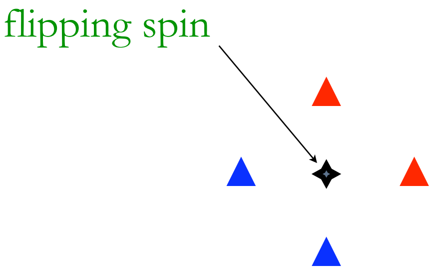
Nonequilibrium stationary states and phase transitions in directed Ising model

Directed Ising models with non conserved dynamics

Example: 2D Ising on the square lattice



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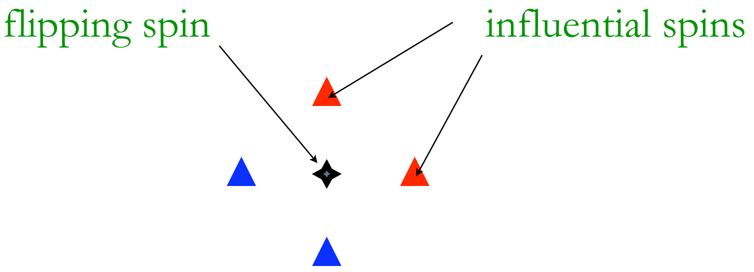

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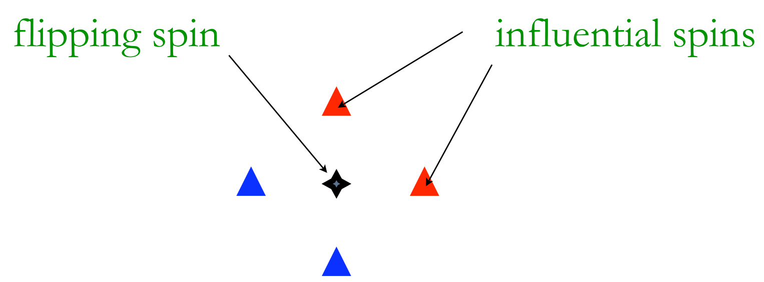

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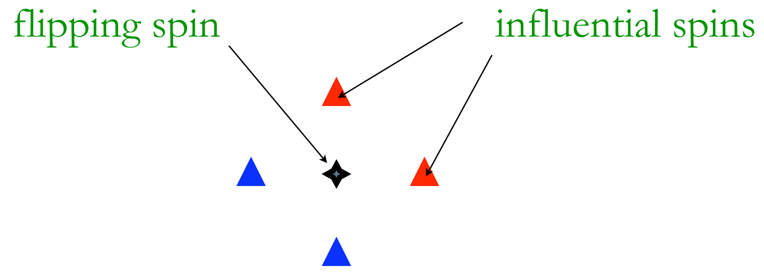

Directedness implies violation of detailed balance

Nature of the stationary state?

Existence of phase transitions?

CG, AJ Bray, JSTAT 2009

- Künsch 1984: 2D square lattice

  if flipping rate $w(\sigma_i \to -\sigma_i) = e^{-\beta\sigma_i 2h_i^+}$ $\quad$ $h_i^+ = J(\sigma_N + \sigma_E)$

  then stationary state is Boltzmann-Gibbs

- Lima & Stauffer 2006: 2D-5D lattices

  if flipping rate $w(\sigma_i \to -\sigma_i) = \frac{1}{2}(1 - \sigma_i \tanh \beta h_i^+)$

  h_i^+ local field due to influential spins

  then apparently no phase transition

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Unicity?

Other dimensions?

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Proof?

(1) For directed Ising models, which rates are compatible with Gibbs stat. measure?

- 1D, 2D square, 2D triangular

$$w(\sigma_i \to -\sigma_i) = e^{-\beta\sigma_i 2h_i^+} \quad (h_i^+ \text{ local field due to influential spins})$$

is unique form of the rate such that stat. measure is Gibbs

- 3D cubic

Gibbs stat. measure imposes detailed balance for the rates

- No lattice with coordination number equal or greater 8 such that stat. measure is Gibbs with rates of the form

$$w(\sigma_i \to -\sigma_i) = e^{-\beta\sigma_i 2h_i^+}$$

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Directed models with Gibbs stat. measure only exist in 1D and 2D

(2) Existence of phase transitions for directed models with «Glauber» rate?

$$w(\sigma_i \to -\sigma_i) = \frac{1}{2}(1 - \sigma_i \tanh \beta h_i^+)$$

- 1D, 2D square: no phase transitions
- 2D triangular, 3D cubic, etc.: open problems
- Cayley trees (each spin is influenced by spins further to the root)

  phase transition provided the branching ratio is equal or larger 3

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For directed Ising models, which rates are compatible with Gibbs measure?

