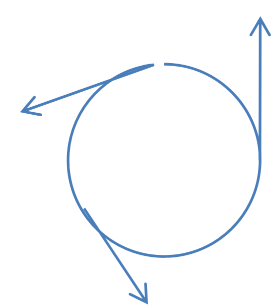
M configuration space, L: TM→ R a Lagrangian,
ξ a vector field on M which is an "infinitesimal symmetry" of L (i.e. $\xi \cdot$ L = 0)

Then the quantity $\xi^i(q)\frac{\partial L}{\partial \dot{q}^i}$ is conserved, i.e. $\frac{d}{dt}\ \xi^i(q)\frac{\partial L}{\partial \dot{q}^i} = 0$

Proof: Chain rule, the Lagrange equations, the assumption $\xi \cdot$ L = 0

Example: A motion in a radial potential (in R^2 or R^3) leads conserves the angular momentum.

$$\begin{aligned} L(q,\dot{q}) &= \frac{1}{2}m|\dot{q}|^2 - V(|x|) \\ \xi(q) &= \begin{pmatrix} -q^2 \\ q^1 \end{pmatrix} \\ \xi^i\frac{\partial L}{\partial \dot{q}^i} &= m(q^1\dot{q}^2 - q^2\dot{q}^1) \end{aligned}$$



Hamiltonian mechanics (gives a more geometric picture):

Instead of working with TM, work with the co-tangent space T^*M.

Additional structures on T^*M: canonical 1 – form α and the symplectic form dα

Example: Let X be a 1-d linear space (it is of course $\sim$ R, but with the ambiguity of choosing a fixed vector)

Let e be a basis of X and e* the basis of X* dual to e, i.e. <e*,e> = 1.

q coordinate in X with respect to e
p coordinate in X* with respect to e*
(p,q) coordinates in $X^*\times X$

pq is independent of the choice of e (by the very construction)

$\alpha = p\,dq$ 1-form on $X^*\times X$ - it is "canonical" (independent of the "arbitrary" choice of e)
$d\alpha = \omega = dp\wedge dq$ canonical 2-form on $X^*\times X$

Conclusion: $X^*\times X$ has a canonical volume element (unlike $X\times X$)

The situation in T*M is very similar:

$q = (q^1, \ldots, q^n)$	coordinates in M
$\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$	basis of T_qM (at a given q)
$dq^1, \ldots, dq^n$	basis of T_q^*M dual to the above basis of T_qM
$(q^1, \ldots, q^n, \xi^1, \ldots, \xi^n)$	coordinates in TM, ξ^i being the coordinates in the basis $\frac{\partial}{\partial q^i}$
$(q^1, \ldots, q^n, p_1, \ldots, p_n)$	coordinates in T^*M p_i being the coordinates in the basis dq^i
$\alpha = p_i dq^i$	canonical 1-form in T^*M
$\omega = d\alpha = dp_i \wedge dq^i$	symplectic form on T^*M

In particular we have a canonical volume form $\omega\wedge \ldots \wedge \omega$ (n times) on T*M, in addition to the anti-symmetric form ω. (And the forms $\omega\wedge\omega$, $\omega\wedge\omega\wedge\omega$, etc.)

Eventually we wish to work in function spaces, and the expressions in local coordinates will not really be suitable, but it is important to understand the finite dimension first.

Nice structures on T*M, but the natural evolution quantity d/dt q(t) (generalized velocity) undoubtedly belongs to TM.

We need a natural map between TM and T*M.
We have the Lagrangian L to provide it!

$$(q, \dot{q}) \in TM \quad \to \quad (q, p) \in T^*M$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

inversion

$$(q, p) \in T^*M \to (q, \dot{q}) \in TM$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

Hamiltonian
(Legendre trasnform of L (at a fixed q))

$$H(p, q) = \inf_v (p_i v^i - L(q, v)) = p_i \dot{q}^i - L(q, \dot{q})$$

The equations in p,q, H

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

Exercise: check this

Our system now "lives" completely in T*M, where we have the benefit of the canonical geometric structures. In particular, the space of parameters which describe the state of our system, has a canonical volume (and much more)

