Integrability by Compensation in the Analysis of Conformally Invariant Problems.

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I Introduction

These lecture notes form the cornerstone between two areas of Mathematics: calculus of variations and conformal invariance theory.

Conformal invariance plays a significant role in many areas of Physics, such as conformal field theory, renormalization theory, turbulence, general relativity. Naturally, it also plays an important role in geometry: theory of Riemannian surfaces, Weyl tensors, Q-curvature, Yang-Mills fields, etc... We shall be concerned with the study of conformal invariance in analysis. More precisely, we will focus on the study of nonlinear PDEs arising from conformally invariant variational problems (e.g. harmonic maps, prescribed mean curvature surfaces, Yang-Mills equations, amongst others).

A transformation is called conformal when it preserves angles, that is, when its differential is a similarity at every point. Unlike in higher dimensions, the group of conformal transformations in two dimensions is very large ; it has infinite dimension. In fact, it contains as many elements as there are holomorphic maps. This particularly rich feature motivates us to restrict our attention on the two-dimensional case. Although we shall not be concerned with higher dimension, the reader should know that many of the results presented in these notes can be generalized to any dimension.

The first historical instance in which calculus of variations encountered conformal invariance took place early in the twentieth century with the resolution of the Plateau problem. Originally posed by J.-L. Lagrange in 1760, it was solved independently over 150 years later by J. Douglas and T. Radó. In recognition of his work, the former was bestowed the first Fields Medal in 1936 (jointly with L. Alhfors).

Plateau Problem. *Given a regular closed connected curve* Γ *in* $\mathbb{R}^3$*, does there exist an immersion* u *of the unit-disk* D^2 *such that* ∂D^2 *is homeomorphically sent onto* Γ *and for which* $u(D^2)$ *has a minimal area?*

One of the most important ideas introduced by Douglas and Radó consists in minimizing the energy of the map u

$$E(u) = \frac{1}{2}\int_{D^2} |\partial_x u|^2 + |\partial_y u|^2 \; dx \wedge dy \quad .$$

It has good coercivity properties and lower semicontinuity in the weak topology of the Sobolev space $W^{1,2}(D^2, \mathbb{R}^3)$, unlike the area functional

$$A(u) = \int_{D^2} |\partial_x u \times \partial_y u| \; dx \wedge dy \quad .$$

One crucial observation is the following inequality, valid for all u dans $W^{1,2}(D^2, \mathbb{R}^3)$,

$$A(u) \le E(u) \quad ,$$

with equality if and only if u is weakly conformal, namely:

$$|\partial_x u| = |\partial_y u| \quad \text{et} \quad \partial_x u \cdot \partial_y u = 0 \quad \text{a.e.} \quad .$$

The energy functional E has another advantage over the area functional A. While A is invariant under the action of the *infinite* group of diffeomorphisms of D^2 into itself[1], the functional E is only invariant through the action of the much smaller group

[1]Indeed, given two distinct positive parametrizations (x, y) and (x', y') of the unit-disk D^2, there holds, for each pair of functions f and g on D^2, the identity

$$df \wedge dg = \partial_x f \partial_y g - \partial_y f \partial_x g \; dx \wedge dy = \partial_{x'} f \partial_{y'} g - \partial_{y'} f \partial_{x'} g \; dx' \wedge dy'$$

so that, owing to $dx \wedge dy$ and $dx' \wedge dy'$ having the same sign, we find

$$|\partial_x f \partial_y g - \partial_y f \partial_x g| \; dx \wedge dy = |\partial_{x'} f \partial_{y'} g - \partial_{y'} f \partial_{x'} g| \; dx' \wedge dy' \quad .$$

This implies that A is invariant through composition with positive diffeomorphisms.

(it is in fact *finite*) of Möbius transformations comprising conformal, degree 1 maps from D^2 into itself[2].

In effect, the idea of Douglas and Radó bears resemblance to that of minimizing, in a normal parametrization $|\dot{\gamma}| = 1$, the energy of a curve $\int_{[0,1]} |\dot{\gamma}|^2 \, dt$, rather than the Lagrangian of the arclength $\int_{[0,1]} |\dot{\gamma}| \, dt$, which is invariant under the too big group of positive diffeomorphisms of the segment $[0, 1]$.

All the disks (D^2, g) are conformally equivalent to the flat disk D^2. Thus, the aforementioned observations enable us to infer that any minimum of the area functional A, if it exists, must be a critical point of the energy functional E. These points are the harmonic maps u in $\mathbb{R}^3$ satisfying

$$\Delta u = 0 \qquad \text{in} \quad \mathcal{D}'(D^2) \quad . \tag{I.1}$$

Leading this process to fruition is however hindered by the boundary data, which is of a "free" Dirichlet type along a curve Γ, and by the non-compacity of the Möbius group, which will thus have to be "broken" by the so-called *three-point method*. Eventually, one reaches the following result.

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[2]The invariance of E under conformal transformations may easily be seen by working with the complex variable $z = x + iy$. Indeed, we note

$$\partial_z := \frac{1}{2} (\partial_x - i\partial_y)$$

et

$$\partial_{\overline{z}} := \frac{1}{2} (\partial_x + i\partial_y)$$

so that $du = \partial_z u \, dz + \partial_{\overline{z}} u \, d\overline{z}$, and thus

$$E(u) = \frac{i}{2} \int_{D^2} |\partial_z u|^2 + |\partial_{\overline{z}} u|^2 \; dz \wedge d\overline{z} \quad .$$

Accordingly, if we compose u with a conformal transformation, i.e. holomorphic, $z = f(w)$, there holds for $\tilde{u}(w) = u(z)$ the identities

$$|\partial_w \tilde{u}|^2 = |f'(w)|^2 \; |\partial_z u|^2 \circ f \qquad \text{and} \qquad |\partial_{\overline{w}} \tilde{u}|^2 = |f'(w)|^2 \; |\partial_{\overline{z}} u|^2 \circ f \quad .$$

Moreover, $dz \wedge d\overline{z} = |f'(w)|^2 \; dw \wedge d\overline{w}$. Bringing altogether these results yields the desired conformal invariance $E(u) = E(\tilde{u})$.

Theorem I.1 *[Douglas-Radó-Courant] Given a regular closed curve Γ in $\mathbb{R}^3$, there exists a continuous minimum u for the energy E within the space of $W^{1,2}(D^2, \mathbb{R}^3)$ functions mapping the boundary of the unit-disk ∂D^2 onto Γ in a monotone fashion, and satisfying*

$$\left\{\begin{array}{l} \Delta u = 0 \qquad in \quad D^2 \\ \\ |\partial_x u|^2 - |\partial_y u|^2 - 2i\, \partial_x u \cdot \partial_y u = 0 \qquad in \quad D^2 \quad . \end{array}\right. \tag{I.2}$$

□

The harmonicity and conformality condition exhibited in (I.2) implies that $u(D^2)$ realizes a minimal surface[3]. R. Osserman showed that it has no branch points in the interior of the unit-disk. This result was subsequently generalized to the boundary of the disk by S. Hildebrandt.

The resolution of the Plateau problem proposed by Douglas and Radó is an example of the use of a conformal invariant Lagrangian E to approach an "extrinsic" problem: minimizing the area of a disk with fixed boundary. The analysis of this problem was eased by the high simplicity of the equation (I.1) satisfied by the critical points of E. It is the Laplace equation. Hence, questions related to unicity, regularity, compactness, etc... can be handled with a direct application of the maximum principle. In these lecture notes, we will be concerned with analogous problems (in particular regularity issues) related to the critical points of conformally invariant, coercive Lagrangians with

[3]Recall the following result from differential geometry. Let u be a positive conformal parametrization from an oriented disk in $\mathbb{R}^3$. The mean curvature vector $\vec{H}$, parallel to the outward unit normal vector $\vec{n}$, is defined as

$$\vec{H} = H\,\vec{n} = 2^{-1}\, e^{-2\lambda}\, \Delta u \quad ,$$

where $e^{\lambda} = |\partial_x u| = |\partial_y u|$ and H is the mean curvature $H = (\kappa_1 + \kappa_2)/2$. Equivalently, there holds

$$\Delta u = 2H\; \partial_x u \times \partial_y u \quad . \tag{I.3}$$

quadratic growth. As we will discover, the maximum principle no longer holds, and one must seek an alternate way to compensate this lack. The conformal invariance of the Lagrangian will generate a very peculiar type of nonlinearities in the corresponding Euler-Lagrange equations. We will see how the specific structure of these nonlinearities enable one to recast the equations in divergence form. This new formulation, combined to the results of *integration by compensation*, will provide the substrate to understanding a variety of problems, such as Willmore surfaces, poly-harmonic and α-harmonic maps, Yang-Mills fields, Hermitte-Einstein equations, wave maps, etc...

II Conformally invariant coercice Lagrangians with quadratic growth, in dimension 2.

We consider a Lagrangian of the form

$$L(u) = \int_{D^2} l(u, \nabla u)\; dx\; dy \quad , \tag{II.4}$$

where the integrand l is a function of the variables $z \in \mathbb{R}^m$ and $p \in \mathbb{R}^2 \otimes \mathbb{R}^m$, which satisfy the following coercivity and "almost quadratic" conditions in p:

$$C^{-1}\; |p|^2 \le l(z, p) \le C\; |p|^2 \quad , \tag{II.5}$$

We further assume that L is conformally invariant: for each positive conformal transformation f of degree 1, and for each map $u \in W^{1,2}(D^2, \mathbb{R}^m)$, there holds

$$\begin{aligned} L(u \circ f) \;&= \int_{f^{-1}(D^2)} l(u \circ f, \nabla(u \circ f))\; dx'\; dy' \\ &= \int_{D^2} l(u, \nabla u)\; dx\; dy = L(u) \quad . \end{aligned} \tag{II.6}$$

Example 1. The Dirichlet energy described in the Introduction,

$$E(u) = \int_{D^2} |\nabla u|^2 \; dx\, dy \quad ,$$

whose critical points satisfy the Laplace equation (I.1), which, owing to the conformal hypothesis, geometrically describes *minimal surfaces*. Regularity and compactness matters relative to this equation are handled with the help of the maximum principle.

Example 2. Let an arbitrary in $\mathbb{R}^m$ be given, namely $(g_{ij})_{i,j\in\mathbb{N}_m} \in C^1(\mathbb{R}^m, \mathcal{S}^+_m)$, where $\mathcal{S}^+_m$ denotes the subset of $M_m(\mathbb{R})$, comprising the symmetric positive definite $m \times m$ matrices. We make the following uniform coercivity and boundedness hypothesis:

$$\exists\; C > 0 \qquad \text{s. t.} \quad C^{-1}\delta_{ij} \le g_{ij} \le C\delta_{ij} \quad \text{on } \mathbb{R}^m.$$

Finally, we suppose that

$$\|\nabla g\|_{L^\infty(\mathbb{R}^m)} < +\infty \quad .$$

With these conditions, the second example of quadratic, coercive, conformally invariant Lagrangian is

$$\begin{aligned} E_g(u) &= \frac{1}{2}\int_{D^2} \langle \nabla u, \nabla u\rangle_g \; dx\, dy \\ &= \frac{1}{2}\int_{D^2} \sum_{i,j=1}^{m} g_{ij}(u)\nabla u^i \cdot \nabla u^j \; dx\, dy \quad . \end{aligned}$$

Note that Example 1 is contained as a particular case.
Verifying that E_g is indeed conformally invariant may be done analogously to the case of the Dirichlet energy, via introducing the complex variable $z = x + iy$. No new difficulty arises, and the details are left to the reader as an exercise.
The weak critical points of E_g are the functions $u \in W^{1,2}(D^2, \mathbb{R}^m)$ which satisfy

$$\forall \xi \in C_0^\infty(D^2, \mathbb{R}^m) \qquad \frac{d}{dt} E_g(u + t\xi)_{|t=0} = 0 \quad .$$

An elementary computation reveals that u is a weak critical point of E_g if and only if the following Euler-Lagrange equation holds in the sense of distributions:

$$\forall i = 1 \cdots m \qquad \Delta u^i + \sum_{k,l=1}^{m} \Gamma^i_{kl}(u)\, \nabla u^k \cdot \nabla u^l = 0 \quad . \tag{II.7}$$

Here, Γ^i_{kl} are the Christoffel symbols corresponding to the metric g, explicitly given by

$$\Gamma^i_{kl}(z) = \frac{1}{2}\sum_{s=1}^{m} g^{is} \left(\partial_{z_l} g_{km} + \partial_{z_k} g_{lm} - \partial_{z_m} g_{kl}\right) \quad ,$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Equation (II.7) bears the name *harmonic map equation*[4] with values in $(\mathbb{R}^m, g)$.

Just as in the flat setting, if we further suppose that u is conformal, then (II.7) is in fact equivalent to $u(D^2)$ being a minimal surface in $(\mathbb{R}^m, g)$.

We note that $\Gamma^i(\nabla u, \nabla u) := \sum_{k,l=1}^m \Gamma^i_{kl} \nabla u^k \cdot \nabla u^l$, so that the harmonic map equation can be recast as

$$\Delta u + \Gamma(\nabla u, \nabla u) = 0 \quad . \tag{II.8}$$

This equation raises several analytical questions:

(i) **Weak limits** : Let u_n be a sequence of solutions of (II.8) with uniformly bounded energy E_g. Can one extract a subsequence converging weakly in $W^{1,2}$ to a harmonic map ?