Symmetric single-spin flip dynamics for the chain

Form of the rates if detailed balance? $E(\mathcal{C}) = -J\sum_i \sigma_i \sigma_{i+1}$

With spin symmetry, unknown rates:

$$w(+++), w(-+-), w(++-), w(-++)$$

Detailed balance relations:

$$\begin{aligned} w(+++) &= e^{-4\beta J} w(-+-) \\ w(++-) &= w(-++) \end{aligned}$$

Hence two remaining unknown rates

Glauber solution:

$$w(\sigma_{i-1}\sigma_i\sigma_{i+1}) = \frac{\alpha}{2}(1 + \delta\sigma_{i-1}\sigma_{i+1} - \frac{1}{2}\gamma(1+\delta)\sigma_i(\sigma_{i-1} + \sigma_{i+1}))$$

Directed dynamics for the chain (pbc)

Form of the rates under the constraint that stat. measure is Gibbs?

With spin symmetry, unknown rates:

$$w(+++), w(-+-), w(++-), w(-++)$$

Method:

- Consider a fixed configuration
- decompose stat. master equations on «motifs» $+++, -+-, ++-, -++,$ and 4 flipped ones
- express number of motifs on independent correlators
- symmetry relationship on correlators (structure of the lattice)
- constraint equations on the rates are coefficients of remaining correlators
- impose symmetry (symmetric dynamics, directed dynamics)
- obtain final constraint equations on the rates

Directed dynamics for the chain (pbc)

Form of the rates under the constraint that stat. measure is Gibbs?

With spin symmetry, unknown rates:

$$w(+++), w(-+-), w(++-), w(-++)$$

Decompose stat. master equation on the «motifs»:

$$\underbrace{N(+++)}_{\text{number of motifs }+++}(w(+++) - e^{-4\beta J} w(-+-)) + 7 \text{ other terms}$$

The number of «motifs» N(...) are expanded on a basis of independent correlators

$$c_1 = \frac{1}{N}\sum_1^N \sigma_i \qquad c_{12} = \frac{1}{N}\sum_1^N \sigma_i\sigma_{i+1}, \ldots$$

which themselves obey symmetry relations (transl. invariance)

$$c_1 = c_2 = c_3, \qquad c_{12} = c_{23}$$

Obtain one constraint equation between two of the four unknown rates

$$w(+++) = e^{-4\beta J} w(-+-)$$

(1) Symmetric dynamics (left and right spin have equal influence on flipping spin):

$$w(++-) = w(-++)$$

Recover Glauber rates: one free parameter (up to a timescale)

$$w(\sigma_{i-1}\sigma_i\sigma_{i+1}) = \frac{\alpha}{2}(1 + \delta\sigma_{i-1}\sigma_{i+1} - \frac{1}{2}\gamma(1+\delta)\sigma_i(\sigma_{i-1} + \sigma_{i+1}))$$

(2) Directed dynamics (only right spin has influence on flipping spin):

$$w(+++) = w(-++), \qquad w(++-) = w(-+-)$$

find a unique solution (up to a timescale)

$$w(\sigma_{i-1}\sigma_i\sigma_{i+1}) = e^{-\beta\sigma_i 2h_i^+}, \qquad h_i^+ = J\sigma_{i+1}$$

Directed Ising models on regular lattices

2D square, 2D triangular, 3D cubic

$$E(\mathcal{C}) = -J \sum_{(i,j)} \sigma_i \sigma_j$$

- 2D square

  $2^5 = 32$ rates, 16 if spin symmetry

  6 constraint equations

  - symmetric dynamics: usual forms recovered
  - directed dynamics: unique solution

$$w(\sigma_i \to -\sigma_i) = e^{-\beta \sigma_i 2 h_i^+} \qquad h_i^+ = J(\sigma_N + \sigma_E)$$

- 2D triangular

  $2^7 = 128$ rates, 64 if spin symmetry

  - directed dynamics: unique solution

$$h_i^+ = J(\sigma_1 + \sigma_2 + \sigma_3)$$

- 3D cubic

  $2^7 = 128$ rates, 64 if spin symmetry

  - directed dynamics: no solution

Remark

On 2D square lattice, if influence on flipping spins comes from West, North, East then unique form of the rates, but not exponential

Explanation: a complementary viewpoint

Up to now:

- impose Gibbs stationary measure,
- impose the lattice,
- rates are unknown

Now

- impose Gibbs stationary measure,
- rates have exponential form with truncated local field,
- the unknown is the lattice

Warming up: check of previous results

- for a given lattice (1D, 2D, 3D),
- write stat. master equation with Gibbs measure,
- replace rates by exponential form
- check success (1D, 2D) or failure (3D)

Stat. master equation

$$\sum_i w(\sigma_i \to -\sigma_i) = \sum_i w(-\sigma_i \to \sigma_i) e^{-\beta \sigma_i 2 h_i}$$

Choice of rate

$$w(\sigma_i \to -\sigma_i) = e^{-\beta \sigma_i 2 h_i^+} \qquad h_i = h_i^+ + h_i^-$$