Another view: the form ω = $dp_i \wedge dq^i$ on X=T*M $\sim$ {(p,q)} provides an isomorhism J: T*(X) and T(X) by

$$<\beta, \xi>= \omega(\xi, J\beta)$$

Letting x=(p,q), we can write the equations

$$\dot{x} = J\ dH(x)$$

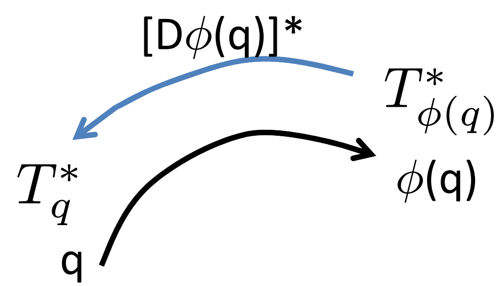
Every transformation $q \to \tilde{q} = \phi(q)$ of the configuration space

extends to a canonical (= symplectic) transformation of the phase space

$$(p, q) \to (\tilde{p}, \tilde{q}) = (\ ([D\phi(q)]^*)^{-1} p, \phi(q)\ )$$

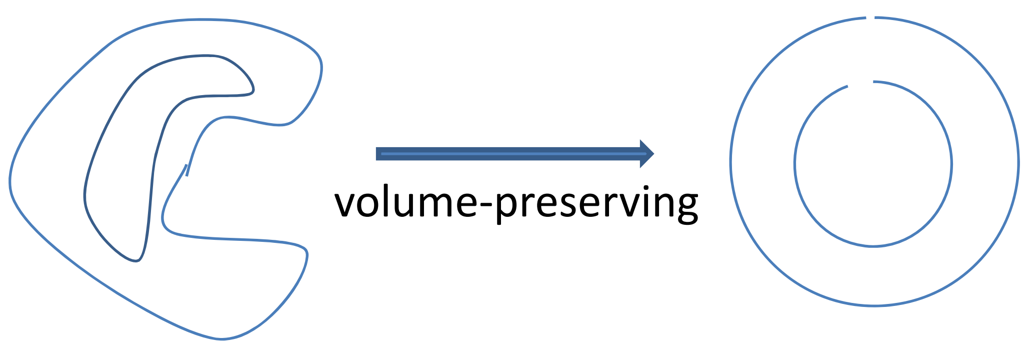
linear in p

In 1d one can see easily how this is volume-preserving:
possible squeezing in q is compensated by streching
in p and vice versa



On the other hand, there are many more symplectic transformation than this:

Example:
dim M = 1
dim T*M = 2



Symmetries in the Hamiltonian picture

1. The Lagrangian picture (confinuration space, etc.) is invarian under the change of coordinates in the configuration space.
   (Example: polar or cartesian coordinates give equivalent equations)

2. The Hamiltonian picture is invariant under transformations of the phase space which preserve the symplectic form:

$$(\tilde{p}, \tilde{q}) = (\tilde{p}(p, q), \tilde{q}(p, q))$$

$$d\tilde{p}_i \wedge d\tilde{q}^i = dp_i \wedge dq^i$$

← "canonical transformation" also called symplectic trans.

$$\tilde{H}(\tilde{p}, \tilde{q}) = H(p, q)$$

$$(p(t), q(t)) \text{ solve for Hamiltonian } H \;\Rightarrow\; (\tilde{p}(t), \tilde{q}(t)) \text{ solve for Hamiltonian } \tilde{H}.$$

$x=(p,q)$, $X = T^*M$, $\omega = dp_i \wedge dq^i$, $J: T^*X \to TX$ induced by ω

Infinitesimal symplectic tranformation:

vector field $\xi(x)$ on X such whose flux is a 1 parameter family of symplectic transformations

Alternatively: $x \to x + \epsilon\xi(x)$ is symplectic modulo $O(\epsilon^2)$, or
$\xi \cdot \omega = 0$ (Lie derivative of ω in the direction ξ)

For any smooth f on X, $\xi(x) = J\, df$ is an infinitesimal symplectic transformation
it is just the vector field generating the evolution by the hamiltonian f

Vice-versa, for any infinitesimal symplectic transformation ξ, there *locally* exists a function f such that $J\, df = \xi$

Proof: We must check that $d J^{-1} \xi = 0$. We can either do it by direct calculation, or use Cartan's formula

$$\xi \cdot \omega = i_\xi d\omega + d(i_\xi \omega)$$

$J^{-1} \xi$

=0 by assumptions
(ω is infinitesima sympl. tr.)

=0 for the
sympl. form

Example: ξ vector field on M

Extend ξ to T*M: (p,q) $\to$ (p - ϵ (Dξ(q))* p , q + ϵ ξ(q))

This vector field is generated by **$f(p,q) = p_i\, \xi^i(q)$**,
the quantity from Noether's theorem:

J df = the extension of ξ to T*M

Noether's theorem:
Assumption: the Hamiltonian H is invariant under (the extension of) ξ
which is the same as: H is invariant under the flow generated by $f=p_i\, \xi^i$
Conclusion: f is invariant under the flow generated by H

Proof: $$(J\, dH)\cdot f = H_{p_i} f_{q^i} - H_{q_i} f_{p^i} = -(J\, df)\cdot H$$

Or: $(\omega)_{ij}$ (anti-sym form on vectors) is non-singular and it also gives $(\omega)^{ij} = [(\omega)_{ij}]^{-1}$, anti-sym. form on co-vectors.