(ii) **Palais-Smale sequences** : Let u_n be a sequence of solutions of (II.8) in $W^{1,2}(D^2, \mathbb{R}^m)$ with uniformly bounded energy E_g, and such that

$$\Delta u_n + \Gamma(\nabla u_n, \nabla u_n) = \delta_n \to 0 \qquad \text{strongly in } H^{-1} \quad .$$

Can one extract a subsequence converging weakly in $W^{1,2}$ to a harmonic map ?

(iii) **Regularity of weak solutions** : Let u be a map in $W^{1,2}(D^2, \mathbb{R}^m)$ which satisfies (II.7) distributionally. How regular is u ? Continuous, smooth, analytic, etc...

The answer to (iii) is strongly tied to that of (i) and (ii). We shall thus restrict our attention in these notes on regularity matters.

[4] One way to interpret (II.7) as the two-dimensional equivalent of the geodesic equation in normal parametrization,

$$\frac{d^2 x^i}{dt^2} + \sum_{k,l=1}^m \Gamma^i_{kl} \frac{dx^k}{dt} \frac{dx^l}{dt} = 0 \quad .$$

Prior to bringing into light further examples of conformally invariant Lagrangians, we feel worthwhile to investigate deeper the difficulties associated with the study of the regularity of harmonic maps in two dimensions.

The harmonic map equation (II.8) belongs to the class of elliptic systems with *quadratic growth*, also known as *natural growth*, of the form

$$\Delta u = f(u, \nabla u) \quad , \tag{II.9}$$

where $f(z,p)$ is an arbitrary continuous function for which there exists constants $C_0 > 0$ and $C_1 > 0$ satisfying

$$\forall z \in \mathbb{R}^m \quad \forall p \in \mathbb{R}^2 \otimes \mathbb{R}^m \qquad f(z,p) \le C_1 |p|^2 + C_0 \quad . \tag{II.10}$$

In dimension two, these equations are critical for the Sobolev space $W^{1,2}$. Indeed,

$$u \in W^{1,2} \Rightarrow \Gamma(\nabla u, \nabla u) \in L^1 \Rightarrow \nabla u \in L^p_{loc}(D^2) \quad \forall p < 2 \quad .$$

In other words, from the regularity standpoint, the demand that ∇u be square-integrable provides the information that[5] ∇u belongs to L^p_{loc} for all $p < 2$. We have thus lost a little bit of information! Had this not been the case, the problem would be "boostrapable", thereby enabling a successful study of the regularity of u. Therefore, in this class of problems, the main difficulty lies in the aforementioned slight loss of information, which we symbolically represent by $L^2 \to L^{2,\infty}$.

[5]Actually, one can show that ∇u belongs to the weak-L^2 Marcinkiewicz space $L^{2,\infty}_{loc}$ comprising those measurable functions f for which

$$\sup_{\lambda>0} \lambda^2 \ \ |\{p \in \omega \ ; \ |f(p)| > \lambda\}| < +\infty \quad , \tag{II.11}$$

where $|\cdot|$ is the standard Lebesgue measure. Note that $L^{2,\infty}$ is a slightly larger space than L^2. However, it possesses the same scaling properties.

There are simple examples of equations with quadratic growth in two dimensions for which the answers to the questions (i)-(iii) are all negative. Consider[6]

$$\Delta u + |\nabla u|^2 = 0 \quad . \tag{II.12}$$

This equation has quadratic growth, and it admits a solution in $W^{1,2}(D^2)$ which is unbounded in L^∞, and thus discontinuous. It is explicitly given by

$$u(x,y) := \log\log \frac{2}{\sqrt{x^2+y^2}} \quad .$$

The regularity issue can thus be answered negatively. Similarly, for the equation (II.12), it takes little effort to devise counter-examples to the weak limit question (i), and thus to the question (ii). To this end, it is helpful to observe that C^2 maps obey the general identity

$$\Delta e^u = e^u \, \left[\Delta u + |\nabla u|^2\right] \quad . \tag{II.13}$$

One easily verifies that if v is a positive solution of

$$\Delta v = -2\pi \sum_i \lambda_i \, \delta_{a_i} \quad ,$$

where $\lambda_i > 0$ and δ_{a_i} are isolated Dirac masses, then $u := \log v$ provides a solution[7] in $W^{1,2}$ of (II.12). We then select a strictly

[6]This equation is conformally invariant. However, as shown by J. Frehse [Fre], it is also the Euler-Lagrange equation derived from a Lagrangian which is *not* conformally invariant:

$$L(u) = \int_{D^2} \left(1 + \frac{1}{1 + e^{12\,u} \; (\log 1/|(x,y)|)^{-12}}\right) \, |\nabla u|^2(x,y) \; dx \; dy \quad .$$

[7]Indeed, per (II.13), we find $\Delta u + |\nabla u|^2 = 0$ away from the points a_i. Near these points, ∇u asymptotically behaves as follows:

$$|\nabla u| = |v|^{-1} \, |\nabla v| \simeq \big(|(x,y) - a_i| \; \log|(x,y) - a_i|\big)^{-1} \; \in L^2 \quad .$$

Hence, $|\nabla u|^2 \in L^1$, so that $\Delta u + |\nabla u|^2$ is a distribution in $H^{-1} + L^1$ supported on the positive regular function f with integral equal to 1, and supported on the ball of radius 1/4 centered on the origin. There exists a sequence of atomic measures with positive weights λ_i^n such that

$$f_n = \sum_{i=1}^{n} \lambda_i^n \; \delta_{a_i^n} \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^n = 1 \quad , \tag{II.14}$$

which converges as Radon measures to f. We next introduce

$$u_n(x,y) := \log \left[\sum_{i=1}^{n} \lambda_i^n \; \log \frac{2}{|(x,y) - a_i^n|} \right] \quad .$$

On D^2, we find that

$$v_n = \sum_{i=1}^{n} \lambda_i^n \; \log \frac{2}{|(x,y) - a_i^n|} > \sum_{i=1}^{n} \lambda_i^n \; \log \frac{8}{5} = \log \frac{8}{5} \quad . \tag{II.15}$$

On the other hand, there holds

$$\begin{aligned} \int_{D^2} |\nabla u_n|^2 &= -\int_{D^2} \Delta u_n = -\int_{\partial D^2} \frac{\partial u_n}{\partial r} \\ &\leq \int_{\partial D^2} \frac{|\nabla v_n|}{|v_n|} \leq \frac{1}{\log \frac{8}{5}} \int_{\partial D^2} |\nabla v_n| \leq C \end{aligned}$$

for some constant C independent of n . Hence, $(u_n)_n$ is a sequence of solutions to (II.12) uniformly bounded in $W^{1,2}$. Since the sequence (f_n) converges as Radon measures to f, it follows that for any $p < 2$, the sequence (v_n) converges strongly in $W^{1,p}$ to

$$v := \log \frac{2}{r} * f \quad .$$

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isolated points a_i. From this, it follows easily that

$$\Delta u + |\nabla u|^2 = \sum_i \mu_i \; \delta_{a_i} \quad .$$

Thus, Δu is the sum of an L^1 function and of Dirac masses. But because Δu lies in H^{-1}, the coefficients μ_i must be zero. Accordingly, u does belong to $W^{1,2}$.

The uniform upper bounded (II.15) paired to the aforementioned strong convergence shows that for each $p < 2$, the sequence $u_n = \log v_n$ converges strongly in $W^{1,p}$ to

$$u := \log \left[\log \frac{2}{r} * f \right]$$

From the hypotheses satisfied by f, we see that $\Delta(e^u) = -2\pi\, f \neq 0$. As f is regular, so is thus e^u, and therefore, owing to (II.13), u cannot fullfill (II.12).

Accordingly, we have constructed a sequence of solutions to (II.12) which converges weakly in $W^{1,2}$ to a map that is *not* a solution to (II.12).

Example 3. We consider a map $(\omega_{ij})_{i,j\in\mathbb{N}_m}$ in $C^1(\mathbb{R}^m, so(m))$, where $so(m)$ is the space antisymmetric square $m \times m$ matrices. We impose the following uniform bound

$$\|\nabla\omega\|_{L^\infty(D^2)} < +\infty \quad .$$

For maps $u \in W^{1,2}(D^2, \mathbb{R}^m)$, we introduce the Lagrangian

$$E^\omega(u) = \frac{1}{2} \int_{D^2} |\nabla u|^2 + \sum_{i,j=1}^{m} \omega_{ij}(u) \partial_x u^i \partial_y u^j - \partial_y u^i \partial_x u^j \ dx\, dy \tag{II.16}$$

The conformal invariance of this Lagrangian arises from the fact that E^ω is made of the conformally invariant Lagrangian E to which is added the integral over D^2 of the 2-form $\omega = \omega_i j dz^i \wedge dz^j$ *pulled back* by u. Composing u by an arbitrary positive diffeomorphism of D^2 will not affect this integral, thereby making E^ω into a conformally invariant Lagrangian.

The Euler-Lagrange equation deriving from (II.16) for variations of the form $u + t\xi$, where ξ is an arbitrary smooth function with compact support in D^2, is found to be

$$\Delta u^i - 2 \sum_{k,l=1}^{m} H^i_{kl}(u)\ \nabla^\perp u^k \cdot \nabla u^l = 0 \qquad \forall\ i = 1, \ldots, m. \tag{II.17}$$

Here, $\nabla^\perp u^l = (-\partial_y u^k, \partial_x u^k)$ [8] while H^i_{kl} is antisymmetric in the indices k et l. It is the coefficient of the $\mathbb{R}^m$-valued two-form H on $\mathbb{R}^m$

$$H^i(z) := \sum_{k,l=1}^m H^i_{kl}(z)\ dz^k \wedge dz^l \quad .$$

The form H appearing in the Euler-Lagrange equation (II.17) is the unique solution of

$$\forall z \in \mathbb{R}^m \quad \forall U, V, W \in \mathbb{R}^m$$

$$\begin{aligned} d\omega_z(U,V,W) &= 4\, U \cdot H(V,W) \\ &= 4 \sum_{i=1}^m U^i\, H^i(V,W) \quad . \end{aligned}$$

For instance, in dimension three, $d\omega$ is a 3-form which can be identified with a function on $\mathbb{R}^m$. More precisely, there exists H such that $d\omega = 4H\ dz^1 \wedge dz^2 \wedge dz^3$. In this notation (II.17) may be recast, for each $i \in \{1, \ldots, m\}$, as

$$\Delta u^i = 2H(u)\ \partial_x u^{i+1} \partial_y u^{i-1} - \partial_x u^{i-1} \partial_y u^{i+1} \quad , \tag{II.18}$$

where the indexing is understood in $\mathbb{Z}_3$. The equation (II.18) may also be written

$$\Delta u = 2H(u)\ \partial_x u \times \partial_y u \quad ,$$

which we recognize as (I.3), the *prescribed mean curvature equation.*

In a general fashion, the equation (II.17) admits the following geometric interpretation. Let u be a conformal solution of

[8] in our notation, $\nabla^\perp u^k \cdot \nabla u^l$ is the Jacobian

$$\nabla^\perp u^k \cdot \nabla u^l = \partial_x u^k \partial_y u^l - \partial_y u^k \partial_x u^l \quad .$$

(II.17), so that $u(D^2)$ is a surface whose mean curvature vector at the point (x,y) is given by

$$e^{-2\lambda}\ u^*H = \left(e^{-2\lambda}\ \sum_{k,l=1}^{m} H^i_{kl}(u)\ \nabla^\perp u^k\cdot\nabla u^l\right)_{i=1\cdots m} \quad , \tag{II.19}$$

where e^λ is the conformal factor $e^\lambda = |\partial_x u| = |\partial_y u|$. As in Example 2, the equation (II.17) forms an elliptic system with quadratic growth, thus critical in dimension two for the $W^{1,2}$ norm. The analytical difficulties relative to this nonlinear system are thus, *a priori*, of the same nature as those arising from the *harmonic map equation*.

Example 4. In this last example, we combine the settings of Examples 2 and 3 to produce a mixed problem. Given on $\mathbb{R}^m$ a metric g and a two-form ω, both C^1 with uniformly bounded Lipschitz norm, consider the Lagrangian

$$E^\omega_g(u) = \frac{1}{2}\int_{D^2}\langle\nabla u,\nabla u\rangle_g\ dx\,dy + u^*\omega \quad .$$

As before, it is a coercive conformally invariant Lagrangian with quadratic growth. Its critical points satisfy the Euler-Lagrangian equation

$$\Delta u^i + \sum_{k,l=1}^{m}\Gamma^i_{kl}(u)\nabla u^k\cdot\nabla u^l - 2\sum_{k,l=1}^{m} H^i_{kl}(u)\nabla^\perp u^k\cdot\nabla u^l = 0 \quad , \tag{II.20}$$

for $i = 1\cdots m$.
Once again, this elliptic system admits a geometric interpretation which generalizes the ones from Examples 2 and 3. Whenever a conformal map u satisfies (II.20), then $u(D^2)$ is a surface in $(\mathbb{R}^m, g)$ whose mean curvature vector is given by (II.19). The equation (II.20) also forms an elliptic system with quadratic growth, and critical in dimension two for the $W^{1,2}$ norm.