Equation to be solved

$$\sum_i e^{-\beta \sigma_i 2 h_i^+} = \sum_i e^{-\beta \sigma_i 2 h_i^-}$$

$$\sum_i \prod_{j \in v^+(i)} (1 + \tau \sigma_i \sigma_j) = \sum_i \prod_{j \in v^-(i)} (1 + \tau \sigma_i \sigma_j)$$

$$\tau = -\tanh 2\beta J$$

2D triangular

- at order τ

$$\sum_i \sigma_i\sigma_{i+e_1} + \sigma_i\sigma_{i+e_2} + \sigma_i\sigma_{i+e_3} = \sum_i \sigma_i\sigma_{i+\bar{e}_1} + \sigma_i\sigma_{i+\bar{e}_2} + \sigma_i\sigma_{i+\bar{e}_3}$$

satisfied

- at order τ^2

$$\sum_i \sigma_{i+e_1}\sigma_{i+e_2} + \sigma_{i+e_1}\sigma_{i+e_3} + \sigma_{i+e_2}\sigma_{i+e_3} =$$
$$\sum_i \sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_2} + \sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_3} + \sigma_{i+\bar{e}_2}\sigma_{i+\bar{e}_3}$$

satisfied

- at order τ^3

$$\sum_i \sigma_i\sigma_{i+e_1}\sigma_{i+e_2}\sigma_{i+e_3} = \sum_i \sigma_i\sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_2}\sigma_{i+\bar{e}_3}$$

satisfied because

$e_2 = e_1 + e_3$

3D cubic

- at order τ

$$\sum_i \sigma_i\sigma_{i+e_1} + \sigma_i\sigma_{i+e_2} + \sigma_i\sigma_{i+e_3} = \sum_i \sigma_i\sigma_{i+\bar{e}_1} + \sigma_i\sigma_{i+\bar{e}_2} + \sigma_i\sigma_{i+\bar{e}_3}$$

satisfied

- at order τ^2

$$\sum_i \sigma_{i+e_1}\sigma_{i+e_2} + \sigma_{i+e_1}\sigma_{i+e_3} + \sigma_{i+e_2}\sigma_{i+e_3} =$$
$$\sum_i \sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_2} + \sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_3} + \sigma_{i+\bar{e}_2}\sigma_{i+\bar{e}_3}$$

satisfied

- at order τ^3

$$\sum_i \sigma_i\sigma_{i+e_1}\sigma_{i+e_2}\sigma_{i+e_3} = \sum_i \sigma_i\sigma_{i+\bar{e}_1}\sigma_{i+\bar{e}_2}\sigma_{i+\bar{e}_3}$$

not satisfied because

$e_2 \neq e_1 + e_3$

Case of lattices with higher coordination number

If coordination number greater or equal 8, no solution (no directed model with Gibbs stat. state)

Directed models with Gibbs stat. measure only exist in 1D and 2D

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Do directed Ising models augmented by «Glauber dynamics» exhibit a phase transition to a ferromagnetic state?

Directed Ising chain

$$w(\sigma_i \to -\sigma_i) = \tfrac{1}{2}(1 - \sigma_i \tanh \beta h_i^+)$$

$$h_i^+ = J\sigma_{i+1}$$

linear equation for magnetisation

$$\dot{m}_i = -m_i + \underbrace{\langle \tanh \beta J \sigma_{i+1} \rangle}_{\tanh \beta J \, m_{i+1}}$$

p.b.c. + translation invariance: $m_i = m \Rightarrow m(t) = m(0)e^{-\mathrm{const}.t}$

no spontaneous magnetisation

Directed 2D square lattice (North-East)

linear equation, no spontaneous magnetisation

Directed Cayley trees

Influence directed from the tips towards the root: each spin only sees the spin further from the root

initial condition: all spins at a given level are independent

- Branching ratio q=2: linear equation for magnetisation between generation n: parent, generation n-1: children (tips counted as level 1)

$$\dot{m}_n = -m_n + \underbrace{\langle \tanh \beta J (S_1^{(n-1)} + S_2^{(n-1)}) \rangle}_{m_{n-1} \tanh \beta J}$$

no spontaneous magnetisation (while undirected Cayley tree with q=2 has a phase transition)

Directed Cayley trees

- Branching ratio q=3: non linear equation for magnetisation

$$\dot{m}_n = -m_n + \underbrace{\langle \tanh \beta J [S_1^{(n-1)} + S_2^{(n-1)} + S_3^{(n-1)}] \rangle}_{A\, m_{n-1} - B\, \underbrace{\langle S_1^{(n-1)}\, S_2^{(n-1)}\, S_3^{(n-1)} \rangle}_{m_{n-1}^3}}$$

spins at the same level are uncorrelated

$$m_n = A\, m_{n-1} - B\, m_{n-1}^3$$

Existence of 2 fixed points

$$m^* = 0, \qquad m^* \neq 0$$

Phase transition at

$$\beta_c J = 0.475\ldots$$

Conclusion

- Boltzmann-Gibbs versus detailed balance

- Directed Ising models carry no mass current: different from KLS 1D, ASEP, ZRP, (with similar property of stat. measure)

- Open problems (phase transition for 2D triangular, 3D cubic? Dynamics?)