Definition: f,g smooth functions on T*M

$$\{f,g\} = f_{p_i} g_{q^i} - g_{p_i} f_{q^i}$$ is called the **Poisson bracket**

Can be defined on any symplectic manifold by {f,g}= (J df) g = $\omega^{ij} f_j g_i$

If H is a Hamiltonian, the evolution of any quantity f is

df/dt = {H,f} (Taking $f=p_i$, or q^i, we get the canonical equations)

Noether's theorem is now clear:
{H,f}=0 means that f is a conserved quantity, but it also means (equivalently) that H is invariant under the infinitesimal transformations generated by f

Properties of the Poisson bracket

$$\{f,g\} = -\{g,f\}$$

$$\{fg,h\} = f\{g,h\} + g\{f,h\}$$ (it is a derivative)

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$ (Jacobi identity)

Corollary: H is a Hamiltonian, f,g are conserved $\Rightarrow$ {f,g} is also conserved:

Proof: $\{H,\{f,g\}\}=\{\{H,f\},g\}+\{f,\{H,g\}\} = 0$

The Lie bracket and the Poisson bracket

ξ , η vector fields on any manifold
we can differentiate functions in those direction: f $\rightarrow \xi \cdot$ f = D_ξ f (often also denoted by L_ξ F)

The Lie bracket [ξ,η] of ξ , η is a vector field given by

$\xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f) = [\xi,\eta] \cdot f$

Remark: [ξ,η]=0
iff the flows given by
ξ and η commute

In coordinates $$[\xi, \eta]^i = \xi^j \eta^i_{,j} - \eta^j \xi^i_{,j}$$

Relation between the Lie bracket and the Poisson bracket

$$[J\,df\;,\;J\,dg] = J\{f, g\}$$

Conceptually, and modulo some corrections which will not be important for us here

the corrections are of course important for symplectic geometry, Lie alg. representations, etc., see e.g. the book of A.A. Kirillov

infinitesimal symplectic transformations, Lie bracket $\sim$ functions, Poisson bracket

↑ symmetries ↑ conserved quantities

$X = T^*M$, Hamiltonian H, group G of symmetries (i.e. leaving invariant H and ω)

$C^\infty(G \backslash X)$ smooth functions invariant under G

$g\cdot f = f$, $g \in G$, where $g\cdot f(x) = f(g^{-1}\cdot x)$

$\{g\cdot f_1, g\cdot f_2\} = g\cdot\{f_1, f_2\}$ (because G preserves ω)

⇓

The algebra $C^\infty(G \backslash X)$ is closed under the Poisson bracket

$Y = G \backslash X$ "manifold of G-orbits" (not quite smooth manifold, but often close) → e.g. smooth near most points

We have the Poisson bracket on $C^\infty(Y)$ inherited from X .

Does the Poisson bracket on Y come from a symplectic form, i.e. does Y also inherit the symplectic structure? → no in general (for example, dim Y can be odd), but the "correction" is in fact beneficial!

Example: motion in a central field ($\sim$ radial potential) in R^3

$X=T^*M = R^3 \times R^3$, coordinates (p,q), $H=|p|^2/2m + V(|q|)$
$G=SO(3)$, $g\cdot (p,q) = (g\cdot p, g\cdot q)$,

Noether's theorem: $q \wedge p$ is conserved → (infinitesimal rotations about the q_3 axis are generated by $q_1 p_2 - q_2 p_1$, similar for the other axes)

This is enough for integrating the equations, but it is instructive to look also at $G \setminus X$

The invariant functions $C^\infty(G\setminus X)$ can be obtained as functions of

$y_0 = p\cdot q$, $y_1 = ½ |q|^2$, and $y_2 = ½ |p|^2$

Calculate the Poisson brackets $\{y_i , y_j\}$

	y_0	y_1	y_2
y_0	0	$2y_1$	$-2y_2$
y_1	$-2y_1$	0	$-y_0$
y_2	$2y_2$	y_0	0

This table also defines the Lie algebra sl(2,R)

For general functions f , g of y_0, y_1,y_2 **$\{ f , g \} = f_{y_i} g_{y_j} \{y_i, y_j\}$**

Equations for y_0, y_1, y_2 are

$$\mathbf{d/dt\ y_j = \{H, y_j\}}$$

The "manifold" Y with the coordinates y_j and the bracket { f , g }
cannot be a symplectic manifold - dim Y = 3.