Interestingly enough, M. Grüter showed that *any* coercive conformally invariant Lagrangian with quadratic growth is of the form E_g^ω for some appropriately chosen g and ω.

Theorem II.2 *[Gr] Let $l(z,p)$ be a real-valued function on $\mathbb{R}^m \times \mathbb{R}^2 \otimes \mathbb{R}^m$, which is C^1 in its first variable and C^2 in its second variable. Suppose that l obeys the coercivity and quadratic growth conditions*

$$\begin{aligned} &\exists C > 0 \quad t.q. \quad \forall z \in \mathbb{R}^m \quad \forall p \in \mathbb{R}^2 \otimes \mathbb{R}^m \\ &C^{-1}|p|^2 \leq l(X,p) \leq C|p|^2 \quad . \end{aligned} \tag{II.21}$$

Let L be the Lagrangian

$$L(u) = \int_{D^2} l(u, \nabla u)(x,y)\ dx\ dy \tag{II.22}$$

acting on $W^{1,2}(D^2, \mathbb{R}^m)$-maps u. We suppose that L is conformally invariant: for every conformal application ϕ positive and of degree 1, there holds

$$L(u \circ \phi) = \int_{\phi^{-1}(D^2)} l(u \circ \phi, \nabla(u \circ \phi))(x,y)\ dx\ dy = L(u) \quad . \tag{II.23}$$

Then there exist on $\mathbb{R}^m$ a C^1 metric g and a C^1 two-form ω such that

$$L = E_g^\omega \quad . \tag{II.24}$$

Maps taking values in a submanifold of $\mathbb{R}^m$.

Up to now, we have restricted our attention to maps from D^2 into a manifold with only one chart $(\mathbb{R}^n, g)$. More generally, it is possible to introduce the Sobolev space $W^{1,2}(D^2, N^n)$, where (N^n, g) is an oriented n-dimensional C^2-manifold. When this manifold is compact without boundary (which we shall henceforth assume, for the sake of simplicity), a theorem by Nash guarantees that it can be isometrically immersed into Euclidean space $\mathbb{R}^m$, for m large enough. We then define

$$W^{1,2}(D^2, N^n) := \left\{u \in W^{1,2}(D^2, \mathbb{R}^m) \ ; \ u(p) \in N^n \quad \text{a.e. } p \in D^2\right\}$$

Given on N^n a C^1 two-form ω, we may consider the Lagrangian

$$E^\omega(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2 \, dx \, dy + u^*\omega \qquad \text{(II.25)}$$

acting on maps $u \in W^{1,2}(D^2, N^n)$. The critical points of E^ω are defined as follows. Let π_N be the orthogonal projection on N^n which to each point in a neighborhood of N associates its nearest orthogonal projection on N^n. For points sufficiently close to N, the map π_N is regular. We decree that $u \in W^{1,2}(D^2, N^n)$ is a critical point of E^ω whenever there holds

$$\frac{d}{dt}E^\omega(\pi_N(u + t\xi))_{t=0} = 0 \quad , \qquad \text{(II.26)}$$

for all $\xi \in C_0^\infty(D^2, \mathbb{R}^m)$.
It can be shown[9] that (II.26) is satisfied by $u \in C_0^\infty(D^2, \mathbb{R}^m)$ if and only if u obeys the Euler-Lagrange equation

$$\Delta u + A(u)(\nabla u, \nabla u) = H(u)(\nabla^\perp u, \nabla u) \quad , \qquad \text{(II.27)}$$

where $A\,(\equiv A_z)$ is the second fundamental form at the point $z \in N^n$ corresponding to the immersion of N^n into $\mathbb{R}^m$. To a pair of vectors in T_zN^n, the map A_z associates a vector orthogonal to T_zN^n. In particular, at a point $(x,y) \in D^2$, the quantity $A_{(x,y)}(u)(\nabla u, \nabla u)$ is the vector of $\mathbb{R}^m$ given by

$$A_{(x,y)}(u)(\nabla u, \nabla u) := A_{(x,y)}(u)(\partial_x u, \partial_x u) + A_{(x,y)}(u)(\partial_y u, \partial_y u) \quad .$$

For notational convenience, we henceforth omit the subscript (x,y).

[9] in codimension 1, this is done below.

Similarly, $H(u)(\nabla^\perp u, \nabla u)$ at the point $(x,y) \in D^2$ is the vector in $\mathbb{R}^m$ given by

$$H(u)(\nabla^\perp u, \nabla u) := H(u)(\partial_x u, \partial_y u) - H(u)(\partial_y u, \partial_x u)$$
$$= 2H(u)(\partial_x u, \partial_y u) \quad ,$$

where $H\,(\equiv\ H_z)$ is the T_zN^n-valued alternating two-form on T_zN^n:

$$\forall\ U, V, W\ \in T_zN^n \qquad d\omega(U,V,W) := U \cdot H_z(V,W) \quad .$$

Note that in the special case when $\omega = 0$, the equation (II.27) reduces to

$$\Delta u + A(u)(\nabla u, \nabla u) = 0 \quad , \tag{II.28}$$

which is known as the *N^n-valued harmonic map equation.*

We now establish (II.27) in the codimension 1 case. Let ν be the normal unit vector to N. The form ω may be naturally extended on a small neighborhood of N^n via the pull-back $\pi_N^*\omega$ of the projection π_N. Infinitesimally, to first order, considering variations for E^ω of the form $\pi_N(u + t\xi)$ is tantamount to considering variations of the kind $u + t\, d\pi_N(u)\xi$, which further amounts to focusing on variations of the form $u + tv$, where $v \in W^{1,2}(D^2, \mathbb{R}^m) \cap L^\infty$ satisfies $v \cdot \nu(u) = 0$ almost everywhere. Following the argument from Example 3, we obtain that u is a critical point of E^ω whenever for all v with $v \cdot \nu(u) = 0$ a.e., there holds

$$\int_{D^2} \sum_{i=1}^m \left[\Delta u^i - 2 \sum_{k,l=1}^m H^i_{kl}(u)\ \nabla^\perp u^k \cdot \nabla u^l \right]\ v^i\ dx\ dy = 0 \quad ,$$

where H is the vector-valued two-form on $\mathbb{R}^m$ given for z on N^n by

$$\forall\ U, V, W\ \in \mathbb{R}^m \qquad d\pi_N^*\omega(U,V,W) := U \cdot H_z(V,W) \quad .$$

In the sense of distributions, we thus find that

$$\Big[\Delta u - H(u)(\nabla^\perp u, \nabla u)\Big] \wedge \nu(u) = 0 \quad . \tag{II.29}$$

Recall, $\nu \circ u \in L^\infty \cap W^{1,2}(D^2, \mathbb{R}^m)$. Accordingly (II.29) does indeed make sense in $\mathcal{D}'(D^2)$.

Note that if any of the vectors U, V, and W is normal to N^n, i.e. parallel to ν, then $d\pi_N^*\omega(U, V, W) = 0$, so that

$$\nu_z \cdot H_z(V, W) = 0 \qquad \forall\ V\,, W\ \in\ \mathbb{R}^m\,.$$

Whence,

$$\begin{aligned} &\Big[\Delta u - H(u)(\nabla^\perp u, \nabla u)\Big] \cdot \nu(u) = \Delta u \cdot \nu(u) \\ &= div(\nabla u \cdot \nu(u)) - \nabla u \cdot \nabla(\nu(u)) = -\nabla u \cdot \nabla(\nu(u)) \end{aligned} \tag{II.30}$$

where we have used the fact that $\nabla u \cdot \nu(u) = 0$ holds almost everywhere, since ∇u is tangent to N^n.

Altogether, (II.29) and (II.30) show that u satisfies in the sense of distributions the equation

$$\Delta u - H(u)(\nabla^\perp u, \nabla u) = -\nu(u)\ \nabla(\nu(u)) \cdot \nabla u \quad . \tag{II.31}$$

In codimension 1, the second fundamental form acts on a pair of vectors (U, V) in $T_z N^n$ via

$$A_z(U, V) = \nu(z)\ \ < d\nu_z U, V > \quad , \tag{II.32}$$

so that, as announced, (II.31) and (II.27) are identical.

We close this section by stating a conjecture formulated by Stefan d'Hildebrandt in the late 1970s.

Conjecture 1 *[Hil] [Hil2] The critical points with finite energy of a coercive conformally invariant Lagrangian with quadractic growth are Hölder continuous.*

The remainder of these lecture notes shall be devoted to establishing this conjecture. Although its resolution is closely related to the compactness questions (i) and (ii) previously formulated on page 9, for lack of time, we shall not dive into the study of this point.
Our proof will begin by recalling the first partial answers to Hildebrandt's conjecture provided by H. Wente and F. Hélein, and the importance in their approach of the rôle played by *conservations laws* and *integration by compensation.*
Then, in the last section, we will investigate the theory of linear elliptic systems with antisymmetric potentials, and show how to apply it to the resolution of Hildebrandt's conjecture.

III Integrability by compensation applied to the regularity of critical points of some conformally invariant Lagrangians

III.1 Constant mean curvature equation (CMC)

Let $H \in \mathbb{R}$ be constant. We study the analytical properties of solutions in $W^{1,2}(D^2, \mathbb{R}^3)$ of the equation

$$\Delta u - 2H\ \partial_x u \times \partial_y u = 0 \quad . \tag{III.1}$$

The Jacobian structure of the right-hand side enable without much trouble, inter alia, to show that **Palais-Smale sequences** converge weakly:

Let F_n be a sequence of distributions converging to zero in $H^{-1}(D^2, \mathbb{R}^3)$, and let u_n be a sequence of functions uniformly bounded in $W^{1,2}$ and satisfying the equation

$$\Delta u_n - 2H\ \partial_x u_n \times \partial_y u_n = F_n \to 0 \text{ strongly in } H^{-1}(D^2) \quad .$$

We use the notation

$$\begin{aligned} (\partial_x u_n \times \partial_y u_n)^i &= \partial_x u_n^{i+1} \partial_y u_n^{i-1} - \partial_x u_n^{i-1} \partial_y u_n^{i+1} \\ &= \partial_x (u_n^{i+1}\ \partial_y u_n^{i-1}) - \partial_y (u_n^{i+1} \partial_x u_n^{i-1}) \quad . \end{aligned} \tag{III.2}$$

The uniform bounded on the $W^{1,2}$-norm of u_n enables the extraction of a subsequence $u_{n'}$ weakly converging in $W^{1,2}$ to some limit u_∞. With the help of the Rellich-Kondrachov theorem, we see that the sequence u_n is strongly compact in L^2. In particular, we can pass to the limit in the following quadratic terms

$$u_n^{i+1}\ \partial_y u_n^{i-1} \to u_\infty^{i+1}\ \partial_y u_\infty^{i-1} \qquad \text{in } \mathcal{D}'(D^2)$$

and

$$u_n^{i+1}\ \partial_x u_n^{i-1} \to u_\infty^{i+1}\ \partial_x u_\infty^{i-1} \qquad \text{in } \mathcal{D}'(D^2) \quad .$$

Combining this to (III.2) reveals that u_∞ is a solution of the CMC equation (III.1).

Obtaining information on the regularity of weak $W^{1,2}$ solutions of the CMC equation (III.2) requires some more elaborate work. More precisely, a result from the theory of integration by compensation due to H. Wente is needed.

Theorem III.1 *[We] Let a and b be two functions in $W^{1,2}(D^2)$, and let ϕ be the unique solution in $W_0^{1,p}(D^2)$ - for $1 \leq p < 2$ - of the equation*

$$\left\{ \begin{array}{ll} -\Delta\phi = \partial_x a\,\partial_y b - \partial_x b\,\partial_y a & \quad \text{in } D^2 \\ \varphi = 0 \qquad \text{on } \partial D^2 & . \end{array} \right. \tag{III.3}$$

Then ϕ belongs to $C^0 \cap W^{1,2}(D^2)$ and

$$\|\phi\|_{L^\infty(D^2)} + \|\nabla\phi\|_{L^2(D^2)} \leq C_0\ \|\nabla a\|_{L^2(D^2)}\ \|\nabla b\|_{L^2(D^2)} \quad . \tag{III.4}$$

where C_0 is a constant independent of a and b.[10] □

Proof of theorem III.1. We shall first assume that a and b are smooth, so as to legitimize the various manipulations which we will need to perform. The conclusion of the theorem for general a and b in $W^{1,2}$ may then be reached through a simple density argument. In this fashion, we will obtain the continuity of ϕ from its being the uniform limit of smooth functions.

Observe first that integration by parts and a simple application of the Cauchy-Schwarz inequality yields the estimate

$$\begin{aligned} \int_{D^2} |\nabla\phi|^2 = -\int_{D^2} \phi\,\Delta\phi \ &\leq \|\phi\|_\infty\ \|\partial_x a\,\partial_y b - \partial_x b\,\partial_y a\|_1 \\ &\leq 2\,\|\phi\|_\infty \|\nabla a\|_2\ \|\nabla b\|_2 \quad . \end{aligned}$$

[10]Actually, one shows that theorem III.1 may be generalized to arbitrary oriented Riemannian surfaces, with a constant C_0 <u>independent of the surface</u>, which is quite a remarkable and useful fact. For more details, see [Ge] and [To].

Accordingly, if ϕ lies in L^∞, then it automatically lies in $W^{1,2}$.

Step 1. given two functions $\tilde{a}$ and $\tilde{b}$ in $C_0^\infty(\mathbb{C})$, which is dense in $W^{1,2}(\mathbb{C})$, we first establish the estimate (III.4) for

$$\tilde{\phi} := \frac{1}{2\pi}\log\frac{1}{r} * \left[\partial_x\tilde{a}\,\partial_y\tilde{b} - \partial_x\tilde{b}\,\partial_y\tilde{a}\right] \quad . \tag{III.5}$$

Owing to the translation-invariance, it suffices to show that

$$|\tilde{\phi}(0)| \le C_0\ \|\nabla\tilde{a}\|_{L^2(\mathbb{C})}\ \|\nabla\tilde{b}\|_{L^2(\mathbb{C})} \quad . \tag{III.6}$$

We have

$$\begin{aligned}
\tilde{\phi}(0) &= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log r\ \partial_x\tilde{a}\,\partial_y\tilde{b} - \partial_x\tilde{b}\,\partial_y\tilde{a}\\
&= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\log\ r\ \frac{\partial}{\partial r}\left(\tilde{a}\ \frac{\partial\tilde{b}}{\partial\theta}\right) - \frac{\partial}{\partial\theta}\left(\tilde{a}\ \frac{\partial\tilde{b}}{\partial r}\right)\ dr\ d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\tilde{a}\ \frac{\partial\tilde{b}}{\partial\theta}\ \frac{dr}{r}\ d\theta
\end{aligned}$$

Because $\int_0^{2\pi}\frac{\partial\tilde{b}}{\partial\theta}\ d\theta = 0$, we may deduct from each circle $\partial B_r(0)$ a constant à $\tilde{a}$ chosen to have average $\overline{\tilde{a}}_r$ on $\partial B_r(0)$. Hence, there holds

$$\tilde{\phi}(0) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}[\tilde{a} - \overline{\tilde{a}}_r]\ \frac{\partial\tilde{b}}{\partial\theta}\ \frac{dr}{r}\ d\theta \quad .$$

Applying successively the Cauchy-Schwarz and Poincaré inequalities on the circle S^1, we obtain

$$\begin{aligned}
|\tilde{\phi}(0)| &\le \frac{1}{2\pi}\int_0^{+\infty}\frac{dr}{r}\left(\int_0^{2\pi}|\tilde{a} - \overline{\tilde{a}}_r|^2\right)^{\frac{1}{2}}\ \left(\int_0^{2\pi}\left|\frac{\partial\tilde{b}}{\partial\theta}\right|^2\right)^{\frac{1}{2}}\\
&\le \frac{1}{2\pi}\int_0^{+\infty}\frac{dr}{r}\left(\int_0^{2\pi}\left|\frac{\partial\tilde{a}}{\partial\theta}\right|^2\right)^{\frac{1}{2}}\ \left(\int_0^{2\pi}\left|\frac{\partial\tilde{b}}{\partial\theta}\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

The sought after inequality (III.6) may then be inferred from the latter via applying once more the Cauchy-Schwarz inequality.

Returning to the disk D^2, the Whitney extension theorem yields the existence of $\tilde{a}$ and $\tilde{b}$ such that

$$\int_{\mathbb{C}} |\nabla \tilde{a}|^2 \le C_1 \int_{D^2} |\nabla a|^2 \quad , \tag{III.7}$$

and

$$\int_{\mathbb{C}} |\nabla \tilde{b}|^2 \le C_1 \int_{D^2} |\nabla b|^2 \quad . \tag{III.8}$$

Let $\tilde{\phi}$ be the function in (III.5). The difference $\phi - \tilde{\phi}$ satisfies the equation

$$\begin{cases} \Delta(\phi - \tilde{\phi}) = 0 & \text{in } D^2 \\ \phi - \tilde{\phi} = -\tilde{\phi} & \text{on } \partial D^2 \end{cases}$$

The *maximum principle* applied to the inequalities (III.6), (III.7) and (III.8) produces

$$\|\phi - \tilde{\phi}\|_{L^\infty(D^2)} \le \|\tilde{\phi}\|_{L^\infty(\partial D^2)} \le C \|\nabla a\|_2 \, \|\nabla b\|_2 \quad .$$

With the triangle inequality $|\|\phi\|_\infty - \|\tilde{\phi}\|_\infty| \le \|\phi - \tilde{\phi}\|_\infty$ and the inequality (III.6), we reach the desired L^∞-estimate of ϕ, and therefore, per the above discussion, the theorem is proved. □

Proof of the regularity of the solutions of the CMC equation.

Our first aim will be to establish the existence of a positive constant α such that

$$\sup_{\rho<1/4,\ p\in B_{1/2}(0)} \rho^{-\alpha} \int_{B_\rho(p)} |\nabla u|^2 < +\infty \quad . \tag{III.9}$$

Owing to a classical result from Functional Analysis[11], the latter implies that $u \in C^{0,\alpha/2}(B_{1/2}(0))$. From this, we deduce that u is locally Hölder continuous in the interior of the disk D^2. We will then explain how to obtain the smoothness of u from its Hölder continuity.

Let $\varepsilon_0 > 0$. There exists some radius $\rho_0 > 0$ such that for every $r < \rho_0$ and every point p in $B_{1/2}(0)$

$$\int_{B_r(p)} |\nabla u|^2 < \varepsilon_0 \quad .$$

We shall in due time adjust the value ε_0 to fit our purposes. In the sequel, $r < \rho_0$. On $B_r(p)$, we decompose $u = \phi + v$ in such a way that

$$\begin{cases} \Delta\phi = H\ \partial_x u \times \partial_y u & \text{in} \quad B_r(p) \\ \phi = 0 & \text{on} \quad \partial B_r(p) \end{cases}$$

Applying theorem III.1 to ϕ yields

$$\begin{aligned} \int_{B_r(p)} |\nabla\phi|^2 &\le C_0|H| \int_{B_r(p)} |\nabla u|^2 \int_{B_r(p)} |\nabla u|^2 \\ &\le C_0|H|\ \varepsilon_0 \int_{B_r(p)} |\nabla u|^2 \quad . \end{aligned} \tag{III.10}$$

The function $v = u - \phi$ is harmonic. To obtain useful estimates on v, we need the following result.

[11] See for instance [Gi].

Lemma III.1 *Let v be a harmonic function on D^2. For every point p in D^2, the function*

$$\rho \longmapsto \frac{1}{\rho^2}\int_{B_\rho(p)} |\nabla v|^2$$

is increasing. □

Proof. Note first that

$$\frac{d}{d\rho}\left[\frac{1}{\rho^2}\int_{B_\rho(p)} |\nabla v|^2\right] = -\frac{2}{\rho^3}\int_{B_\rho(p)} |\nabla v|^2 + \frac{1}{\rho^2}\int_{\partial B_\rho(p)} |\nabla v|^2 \quad . \tag{III.11}$$

Denote by $\overline{v}$ the average of v on $\partial B_\rho(p)$: $\overline{v} := |\partial B_\rho(p)|^{-1}\int_{\partial B_\rho(p)} v$. Then, there holds

$$0 = \int_{B_\rho(p)} (v - \overline{v})\ \Delta v = -\int_{B_\rho(p)} |\nabla v|^2 + \int_{\partial B_\rho(p)} (v - \overline{v})\ \frac{\partial v}{\partial \rho} \quad .$$

This implies that

$$\frac{1}{\rho}\int_{B_\rho(p)} |\nabla v|^2 \leq \left(\frac{1}{\rho^2}\int_{\partial B_\rho(p)} |v - \overline{v}|^2\right)^{\frac{1}{2}} \left(\int_{\partial B_\rho(p)} \left|\frac{\partial v}{\partial \rho}\right|^2\right)^{\frac{1}{2}} \quad . \tag{III.12}$$

In Fourier space, v satisfies $v = \sum_{n\in\mathbb{Z}} a_n\, e^{in\theta}$ and $v - \overline{v} = \sum_{n\in\mathbb{Z}^*} a_n\, e^{in\theta}$. Accordingly,

$$\frac{1}{2\pi\rho}\int_{\partial B_\rho(p)} |v - \overline{v}|^2 = \sum_{n\in\mathbb{Z}^*} |a_n|^2 \leq \sum_{n\in\mathbb{Z}^*} |n|^2 |a_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\partial v}{\partial \theta}\right|^2\, d\theta \quad .$$

Combining the latter with (III.12) then gives

$$\frac{1}{\rho}\int_{B_\rho(p)} |\nabla v|^2 \leq \left(\int_{\partial B_\rho(p)} \left|\frac{1}{\rho}\frac{\partial v}{\partial \theta}\right|^2\right)^{\frac{1}{2}} \left(\int_{\partial B_\rho(p)} \left|\frac{\partial v}{\partial \rho}\right|^2\right)^{\frac{1}{2}} \quad . \tag{III.13}$$

If we multiply the Laplace equation throughout by $(x-x_p)\,\partial_x v+(y-y_p)\,\partial_y v$, and then integrate by parts over $B_\rho(p)$, we reach the **Pohozaev identity** :

$$2\int_{\partial B_\rho(p)}\left|\frac{\partial v}{\partial\rho}\right|^2=\int_{\partial B_\rho(p)}|\nabla v|^2 \quad . \tag{III.14}$$

Altogether with (III.13), this identity implies that the right-hand side of (III.11) is positive, thereby concluding the proof. □

We now return to the proof of the regularity of the solutions of the CMC equation. Per the above lemma, there holds

$$\int_{B_{\rho/2}(p)}|\nabla v|^2\le\frac{1}{4}\int_{B_\rho(p)}|\nabla v|^2 \quad . \tag{III.15}$$

Since $\Delta v=0$ on $B_\rho(p)$, while $\phi=0$ on $\partial B_\rho(p)$, we have

$$\int_{B_\rho(p)}\nabla v\cdot\nabla\phi=0 \quad .$$

Combining this identity to the inequality in (III.15), we obtain

$$\begin{aligned}\int_{B_{\rho/2}(p)}|\nabla(v+\phi)|^2 \;&\le\frac{1}{2}\int_{B_\rho(p)}|\nabla(v+\phi)|^2\\ &\quad+3\int_{B_\rho(p)}|\nabla\phi|^2 \quad .\end{aligned} \tag{III.16}$$

which, accounting for (III.10), yields

$$\int_{B_{\rho/2}(p)}|\nabla u|^2\le\left(\frac{1}{2}+3\;C_0\;|H|\;\varepsilon_0\right)\int_{B_\rho(p)}|\nabla u|^2 \quad . \tag{III.17}$$

If we adjust ε_0 sufficiently small as to have $3\;C_0\;|H|\;\varepsilon_0<1/4$, it follows that

$$\int_{B_{\rho/2}(p)}|\nabla u|^2\le\frac{3}{4}\int_{B_\rho(p)}|\nabla u|^2 \quad . \tag{III.18}$$

Iterating this inequality gives the existence of a constant $\alpha > 0$ such that for all $p \in B_{1/2}(0)$ and all $r < \rho$, there holds

$$\int_{B_r(p)} |\nabla u|^2 \leq \left(\frac{r}{\rho_0}\right)^\alpha \int_{D^2} |\nabla u|^2 \quad ,$$

which implies (III.9). Accordingly, the solution u of the CMC equation is Hölder continuous.
Next, we infer from (III.9) and (III.1) the bound

$$\sup_{\rho<1/2,\ p\in B_{1/2}(0)} \rho^{-\alpha} \int_{B_\rho(p)} |\Delta u| < +\infty \quad . \tag{III.19}$$

A classical estimate on Riesz potentials gives

$$|\nabla u|(p) \leq C\frac{1}{|x|} * \chi_{B_{1/2}} \, |\Delta u| + C \qquad \forall \ p \in B_{1/4}(0) \quad ,$$

where $\chi_{B_{1/2}}$ is the characteristic function of the ball $B_{1/2}(0)$. Together with injections proved by Adams in [Ad], the latter shows that $u \in W^{1,q}(B_{1/4}(0))$ for any $q > (2-\alpha)/(1-\alpha)$. Substituted back into (III.1), this fact implies that $\Delta u \in L^r$ for some $r > 1$. The equation the becomes subcritical, and a standard bootstrapping argument eventually yields that $u \in C^\infty$. This conclusdes the proof of the regularity of solutions of the CMC equation.