A special function C on Y :

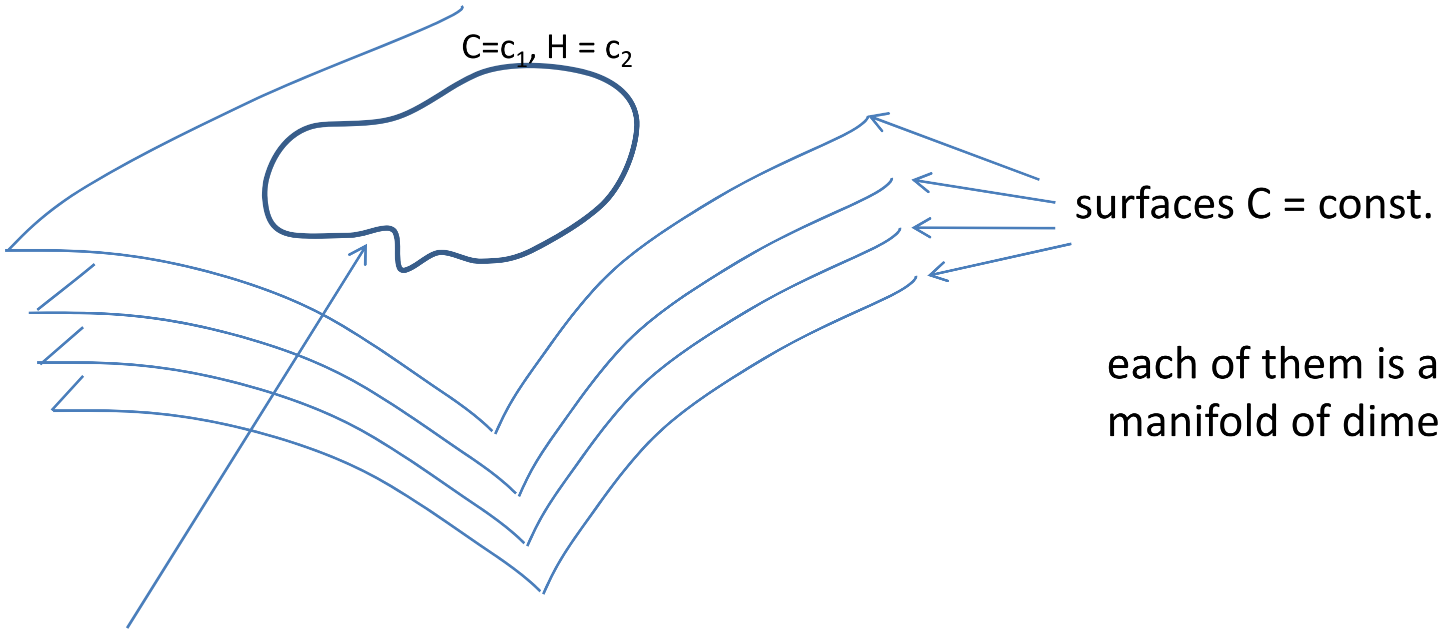
$$C(y_0,y_1,y_2) = 4y_1y_2 - y_0^{\,2}$$

satisfies { C , f }=0 for each f (check that $\{ C , y_j \} = 0$, j=0,1,2)

The evolution by *any* Hamiltonian always preserves C

So the evolution $d/dt\, y_j = \{H, y_j\}$ takes place on C=const.,
and we are dealing with a system with 1 degree of freedom.

The solutions curves are given by C=const. and H=const.,
time-dependence is calculated by integrating along the curves.



each of them is a symplectic manifold of dimension 2

Solution of $d/dt\ y_j = \{H, y_j\}$ moving along C=const. H=const.

In general, the structure of the manifolds $Y = G \backslash X$ is similar:
Y is foliated into “symplectic leaves”, the leaves “do not interact”.

In general, Y is not a manifold, the foliation can have singularities, etc.

The conservation law C=const. looks first unexpected:
naively we expect to reduce the dimension of the system by the dimension of the group G, but the symplectic structure gives often more!

Example: general ODE system with a 1d symmetry group G on a Manifold X

$$\dot{x} = f(x)$$

symmetry $$f(g \cdot x) = g \cdot f(x)$$

the reduced system is on $Y=G \setminus X$, dim Y = dim X – dim G

Symplectic situation: we still have the same reduction as in the general case.

In addition: the equations on Y are again $dy_i/dt = \{ H, y_i\}$
and (given dim G=1), we have an additional reduction:
There is at least one Casimir function C
and the evolution in Y takes place on C=const.

Typically C is given by the function generating the conservation law

Example: we can see without calculation that a rigid body rotating about a given point in the absence of external forces should be integrable:

Configuration space: $M = SO(3)$
Phase space: $X = T^*M$
Symmetry groups: $G = SO(3)$ (acting on M by left multiplication)
Reduced space: $Y = G \backslash X$, dim $Y = 3$ (it turns out $Y = so(3) \sim R^3$)

↓

But symplectic leaves must have even dimension,
so there should be at least one Casimir function

↓

2d symplectic leaves $\Rightarrow$ integrable