III.2 Harmonic maps with values in the sphere S^n

When the target manifold N^n has codimension 1, the *harmonic map equation* (II.28) becomes (cf. (II.32))

$$-\Delta u = \nu(u)\ \nabla(\nu(u)) \cdot \nabla u \quad , \tag{III.20}$$

where u still denotes the normal unit-vector to the submanifold $N^n \subset \mathbb{R}^{n+1}$. In particular, if N^n is the sphere S^n, there holds $\nu(u) = u$, and the equation reads

$$-\Delta u = u\ |\nabla u|^2 \quad . \tag{III.21}$$

Another characterization of (III.21) states that the function $u \in W^{1,2}(D^2, S^n)$ satisfies (III.21) if and only if

$$u \wedge \Delta u = 0 \qquad \text{in } \mathcal{D}'(D^2) \quad . \tag{III.22}$$

Indeed, any S^n-valued map u obeys

$$0 = \Delta \frac{|u|^2}{2} = div(u\,\nabla u) = |\nabla u|^2 + u\,\Delta u$$

so that Δu is parallel to u as in (III.22) if and only if the proportionality is $-|\nabla u|^2$. This is equivalent to (III.21). Interestingly enough, J. Shatah [Sha] observed that (III.22) is tantamount to

$$\forall i, j = 1 \cdots n+1 \qquad div(u^i\,\nabla u_j - u_j\,\nabla u^i) = 0 \quad . \tag{III.23}$$

This formulation of the equation for S^n-valued harmonic maps enables one to pass to the weak limit, just as we previously did in the CMC equation.

The **regularity of S^n-valued harmonic maps** was obtained by F.Hélein, [He]. It is established as follows.

For each pair of indices (i,j) in $\{1 \cdots n+1\}^2$, the equation (III.23) reveals that the vector field $u^i\,\nabla u^j - u^j\,\nabla u^i$ forms a curl term, and hence there exists $B^i_j \in W^{1,2}$ with

$$\nabla^\perp B^i_j = u^i\,\nabla u_j - u_j\,\nabla u^i \quad .$$

In local coordinates, (III.21) may be written

$$-\Delta u^i = \sum_{j=1}^{n+1} u^i\,\nabla u_j \cdot \nabla u^j \quad . \tag{III.24}$$

We then make the field $\nabla^\perp B^i_j$ appear on the right-hand side by observing that

$$\sum_{j=1}^{n+1} u_j\,\nabla u^i \cdot \nabla u^j = \nabla u^i \cdot \nabla \left(\sum_{j=1}^{n+1} |u^j|^2/2 \right) = \nabla u^i \cdot \nabla |u|^2/2 = 0 \quad .$$

Deducting this null term from the right-hand side of (III.24) yields that for all $i = 1 \cdots n+1$, there holds

$$
\begin{aligned}
-\Delta u^i &= \sum_{j=1}^{n+1} \nabla^\perp B^i_j \cdot \nabla u^j \\
&= \sum_{j=1}^{n+1} \partial_x B^i_j \, \partial_y u^j - \partial_y B^i_j \, \partial_x u^i \quad .
\end{aligned}
\tag{III.25}
$$

We recognize the same Jacobian structure which we previously employed to establish the regularity of solutions of the CMC equation. It is thus possible to adapt mutatis mutandis our argument to (III.25) so as to infer that S^n-valued harmonic maps are regular.

III.3 Hélein's moving frames method and the regularity of harmonic maps mapping into a manifold.

When the target manifold is no longer a sphere (or, more generally, when it is no longer homogeneous), the aforementioned Jacobian structure disappears, and the techniques we employed no longer seem to be directly applicable.

To palliate this lack of structure, and thus extend the regularity result to harmonic maps mapping into an arbitrary manifold, F. Hélein devised the *moving frames method*. The divergence-form structure being the result of the global symmetry of the target manifold, Hélein's idea consists in expressing the harmonic map equation in preferred moving frames, called *Coulomb frames*, thereby compensating for the lack of global symmetry with "infinitesimal symmetries".

This method, although seemingly unnatural and rather mysterious, has subsequently proved very efficient to answer regularity and compactness questions, such as in the study of nonlinear wave maps (see [FMS], [ShS], [Tao1], [Tao2]). For this reason, it is worthwhile to dwell a bit more on Hélein's method.

We first recall the main result of F. Hélein.

Theorem III.2 *[He] Let N^n be a closed C^2-submanifold of $\mathbb{R}^m$. Suppose that u is a harmonic map in $W^{1,2}(D^2, N^n)$ that weakly satisfies the harmonic map equation (II.28). Then u lies in $C^{1,\alpha}$ for all $\alpha < 1$.*

Proof of theorem III.2 when N^n is a two-torus.

We will consider the case when N^n is a two-dimensional parallelizable manifold (i.e. admitting a global basis of tangent vectors for the tangent space), namely a torus T^2 arbitrarily immersed into Euclidean space $\mathbb{R}^m$, for m large enough. The case of the two-torus is distinguished. Indeed, in general, if a harmonic map u takes its values in an immersed manifold N^n, then it is possible to lift u to a harmonic map $\tilde{u}$ taking values in a parallelizable torus $(S^1)^q$ of higher dimension. Accordingly, the argument which we present below can be analogously extended to a more general setting[12].

Let $u \in W^{1,2}(D^2, T^2)$ satisfy weakly (II.28). We equip T^2 with a global, regular, positive orthonormal tangent frame field $(\varepsilon_1, \varepsilon_2)$. Let $\tilde{e} := (\tilde{e}_1, \tilde{e}_2) \in W^{1,2}(D^2, \mathbb{R}^m \times \mathbb{R}^m)$ be defined by the composition

$$\tilde{e}_i(x, y) := \varepsilon_i(u(x, y)) \quad .$$

The map $(\tilde{e})$ is defined on D^2 and it takes its values in the tangent frame field to T^2. Define the energy

$$\min_{\psi \in W^{1,2}(D^2,\mathbb{R})} \int_{D^2} |(e_1, \nabla e_2)|^2 \, dx\, dy \quad , \tag{III.26}$$

[12] although the lifting procedure is rather technical. The details are presented in Lemma 4.1.2 from [He].

where $(\cdot,\cdot)$ is the standard scalar product on $\mathbb{R}^m$, and

$$e_1(x,y) + i e_2(x,y) := e^{i\psi(x,y)}\,(\tilde{e}_1(x,y) + i\tilde{e}_2(x,y)) \quad .$$

We seek to optimize the map $(\tilde{e})$ by minimizing this energy over the $W^{1,2}(D^2)$-maps taking values in the space of rotations of the plane $\mathbb{R}^2 \simeq T_{u(x,y)}T^2$. Our goal is to seek a frame field as regular as possible in which the harmonic map equation will be recast. The variational problem (III.26) is well-posed, and it further admits a solution in $W^{1,2}$. Indeed, there holds

$$|(e_1, \nabla e_2)|^2 = |\nabla\psi + (\tilde{e}_1, \nabla\tilde{e}_2)|^2 \quad .$$

Hence, there exists a unique minimizer in $W^{1,2}$ which satisfies

$$0 = div\,(\nabla\psi + (\tilde{e}_1, \nabla\tilde{e}_2)) = div((e_1, \nabla e_2)) \quad . \tag{III.27}$$

A priori, $(e_1, \nabla e_2)$ belongs to L^2. But actually, thanks to the careful selection brought in by the variational problem (III.26), we shall discover that the frame field $(e_1, \nabla e_2)$ over D^2 lies in $W^{1,1}$, thereby improving the original L^2 belongingness[13]. Because the vector field $(e_1, \nabla e_2)$ is divergence-free, there exists some function $\phi \in W^{1,2}$ such that

$$(e_1, \nabla e_2) = \nabla^{\perp}\phi \quad . \tag{III.29}$$

On the other hand, ϕ satisfies by definition

$$-\Delta\phi = (\nabla e_1, \nabla^{\perp} e_2) = \sum_{j=1}^{m} \partial_y e_1^j \partial_x e_2^j - \partial_x e_1^j \partial_y e_2^j \quad . \tag{III.30}$$

[13]Further yet, owing to a result of Luc Tartar [Tar2], we know that $W^{1,1}(D^2)$ is continuously embedded in the Lorentz space $L^{2,1}(D^2)$, whose dual is the Marcinkiewicz weak-L^2 space $L^{2,\infty}(D^2)$, whose definition was recalled in (II.11). A measurable function f is an element of $L^{2,1}(D^2)$ whenever

$$\int_0^{+\infty} \left|\{p \in D^2\ ;\ |f(p)| > \lambda\}\right|^{\frac{1}{2}}\ d\lambda \quad . \tag{III.28}$$

The right-hand side of this elliptic equation comprises only Jacobians of elements of $W^{1,2}$. This configuration is identical to those previously encountered in our study of the constant mean curvature equation and of the equation of S^n-valued harmonic maps. In order to capitalize on this particular structure, we call upon an extension of Wente's theorem III.1 due to Coifman, Lions, Meyer, and Semmes.

Theorem III.3 *[CLMS] Let a and b be two functions in $W^{1,2}(D^2)$, and let ϕ be the unique solution in $W^{1,p}_0(D^2)$, for $1 \leq p < 2$, of the equation*

$$\left\{\begin{array}{ll} -\Delta\phi = \partial_x a\,\partial_y b - \partial_x b\,\partial_y a & \text{in } D^2 \\ \phi = 0 & \text{on } \partial D^2 \quad . \end{array}\right. \tag{III.31}$$

Then ϕ lies in $W^{2,1}$ and

$$\|\nabla^2\phi\|_{L^1(D^2)} \leq C_1\ \|\nabla a\|_{L^2(D^2)}\ \|\nabla b\|_{L^2(D^2)} \quad . \tag{III.32}$$

where C_1 is a constant independent of a and b.[14] □

Applying this result to the solution ϕ of (III.30) then reveals that $(e_1, \nabla e_2)$ is indeed an element of $W^{1,1}$.

We will express the harmonic map equation (II.28) in this particular Coulomb frame field, distinguished by its increased regularity. Note that (II.28) is equivalent to

$$\left\{\begin{array}{l} (\Delta u, e_1) = 0 \\ (\Delta u, e_2) = 0 \end{array}\right. \tag{III.33}$$

[14] Theorem III.1 is a corollary of theorem III.3 owing to the Sobolev embedding $W^{2,1}(D^2) \subset W^{1,2} \cap C^0$. In the same vein, theorem III.3 was preceded by two intermediary results. The first one, by Luc Tartar [Tar1], states that the Fourier transform of $\nabla\phi$ lies in the Lorentz space $L^{2,1}$, which also implies theorem III.1. The second one, due to Stefan Müller, obtains the statement of theorem III.3 under the additional hypothesis that the Jacobian $\partial_x a\,\partial_y b - \partial_x b\,\partial_y a$ be positive. One should also recall that in [CLMS] it is also explain how to deduce theorem III.3 from an important older result by R.Coifman, R.Rochberg and Guido Weiss [CRW] which is maybe the founding result in the integrability by compensation theory.

Using the fact that

$$\partial_x u, \partial_y u \in T_u N^n = \mathrm{vec}\{e_1, e_2\}$$

$$(\nabla e_1, e_1) = (\nabla e_2, e_2) = 0$$

$$(\nabla e_1, e_2) + (e_1, \nabla e_2) = 0$$

we obtain that (III.33) may be recast in the form

$$\left\{ \begin{array}{l} div((e_1, \nabla u)) = -(\nabla e_2, e_1) \cdot (e_2, \nabla u) \\[2mm] div((e_2, \nabla u)) = (\nabla e_2, e_1) \cdot (e_1, \nabla u) \end{array} \right. \tag{III.34}$$

On the other hand, there holds

$$\left\{ \begin{array}{l} rot((e_1, \nabla u)) = -(\nabla^{\perp} e_2, e_1) \cdot (e_2, \nabla u) \\[2mm] rot((e_2, \nabla u)) = (\nabla^{\perp} e_2, e_1) \cdot (e_1, \nabla u) \end{array} \right. \tag{III.35}$$

We next proceed by introducing the Hodge decompositions in L^2 of the frames $(e_i, \nabla u)$, for $i \in \{1, 2\}$. In particular, there exist four functions C_i and D_i in $W^{1,2}$ such that

$$(e_i, \nabla u) = \nabla C_i + \nabla^{\perp} D_i \quad .$$

Setting $W := (C_1, C_2, D_1, D_2)$, the identities (III.34) et (III.35) become

$$-\Delta W = \Omega \cdot \nabla W \quad , \tag{III.36}$$

where Ω is the vector field valued in the space of 4×4 matrices defined by

$$\Omega = \left(\begin{array}{cccc} 0 & -\nabla^{\perp}\phi & 0 & -\nabla\phi \\[2mm] \nabla^{\perp}\phi & 0 & \nabla\phi & 0 \\[2mm] 0 & \nabla\phi & 0 & -\nabla^{\perp}\phi \\[2mm] -\nabla\phi & 0 & \nabla^{\perp}\phi & 0 \end{array} \right) \tag{III.37}$$

Since $\phi \in W^{2,1}$, the following theorem III.4 implies that ∇W, and hence ∇u, belong to L^p for some $p > 2$, thereby enabling the initialization of a bootstrapping argument analogous to that previously encountered in our study of the CMC equation. This procedure yields that u lies in $W^{2,q}$ for all $q < +\infty$. Owing to the standard Sobolev embedding theorem, it follows that $u \in C^{1,\alpha}$, which concludes the proof of the desired theorem III.2 in the case when the target manifold of the harmonic map u is the two-torus. □

Theorem III.4 *Let W be a solution in $W^{1,2}(D^2, \mathbb{R}^n)$ of the linear system*

$$-\Delta W = \Omega \cdot \nabla W \quad , \tag{III.38}$$

where Ω is a $W^{1,1}$ vector field on D^2 taking values in the space of $n \times n$ matrices. Then W belongs to $W^{1,p}(B_{1/2}(0))$, for some $p > 2$. In particular, W is Hölder continuous[15 16]*.* □

Proof of theorem III.4.

Just as in the proof of the regularity of solutions of the CMC equation, we seek to obtain a Morrey-type estimate via the existence of some constant $\alpha > 0$ such that

$$\sup_{p \in B_{1/2}(0)\ ,\ 0<\rho<1/4} \rho^{-\alpha} \int_{B_\rho(p)} |\Delta W| < +\infty \quad . \tag{III.39}$$

The statement of the theorem is then a corollary of an inequality involving Riesz potentials (cf. [Ad] and the CMC equation case on page 28 above).

[15]The statement of theorem III.4 is optimal. To see this, consider $u = \log\log 1/r = W$. One verifies easily that $u \in W^{1,2}(D^2, T^2)$ satisfies weakly (II.28). Yet, $\Omega \equiv \nabla u$ fails to be $W^{1,1}$, owing to

$$\int_0^1 \frac{dr}{r \log \frac{1}{r}} = +\infty \quad .$$

[16]The hypothesis $\Omega \in W^{1,1}$ may be replaced by the condition that $\Omega \in L^{2,1}$.

Let $\varepsilon_0 > 0$ be some constant whose size shall be in due time adjusted to fit our needs. There exists some radius ρ_0 such that for every $r < \rho_0$ and every point $p \in B_{1/2}(0)$, there holds

$$\|\Omega\|_{L^{2,1}(B_r(p))} < \varepsilon_0 \quad .$$

Note that we have used the aforementioned continuous injection $W^{1,1} \subset L^{2,1}$.

Henceforth, we consider $r < \rho_0$. On $B_r(p)$, we introduce the decomposition $W = \Phi + V$, with

$$\left\{ \begin{array}{ll} \Delta\Phi = \Omega \cdot \nabla W & \quad \text{in } B_r(p) \\ \\ \Phi = 0 & \quad \text{on } \partial B_r(p) \quad . \end{array} \right.$$

A classical result on Riesz potentials (cf. [Ad]) grants the existence of a constant C_0 independent of r and such that

$$\begin{aligned} \|\nabla\Phi\|_{L^{2,\infty}(B_r(p))} &\le C_0 \int_{B_r(p)} |\Omega \cdot \nabla W| \\ &\le C_0 \|\Omega\|_{L^{2,1}(B_r(p))} \, \|\nabla W\|_{L^{2,\infty}(B_r(p))} \\ &\le C_0 \, \varepsilon_0 \, \|\nabla W\|_{L^{2,\infty}(B_r(p))} \end{aligned} \tag{III.40}$$

As for the function V, since it is harmonic, we can call upon lemma III.1 to deduce that for every $0 < \delta < 1$ there holds

$$\begin{aligned} \|\nabla V\|^2_{L^{2,\infty}(B_{\delta r}(p))} &\le \|\nabla V\|^2_{L^2(B_{\delta r}(p))} \\ &\le \left(\frac{4\delta}{3}\right)^2 \, \|\nabla V\|^2_{L^2(B_{3r/4}(p))} \\ &\le C_1 \, \left(\frac{4\delta}{3}\right)^2 \, \|\nabla V\|^2_{L^{2,\infty}(B_r(p))} \quad , \end{aligned} \tag{III.41}$$

where C_1 is a constant independent of r. Indeed, the $L^{2,\infty}$-norm of a harmonic function on the unit ball controls all its other norms on balls of radii inferior to $3/4$.

We next choose δ independent of r and so small as to have $C_1\left(\frac{4\delta}{3}\right)^2 < 1/16$. We also adjust ε_0 to satisfy $C_0\varepsilon_0 < 1/8$. Then, combining (III.40) and (III.41) yields the following inequality

$$\|\nabla W\|_{L^{2,\infty}(B_{\delta r}(p))} \le \frac{1}{2}\|\nabla W\|_{L^{2,\infty}(B_r(p))} \quad , \tag{III.42}$$

valid for all $r < \rho_0$ and all $p \in B_{1/2}(0)$.
Just as in the regularity proof for the CMC equation, the latter is iterated to eventually produce the estimate

$$\sup_{p\in B_{1/2}(0)\ ,\ 0<\rho<1/4} \rho^{-\alpha}\|\nabla W\|_{L^{2,\infty}(B_\rho(p))} < +\infty \quad . \tag{III.43}$$

Calling once again upon the duality $L^{2,1} - L^{2,\infty}$, and upon the upper bound on $\|\Omega\|_{L^{2,1}(D^2)}$ provided in (III.43), we infer that

$$\sup_{p\in B_{1/2}(0)\ ,\ 0<\rho<1/4} \rho^{-\alpha}\|\Omega\cdot\nabla W\|_{L^1(B_\rho(p))} < +\infty \quad , \tag{III.44}$$

thereby giving (III.39). This concludes the proof of the desired statement. □

IV Schrödinger systems with antisymmetric potentials, and the proof of Hildebrandt's conjecture.

The methods which we have used up to now to approach Hildebrandt's conjecture and obtain the regularity of $W^{1,2}$ solutions of the generic system

$$\Delta u + A(u)(\nabla u, \nabla u) = H(u)(\nabla^{\perp} u, \nabla u) \tag{IV.1}$$

rely on two main ideas:

i) recast, as much as possible, quadratic nonlinear terms as linear combinations of Jacobians or as *null forms* ;

ii) project equation (IV.1) on a *moving frame* $(e_1 \cdots e_n)$ satisfying the *Coulomb gauge condition*

$$\forall i, j = 1 \cdots m \qquad div((e_j, \nabla e_i)) = 0 \quad .$$

Both approaches can be combined to establish the Hölder continuity of $W^{1,2}$ solutions of (IV.1) when the target manifold N^n is C^2, and when the prescribed mean curvature H is Lipschitz continuous (see [Bet1], [Cho], and [He]). Seemingly, these are the weakest possible hypotheses required to carry out the above strategy.

However, to fully solve Hildebrandt's conjecture, one must replace the Lipschitzean condition on H by its being an element of L^∞. This makes quite a difference!

Despite its evident elegance and verified usefulness, Hélein's moving frames method suffers from a relative opacity:[17] what

[17] Yet another drawback of the moving frames method is that it lifts an N^n-valued harmonic map, with $n > 2$, to another harmonic map, valued in a parallelizable manifold $(S^1)^q$ of higher dimension. This procedure requires that N^n have a higher regularity than the "natural" one (namely, C^5 in place of C^2). It is only under this more stringent assumption that the regularity of N^n-valued harmonic maps was obtained in [Bet2] and [He]. The introduction of Schrödinger systems with antisymmetric potentials in [RiSt] enabled to improve these results.

makes nonlinearities of the form

$$A(u)(\nabla u, \nabla u) - H(u)(\nabla^{\perp} u, \nabla u) \quad ,$$

so special and more favorable to treating regularity/compactness matters than seemingly simpler nonlinearities, such as

$$|\nabla u|^2 \qquad ,$$

which we encountered in Section 1 ?

The moving frames method does not address this question.

We consider a weakly harmonic map u with finite energy, on D^2 and taking values in a regular oriented closed submanifold $N^n \subset \mathbb{R}^{n+1}$ of codimension 1. We saw at the end of Section 2 that u satisfies the equation

$$-\Delta u = \nu(u)\ \nabla(\nu(u)) \cdot \nabla u \quad , \tag{IV.2}$$

where ν is the normal unit-vector to N^n relative to the orientation of N^n.

In local coordinates, (IV.2) may be recast as

$$-\Delta u^i = \nu(u)^i \sum_{j=1}^{n+1} \nabla(\nu(u))_j \cdot \nabla u^j \qquad \forall\, i = 1 \cdots n+1 \quad . \tag{IV.3}$$

In this more general framework, we may attempt to adapt Hélein's operation which changes (III.24) into (III.25). The first step of this process is easily accomplished. Indeed, since ∇u is orthogonal to $\nu(u)$, there holds

$$\sum_{j=1}^{n+1} \nu_j(u)\, \nabla u^j = 0 \quad .$$

Substituting this identity into (IV.4) yields another equivalent formulation of the equation satisfies by N^n-valued harmonic maps, namely

$$-\Delta u^i = \sum_{j=1}^{n+1} \big(\nu(u)^i \ \nabla(\nu(u))_j - \nu(u)_j \ \nabla(\nu(u))^i\big) \cdot \nabla u^j \quad . \qquad \text{(IV.4)}$$

On the contrary, the second step of the process can not *a priori* be extended. Indeed, one cannot justify that the vector field

$$\nu(u)^i \ \nabla(\nu(u))_j - \nu(u)_j \ \nabla(\nu(u))^i$$

is divergence-free. This was true so long as N^n was the sphere S^n, but it fails so soon as the metric is ever so slightly perturbed. What remains however robust is the antisymmetry of the matrix

$$\Omega := \big(\nu(u)^i \ \nabla(\nu(u))_j - \nu(u)_j \ \nabla(\nu(u))^i\big)_{i,j=1\cdots n+1} \quad . \qquad \text{(IV.5)}$$

It turns out that the antisymmetry of Ω lies in the heart of the problem we have been tackling in these lecture notes. The following result sheds some light onto this claim.

Theorem IV.1 *[Riv1] Let Ω be a vector field in $L^2(\wedge^1 D^2 \otimes so(m))$, thus takings values in the space antisymmetric $m \times m$ matrices $so(m)$. Suppose that u is a map in $W^{1,2}(D^2, \mathbb{R}^m)$ satisfying the equation*[18]

$$-\Delta u = \Omega \cdot \nabla u \qquad in \quad \mathcal{D}'(D^2) \quad . \qquad \text{(IV.6)}$$

Then there exists some $p > 2$ such that $u \in W^{1,p}_{loc}(D^2, \mathbb{R}^m)$. In particular, u is Hölder continuous. □

Prior to delving into the proof of this theorem, let us first examine some of its implications towards answering the questions we aim to solve.

[18] In local coordinates, (IV.6) reads

$$-\Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j \qquad \forall \ i = 1 \cdots m \quad .$$

First of all, it is clear that theorem IV.1 is applicable to the equation (IV.4) so as to yield the regularity of harmonic maps taking values in a manifold of codimension 1.

Another rather direct application of theorem IV.1 deals with the solutions of the prescribed mean curvature equation in $\mathbb{R}^3$,

$$\Delta u = 2H(u)\;\partial_x u \times \partial_y u \qquad \text{dans} \quad \mathcal{D}'(D^2) \quad .$$

This equation can be recast in the form

$$\Delta u = H(u)\nabla^\perp u \times \nabla u \quad ,$$

Via introducing

$$\Omega := H(u) \begin{pmatrix} 0 & -\nabla^\perp u_3 & \nabla^\perp u_2 \\ \nabla^\perp u_3 & 0 & -\nabla^\perp u_1 \\ -\nabla^\perp u_2 & \nabla^\perp u_1 & 0 \end{pmatrix}$$

we observe successively that Ω is antisymmetric, that it belongs to L^2 whenever H belongs to L^∞, and that u satisfies (IV.6). The hypotheses of theorem IV.1 are thus all satisfied, and so we conclude that that u is Hölder continuous.

This last example outlines clearly the usefulness of theorem IV.1 towards solving Hildebrandt's conjecture. Namely, it enables us to weaken the Lipschitzean assumption on H found in previous works ([Hei1], [Hei2], [Gr2], [Bet1], ...), by only requiring that H be an element of L^∞. This is precisely the condition stated in Hildebrandt's conjecture. By all means, we are in good shape.

In fact, Hildebrandt's conjecture will be completely resolved with the help of the following result.

Theorem IV.2 *[Riv1] Let N^n be an arbitrary closed oriented C^2-submanifold of $\mathbb{R}^m$, with $1 \leq n < m$, and let ω be a C^1 two-form on N^n. Suppose that u is a critical point in $W^{1,2}(D^2, N^n)$ of the energy*

$$E^\omega(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2(x,y)\ dx\ dy + u^*\omega \quad .$$

Then u fulfills all of the hypotheses of theoreme IV.1, and therefore is Hölder continuous. □

Proof of theorem IV.2.

The critical points of E^ω satisfy the equation (II.27), which, in local coordinates, takes the form

$$\Delta u^i = -\sum_{j,k=1}^{m} H^i_{jk}(u)\ \nabla^\perp u^k \cdot \nabla u^j - \sum_{j,k=1}^{m} A^i_{jk}(u)\ \nabla u^k \cdot \nabla u^j \quad , \tag{IV.7}$$

for $i = 1 \cdots m$. Denoting by $(\varepsilon_i)_{i=1\cdots m}$ the canonical basis of $\mathbb{R}^m$, we first observe that since

$$H^i_{jk}(z) = d\omega_z(\varepsilon_i, \varepsilon_j\varepsilon_k)$$

the antisymmetry of the 3-forme $d\omega$ yields for every $z \in \mathbb{R}^m$ the identity $H^i_{jk}(z) = -H^j_{ik}(z)$. Then, (IV.7) becomes

$$\Delta u^i = -\sum_{j,k=1}^{m} (H^i_{jk}(u) - H^j_{ik}(u))\ \nabla^\perp u^k \cdot \nabla u^j - \sum_{j,k=1}^{m} A^i_{jk}(u)\ \nabla u^k \cdot \nabla u^j \quad . \tag{IV.8}$$

On the other hand, $A(u)(U,V)$ is orthogonal to the tangent plane for every choice of vectors U et V[19]. In particular, there

[19]Rigorously speaking, A is only defined for pairs of vectors which are tangent to the surface. Nevertheless, A can be extended to all pairs of vectors in $\mathbb{R}^m$ in a neighborhood of N^n by applying the pull-back of the projection on N^n. This extension procedure is analogous to that outlined on page 18.

holds

$$\sum_{j=1}^{m} A^j_{ik}\ \nabla u^j = 0 \qquad \forall\ i,k = 1\cdots m \quad . \tag{IV.9}$$

Inserting this identity into (IV.8) produces

$$\begin{aligned}\Delta u^i\ &= -\sum_{j,k=1}^{m} (H^i_{jk}(u) - H^j_{ik}(u))\ \nabla^\perp u^k \cdot \nabla u^j \\ &\quad -\sum_{j,k=1}^{m} (A^i_{jk}(u) - A^j_{ik}(u))\ \nabla u^k \cdot \nabla u^j \quad .\end{aligned} \tag{IV.10}$$

The $m \times m$ matrix $\Omega := (\Omega^i_j)_{i,j=1\cdots m}$ defined via

$$\Omega^i_j := \sum_{k=1}^{m}(H^i_{jk}(u) - H^j_{ik}(u))\ \nabla^\perp u^k + \sum_{k=1}^{m}(A^i_{jk}(u) - A^j_{ik}(u))\ \nabla u^k \quad ,$$

is evidently antisymmetric, and it belongs to L^2. With this notation, (IV.10) is recast in the form (IV.6), and thus all of the hypotheses of theorem IV.1 are fulfilled, thereby concluding the proof of theorem IV.2. □

On the conservation laws for Schrödinger systems with antisymmetric potentials.

Per the above discussion, there only remains to establish theorem IV.1 in order to reach our goal. To this end, we will express the Schrödinger systems with antisymmetric potentials in the form of *conservation laws*. More precisely, we have

Theorem IV.3 *[Riv1] Let Ω be a vector field in $L^2(D^2, so(m))$. Suppose that A and B are two $W^{1,2}$ functions on D^2 taking their values in the same of square $m \times m$ matrices which satisfy the equation*

$$\nabla A - A\Omega = -\nabla^\perp B \quad . \tag{IV.11}$$

If A is almost everywhere invertible, and if it has the bound

$$\|A\|_{L^\infty(D^2)} + \|A^{-1}\|_{L^\infty(D^2)} < +\infty \quad , \tag{IV.12}$$

then u is a solution of the Schrödinger system (IV.6) if and only if it satisfies the conservation law

$$div(A\nabla u - B\nabla^{\perp}u) = 0 \quad . \tag{IV.13}$$

If (IV.13) holds, then $u \in W^{1,p}_{loc}(D^2, \mathbb{R}^m)$ for some $p > 2$, and therefore u is Hölder continuous in the interior of D^2. □

We note that the conservation law (IV.13), when it exists, generalizes the conservation laws previously encountered in the study of problems with symmetry, namely:

1) In the case of the constant mean curvature equation, the conservation law (III.1) is (IV.13) with the choice

$$A_{ij} = \delta_{ij} \quad ,$$

and

$$B = \begin{pmatrix} 0 & -H\,u_3 & H\,u_2 \\ H\,u_3 & 0 & -H\,u_1 \\ -H\,u_2 & H\,u_1 & 0 \end{pmatrix}$$

2) In the case of S^n-valued harmonic maps, the conservation law (III.25) is (IV.13) for

$$A_{ij} = \delta_{ij} \quad ,$$

and $B = (B^i_j)$ with

$$\nabla^{\perp} B^i_j = u^i\,\nabla u_j - u_j\,\nabla u^i \quad .$$

The ultimate part of these notes will be devoted to constructing A and B, for any given antisymmetric Ω, with sufficiently small L^2-norms (cf. theorem IV.4 below). As a result, all coercive conformally invariant Lagrangians with quadratic growth will yield conservation laws written in divergence form. This is quite an amazing fact. Indeed, while in cases of the CMC and S^n-valued harmonic map equations the existence of conservation laws can be explained by Noether's theorem[20], one may wonder **which hidden symmetries yield the existence of the general divergence form (IV.13) ?** This profound question shall unfortunately not be addressed here.

Prior to constructing A and B in the general case, we first establish theorem IV.3.

Proof of theorem IV.3.

The first part of the theorem is the result of the elementary calculation,

$$\begin{aligned} div(A\,\nabla u - B\,\nabla^\perp u) &= A\,\Delta u + \nabla A\cdot\nabla u - \nabla B\cdot\nabla^\perp u \\ &= A\ \Delta u + (\nabla A + \nabla^\perp B)\cdot\nabla u \\ &= A(\Delta u + \Omega\cdot\nabla u) = 0 \end{aligned}$$

Regularity matters are settled as follows. Just as in the previously encountered problems, we seek to employ a Morrey-type argument via the existence of some constant $\alpha > 0$ such that

$$\sup_{p\in B_{1/2}(0)\,,\,0<\rho<1/4} \rho^{-\alpha}\int_{B_\rho(p)} |\Delta u| < +\infty \quad . \tag{IV.14}$$

The statement of the theorem is then deduced through calling upon the inequalities in [Ad], exactly in the same manner as we previously outlined.

Let $\varepsilon_0 > 0$ be some constant whose value will be adjusted in due time to fit our needs. There exists a radius ρ_0 such that for

[20] roughly speaking, symmetries give rise to conservation laws. In both the CMC and S^n-harmonic map equations, the said symmetries are tantamount to the corresponding Lagrangians being invariant under the group of isometries of the target space $\mathbb{R}^m$.

every $r < \rho_0$ and every point p dans $B_{1/2}(0)$, there holds

$$\int_{B_r(p)} |\nabla A|^2 + |\nabla B|^2 < \varepsilon_0 \quad . \tag{IV.15}$$

Henceforth, we consider only radii $r < \rho_0$.

Note that $A\nabla u$ satisfies the elliptic system

$$\left\{ \begin{array}{ll} div(A\nabla u) & = \nabla B \cdot \nabla^\perp u = \partial_y B \, \partial_x u - \partial_x B \, \partial_y u \\ & \\ rot(A\nabla u) & = -\nabla A \cdot \nabla^\perp u = \partial_x A \, \partial_y u - \partial_y A \, \partial_x u \end{array} \right.$$

We proceed by introducing on $B_r(p)$ the linear Hodge decomposition in L^2 of $A\nabla u$. Namely, there exist two functions C and D, unique up to additive constants, elements of $W^{1,2}_0(B_r(p))$ and $W^{1,2}(B_r(p))$ respectively, and such that

$$A\nabla u = \nabla C + \nabla^\perp D \quad . \tag{IV.16}$$

To see why such C and D do indeed exist, consider first the equation

$$\left\{ \begin{array}{l} \Delta C = div(A\nabla u) = \partial_y B \, \partial_x u - \partial_x B \, \partial_y u \\ \\ \quad C = 0 \quad . \end{array} \right. \tag{IV.17}$$

Wente's theorem (III.1) guarantees that C lies in $W^{1,2}$, and moreover

$$\int_{D^2} |\nabla C|^2 \leq C_0 \int_{D^2} |\nabla B|^2 \int_{D^2} |\nabla u|^2 \quad . \tag{IV.18}$$

By construction, $div(A\nabla u - \nabla C) = 0$. Poincaré's lemma thus yields the existence of D in $W^{1,2}$ with $\nabla^\perp D := A\nabla u - \nabla C$, and

$$\begin{aligned} \int_{D^2} |\nabla D|^2 &\leq 2 \int_{D^2} |A\nabla u|^2 + |\nabla C|^2 \\ &\leq 2\|A\|_\infty \int_{D^2} |\nabla u|^2 + 2C_0 \int_{D^2} |\nabla B|^2 \int_{D^2} |\nabla u|^2 \quad . \end{aligned} \tag{IV.19}$$

The function D satisfies the identity

$$\Delta D = -\nabla A \cdot \nabla^\perp u = \partial_x A\, \partial_y u - \partial_y A\, \partial_x u \quad .$$

Just as we did in the case of the CMC equation, we introduce the decomposition $D = \phi + v$, with ϕ fulfilling

$$\left\{ \begin{array}{l} \Delta\phi = \partial_x A\, \partial_y u - \partial_y A\, \partial_x u \qquad \text{in} \quad B_r(p) \\[2ex] \quad \phi = 0 \qquad\qquad \text{on} \quad \partial B_r(p) \quad , \end{array} \right. \tag{IV.20}$$

and with v being harmonic. Once again, Wente's theorem III.1 gives us the estimate

$$\int_{B_r(p)} |\nabla\phi|^2 \leq C_0 \int_{B_r(p)} |\nabla A|^2 \int_{B_r(p)} |\nabla u|^2 \quad . \tag{IV.21}$$

The arguments which we used in the course of the regularity proof for the CMC equation may be recycled here so as to obtain the analogous version of (III.16), only this time on the ball $B_{\delta r}(p)$, where $0 < \delta < 1$ will be adjusted in due time. More precisely, we find

$$\begin{aligned} \int_{B_{\delta r}(p)} |\nabla D|^2 \ &\leq 2\delta^2 \int_{B_r(p)} |\nabla D|^2 \\ &+3 \int_{B_r(p)} |\nabla\phi|^2 \quad . \end{aligned} \tag{IV.22}$$

Bringing altogether (IV.15), (IV.18), (IV.19), (IV.21) et (IV.22) produces

$$\begin{aligned} \int_{B_{\delta r}(p)} |A\, \nabla u|^2 \ &\leq 3\delta^2 \int_{B_r(p)} |A\, \nabla u|^2 \\ &+C_1\, \varepsilon_0 \int_{B_r(p)} |\nabla u|^2 \end{aligned} \tag{IV.23}$$

Using the hypotheses that A and A^{-1} are bounded in L^∞, it follows from (IV.23) that for all $1 > \delta > 0$, there holds the estimate

$$
\begin{aligned}
\int_{B_{\delta\, r}(p)} |\nabla u|^2 \ &\leq 3\|A^{-1}\|_\infty \, \|A\|_\infty \delta^2 \int_{B_r(p)} |\nabla u|^2 \\
&\quad + C_1 \, \|A^{-1}\|_\infty \varepsilon_0 \int_{B_r(p)} |\nabla u|^2 \quad .
\end{aligned}
\tag{IV.24}
$$

Next, we choose ε_0 and δ strictly positive, independent of r et p, and such that

$$
3\|A^{-1}\|_\infty \, \|A\|_\infty \delta^2 + C_1 \, \|A^{-1}\|_\infty \varepsilon_0 = \frac{1}{2} \quad .
$$

For this particular choice of δ, we have thus obtained the inequality

$$
\int_{B_{\delta\, r}(p)} |\nabla u|^2 \leq \frac{1}{2} \int_{B_r(p)} |\nabla u|^2 \quad .
$$

Iterating this inequality as in the previous regularity proofs yields the existence of some constant $\alpha > 0$ for which

$$
\sup_{p \in B_{1/2}(0)\ ,\ 0<\rho<1/4} \rho^{-2\alpha} \int_{B_\rho(p)} |\nabla u|^2 < +\infty \quad .
$$

Since $|\Delta u| \leq |\Omega| \, |\nabla u|$, the latter gives us (IV.14), thereby concluding the proof of theorem IV.3. □

There only now remains to establish the existence of the functions A and B in $W^{1,2}$ satisfying the equation (IV.11) and the hypothesis (IV.12).

The construction of conservation laws for systems with antisymmetric potentials, and the proof of theorem IV.1.

The following result, combined to theorem IV.3, implies theorem IV.1, itself yielding theorem IV.2, and thereby providing a proof of Hildebrandt's conjecture, as we previously explained.

Theorem IV.4 *[Riv1] There exists a constant $\varepsilon_0(m) > 0$ depending only on the integer m, such that for every vector field $\Omega \in L^2(D^2, so(m))$ with*

$$\int_{D^2} |\Omega|^2 < \varepsilon_0(m) \quad , \tag{IV.25}$$

it is possible to construct $A \in L^\infty(D^2, Gl_m(\mathbb{R})) \cap W^{1,2}$ and $B \in W^{1,2}(D^2, M_m(\mathbb{R}))$ with the properties

i)

$$\int_{D^2} |\nabla A|^2 + \|dist(A, SO(m))\|_{L^\infty(D^2)} \le C(m) \int_{D^2} |\Omega|^2 \quad , \tag{IV.26}$$

ii)

$$\int_{D^2} |\nabla B|^2 \le C(m) \int_{D^2} |\Omega|^2 \quad , \tag{IV.27}$$

iii)

$$\nabla_\Omega A := \nabla A - A\Omega = -\nabla^\perp B \tag{IV.28}$$

where $C(m)$ is a constant depending only on the dimension m. □

Prior to delving into the proof of theorem IV.4, a few comments and observations are in order.

Glancing at the statement of the theorem, one question naturally arises: **why is the antisymmetry of Ω so important ?**

It can be understood as follow.

In the simpler case when Ω is **divergence-free**, we can write Ω in the form

$$\Omega = \nabla^\perp \xi \quad ,$$

for some $\xi \in W^{1,2}(D^2, so(m))$. In particular, the statement of theorem IV.4 is settled by choosing

$$A_{ij} \;=\; \delta_{ij} \qquad and \qquad B_{ij} \;=\; \xi_{ij} \quad . \tag{IV.29}$$

Accordingly, it seems reasonable in the general case to seek a solution pair (A, B) which comes as "close" as can be to (IV.29). A first approach consists in performing a **linear Hodge decomposition** in L^2 of Ω. Hence, for some ξ and P in $W^{1,2}$, we write

$$\Omega = \nabla^\perp \xi - \nabla P \quad . \tag{IV.30}$$

In this case, we see that if A exists, then it must satisfy the equation

$$\Delta A = \nabla A \cdot \nabla^\perp \xi - div(A\nabla P) \quad . \tag{IV.31}$$

This equation is critical in $W^{1,2}$. The first summand $\nabla A \cdot \nabla^\perp \xi$ on the right-hand side of (IV.31) is a Jacobian. This is a desirable feature with many very good analytical properties, as we have previously seen. In particular, using integration by compensation (Wente's theorem III.1), we can devise a bootstrapping argument beginning in $W^{1,2}$. On the other hand, the second summand $div(A\nabla P)$ on the right-hand side of (IV.31) displays no particular structure. All which we know about it, is that it belongs to $W^{1,2}$. But this space is not embedded in L^∞, and so we cannot a priori conclude that $A\nabla P$ lies in L^2, thereby obstructing a successful analysis...

However, not all hope is lost for the **antisymmetric structure** of Ω still remains to be used. The idea is to perform a **nonlinear** Hodge decomposition[21] in L^2 of Ω. Thus, let $\xi \in W^{1,2}(D^2, so(m))$ and P be a $W^{1,2}$ map taking values in the group $SO(m)$ of proper rotations of $\mathbb{R}^m$, such that

$$\Omega = P\,\nabla^\perp \xi\, P^{-1} - \nabla P\, P^{-1} \quad . \tag{IV.32}$$

[21] which is tantamount to a **change of gauge**.

At first glance, the advantage of (IV.32) over (IV.31) is not obvious. If anything, it seems as though we have complicated the problem by having to introduce left and right multiplications by P and P^{-1}. On second thought, however, since rotations are always bounded, the map P in (IV.32) is an element of $W^{1,2} \cap L^\infty$, whereas in (IV.31), the map P belonged only to $W^{1,2}$. This slight improvement will actually be sufficient to successfully carry out our proof. Furthermore, (IV.32) has yet another advantage over (IV.31). Indeed, whenever A and B are solutions of (IV.28), there holds

$$\begin{aligned}
\nabla_{\nabla^\perp\xi}(AP) &= \nabla(AP) - (AP)\,\nabla^\perp\xi \\
&= \nabla A\,P + A\,\nabla P - A\,P\,(P^{-1}\Omega P + P^{-1}\nabla P) \\
&= (\nabla_\Omega A)P = -\nabla^\perp B\ P \quad .
\end{aligned}$$

Hence, via setting $\tilde{A} := AP$, $\tilde{A}$, we find

$$\Delta\tilde{A} = \nabla\tilde{A}\cdot\nabla^\perp\xi + \nabla^\perp B\cdot\nabla P \quad . \tag{IV.33}$$

Unlike (IV.31), the second summand on the right-hand side of (IV.33) is a linear combination of Jacobians of terms which lie in $W^{1,2}$. Accordingly, calling upon theorem III.1, we can control $\tilde{A}$ in $L^\infty \cap W^{1,2}$. This will make a bootstrapping argument possible.

One point still remains to be verified. Namely, that the nonlinear Hodge decomposition (IV.32) does exist. This can be accomplished with the help of a result of Karen Uhlenbeck[22].

Theorem IV.5 *[Uhl], [Riv1] Let $m \in \mathbb{N}$. There are two constants $\varepsilon(m) > 0$ and $C(m) > 0$, depending only on m, such that for each vector field $\Omega \in L^2(D^2, so(m))$ with*

$$\int_{D^2} |\Omega|^2 < \varepsilon(m) \quad ,$$

[22] In reality, this result, as it is stated here, does not appear in the original work of Uhlenbeck. In [Riv1], it is shown how to deduce theorem IV.5 from Uhlenbeck's approach.

there exist $\xi \in W^{1,2}(D^2, so(m))$ *and* $P \in W^{1,2}(D^2, SO(m))$ *satisfying*

$$\Omega = P\,\nabla^\perp\xi\,P^{-1} - \nabla P\,P^{-1} \quad , \tag{IV.34}$$

$$\xi = 0 \qquad \text{on} \quad \partial D^2 \quad , \tag{IV.35}$$

and

$$\int_{D^2} |\nabla\xi|^2 + \int_{D^2} |\nabla P|^2 \le C(m) \int_{D^2} |\Omega|^2 \quad . \tag{IV.36}$$

□

Proof of theorem IV.4.

Let P and ξ be as in theorem IV.5. To each $A \in L^\infty \cap W^{1,2}(D^2, M_m(\mathbb{R}))$ we associate $\tilde{A} = AP$. Suppose that A and B are solutions of (IV.28). Then $\tilde{A}$ and B satisfy the elliptic system

$$\left\{\begin{array}{l} \Delta\tilde{A} = \nabla\tilde{A}\cdot\nabla^\perp\xi + \nabla^\perp B\cdot\nabla P \\[2mm] \Delta B = -div(\tilde{A}\,\nabla\xi\ P^{-1}) + \nabla^\perp\tilde{A}\cdot\nabla P^{-1} \quad . \end{array}\right. \tag{IV.37}$$

We first consider the invertible elliptic system

$$\left\{\begin{array}{l} \Delta\tilde{A} = \nabla\hat{A}\cdot\nabla^\perp\xi + \nabla^\perp\hat{B}\cdot\nabla P \\[2mm] \Delta B = -div(\hat{A}\,\nabla\xi\ P^{-1}) + \nabla^\perp\hat{A}\cdot\nabla P^{-1} \\[2mm] \dfrac{\partial\tilde{A}}{\partial\nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial D^2 \\[2mm] \displaystyle\int_{D^2} \tilde{A} = \pi^2\ Id_m \end{array}\right. \tag{IV.38}$$

where $\hat{A}$ and $\hat{B}$ are arbitrary functions in $L^\infty \cap W^{1,2}$ and in $W^{1,2}$ respectively. An analogous version[23] of theorem III.1 with Neuman boundary conditions in place of Dirichlet conditions,

[23] whose proof is left as an exercise.

we deduce that the unique solution $(\tilde{A}, B)$ of (IV.38) satisfies the estimates

$$\begin{aligned}\int_{D^2}|\nabla\tilde{A}|^2+\|\tilde{A}-Id_m\|^2_\infty\ &\le C\ \int_{D^2}|\nabla\hat{A}|^2\ \int_{D^2}|\nabla\xi|^2\\ &+C\ \int_{D^2}|\nabla\hat{B}|^2\ \int_{D^2}|\nabla P|^2\quad ,\end{aligned} \tag{IV.39}$$

and

$$\begin{aligned}\int_{D^2}|\nabla(\tilde{B}-B_0)|^2\ &\le C\ \|\hat{A}-Id_m\|^2_\infty\ \int_{D^2}|\nabla\xi|^2\\ &+C\ \int_{D^2}|\nabla\hat{A}|^2\ \int_{D^2}|\nabla P|^2\quad ,\end{aligned} \tag{IV.40}$$

where B_0 is the solution in $W^{1,2}$ of

$$\left\{\begin{array}{l}\Delta B_0=-div(\nabla\xi\ P^{-1})\qquad \text{in}\quad D^2\\[2mm] \quad B_0=0\qquad\quad \text{on}\quad \partial D^2\end{array}\right. \tag{IV.41}$$

Hence, if

$$\int_{D^2}|\nabla P|^2+|\nabla\xi|^2$$

is sufficiently small (this can always be arranged owing to (IV.36) and the hypothesis (IV.25)), then a standard fixed point argument in the space $(L^\infty\cap W^{1,2}(D^2,M_m(\mathbb{R})))\times W^{1,2}(D^2,M_m(\mathbb{R}))$ yields the existence of the solution $(\tilde{A},B)$ of the system

$$\left\{\begin{array}{l}\Delta\tilde{A}=\nabla\tilde{A}\cdot\nabla^\perp\xi+\nabla^\perp B\cdot\nabla P\\[2mm] \Delta B=-div(\tilde{A}\,\nabla\xi\ P^{-1})+\nabla^\perp\tilde{A}\cdot\nabla P^{-1}\\[2mm] \dfrac{\partial\tilde{A}}{\partial\nu}=0\quad \text{and}\quad B=0\quad\text{on}\quad\partial D^2\\[2mm] \displaystyle\int_{D^2}\tilde{A}=\pi^2\ Id_m\end{array}\right. \tag{IV.42}$$

By construction, this solution satisfies the estimates (IV.26) and (IV.27), with $A = \tilde{A}\, P^{-1}$.

The proof of theorem IV.4 will then be finished once it is established that (A, B) is a solution of (IV.28).

To do so, we introduce the following linear Hodge decomposition in L^2 :

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi + \nabla^\perp B\ P = \nabla C + \nabla^\perp D$$

where $C = 0$ on ∂D^2. The first equation in (IV.42) states that $\Delta C = 0$, so that $C \equiv 0$ sur D^2. The second equation in (IV.42) along with the boundary conditions imply that D satisfies

$$\left\{\begin{array}{l} div(\nabla D\ P^{-1}) = 0 \qquad \text{in} \quad \partial D^2 \\[3mm] \quad D = 0 \qquad \text{on} \quad \partial D^2 \quad . \end{array}\right. \tag{IV.43}$$

Thus, there exists $E \in W^{1,2}(D^2, M_n(\mathbb{R}))$ such that

$$\left\{\begin{array}{l} -\Delta E = \nabla^\perp D\cdot\nabla P^{-1} \qquad \text{in} \quad D^2 \\[3mm] \quad \dfrac{\partial E}{\partial\nu} = 0 \qquad \text{on} \quad \partial D^2 \end{array}\right. \tag{IV.44}$$

The analogous version of theorem III.1 with Neuman boundary conditions yields the estimate

$$\int_{D^2}|\nabla E| \le C_0\int_{D^2}|\nabla D|^2\int_{D^2}|\nabla P^{-1}|^2 \quad . \tag{IV.45}$$

Moreover, because $\nabla D = \nabla^\perp E\ P$, there holds $|\nabla D| \le |\nabla E|$. Put into (IV.45), this shows that if $\int_{D^2}|\nabla P|^2$ is chosen sufficiently small (i.e. for $\varepsilon_0(m)$ in (IV.25) small enough), then $D \equiv 0$. Whence, we find

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi + \nabla^\perp B\ P = 0 \qquad \text{in} \quad D^2 \quad ,$$

thereby ending the proof of theorem IV.4. □

V Conclusion

The regularity results we presented have been extended to partial regularity results in higher dimension in a work in collaboration with M. Struwe (see [RiSt]).

There is presently no really satisfying answer why conformal invariance in 2 dimension generates antisymmetric potentials. This matter of fact is also not specific to dimension 2. In dimension 4 for instance we showed together with T.Lamm that intrinsic and extrinsic bi-harmonic map equations, which are also dilation invariant, can be written in the form

$$\Delta^2 u = \Delta\,(V \cdot \nabla u) + div\,(w\ \nabla u) + W \cdot \nabla u \quad ,$$

where the *a-priori* less regular part Ω in W is again antisymmetric. This antisymmetry can also be exploited in order to show the Hölder-continuity on u under assumptions on V, w and W which makes the system critical see [LaRi].

More recently, in collaboration with F. Da Lio we observed that the antisymmetry of the potential plays also a decisive role, regarding compactness and regularity issues, for *Non-local Schrödinger* type equation of the form

$$\Delta^{1/4} v = \Omega\, v \quad ,$$

where for instance $\Omega \in L^2(\mathbb{R}, so(n))$ and $v \in L^2(\mathbb{R}, \mathbb{R}^n)$. Under these assumptions the equation is critical and the antisymmetry of Ω implies a gain of integrability : $v \in L^q_{loc}$ for some $q > 2$ ([DR2]). This later fact can be applied in order to show the regularity of $1/2-$harmonic maps into manifolds (see [DR1], [DR2]).

It would have been interesting to study further the possible special interaction between antisymmetric potentials and other kind of operators - non necessarily elliptic - such as $\Box = \partial_t^2 - \Delta$...

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