Local scale-invariance and ageing phenomena: where do we stand?

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M. Pleimling, S. Stoimenov

Workshop NEQ,
University of Warwick, the 11th of January, 2010
I. Why local dynamical scaling?
   physical ageing; scaling behaviour and exponents; tests of dynamical scaling; theoretical formulation

II. Local scale-invariance for $z \neq 2$
   axioms of LSI; classifications; mass terms; relation to integrability; computation of responses and correlators

III. How to test the foundations of LSI
   kinds of tests; Ising model; in which models responses and correlators were compared with LSI-predictions?

IV. Conclusions

MH & Pleimling, Non-equilibrium phase transitions 2 (2010)
I. Why local dynamical scaling?

- non-equilibrium systems naturally display **dynamical scaling**
- a common example: **ageing** phenomena
  1. slow relaxation (non-exponential)
  2. breaking of time-translation-invariance
  3. dynamical scaling

- which (reversible) microscopic processes lead to such macroscopic effects?

- **physical ageing** known since (pre-)historical times, but systematic studies first in glassy systems
  - *a priori*, behaviour prehistory-dependent
  - *but* evidence for **reproducible** and **universal** behaviour

- for better conceptual understanding: study ageing in simpler systems without disorder (i.e. ferromagnets)

**Question**: what is the current evidence for larger, **local scaling symmetries**?
for symmetry analysis: simple ageing systems without disorder
consider a simple magnet (ferromagnet, i.e. Ising model)

1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c$ (or $T = T_c$)
   $\rightarrow$ non-equilibrium state
3. fix $T$ and observe dynamics

competition:
- at least 2 equivalent ground states
- local fields lead to rapid local ordering
- no global order, relaxation time $\infty$

formation of ordered domains, of linear size $L = L(t) \sim t^{1/z}$

dynamical exponent $z$
Scaling behaviour & exponents

**single** relevant time-dependent length scale \( L(t) \sim t^{1/z} \)

Bray 94, Janssen et al. 92, Cugliandolo & Kurchan 90s, Godrèche & Luck 00, ...

\( \phi(t, r) \) – space-time-dependent order-parameter (magnetisation)

correlator \( C(t, s; r) := \langle \phi(t, r)\phi(s, 0) \rangle = s^{-b}f_C(t/s, |r|^z/(t-s)) \)

response \( R(t, s; r) := \left. \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, 0)} \right|_{h=0} = s^{-1-a}f_R(t/s, |r|^z/(t-s)) \)

**No** fluctuation-dissipation theorem: \( R(t, s; r) \neq T \partial C(t, s; r)/\partial s \)

values of exponents: equilibrium correlator \( \rightarrow \) classes S and L

\[ C_{eq}(r) \sim \begin{cases} \exp(-|r|/\xi) & \text{class } S \\ |r|^{-(d-2+\eta)} & \text{class } L \end{cases} \]

if \( T < T_c \): \( z = 2 \) and \( b = 0 \)

for \( y \to \infty \): \( f_{C,R}(y) \sim y^{-\lambda_{C,R}/z} \), \( \lambda_{C,R} \) independent exponents

if \( T = T_c \): \( z = z_c \) and \( b = a \)
Test of dynamical scaling: 3D Ising model, $T < T_c$

no time-translation invariance

\[ C(t,s) : \text{autocorrelation function, quenched to } T < T_c \]

**scaling regime:** $t, s \gg \tau_{\text{micro}}$ and $t - s \gg \tau_{\text{micro}}$

**Question:** how to find the scaling functions $f_R(y)$ and $f_C(y)$?
How to understand these scaling forms \(\rightarrow\) mean-field

**Langevin eq. for order parameter** \(m(t)\)

\[
\frac{dm(t)}{dt} = 3\lambda^2 m(t) - m(t)^3 + \eta(t) \quad \langle \eta(t)\eta(s) \rangle = 2T \delta(t - s)
\]

**contrôlé parameter** \(\lambda^2\):

- (1) \(\lambda^2 > 0: T < T_c\)
- (2) \(\lambda^2 = 0: T = T_c\)
- (3) \(\lambda^2 < 0: T > T_c\)

**two-time observables**:
- **response** \(R(t, s)\), **correlation** \(C(t, s)\)

\[
R(t, s) = \left. \frac{\delta \langle m(t) \rangle}{\delta h(s)} \right|_{h=0} = \frac{1}{2T} \langle m(t)\eta(s) \rangle \quad C(t, s) = \langle m(t)m(s) \rangle
\]

**mean-field** equation of motion (cumulants neglected):

\[
\partial_t R(t, s) = 3 \left( \lambda^2 - v(t) \right) R(t, s) + \delta(t - s)
\]

\[
\partial_s C(t, s) = 3 \left( \lambda^2 - v(s) \right) C(t, s) + 2T R(t, s)
\]

with variance \(v(t) = \langle m(t)^2 \rangle\),

\[
\dot{v}(t) = 6(\lambda^2 - v(t))v(t)
\]
if $\lambda^2 \geq 0$ : fluctuations persist
if $\lambda^2 < 0$ : fluctuations disappear

in the long-time limit $t, s \to \infty : (t > s)$

$$R(t, s) \simeq \begin{cases} 
1 \sqrt{s/t} \
\frac{1}{e^{-3|\lambda^2|(t-s)}} 
\end{cases} ; 
C(t, s) \simeq T \begin{cases} 
2 \min(t, s) \quad ; \quad \lambda^2 > 0 \\
\frac{2}{3} \quad ; \quad \lambda^2 = 0 \\
\frac{1}{(3|\lambda^2|)} e^{-3|\lambda^2||t-s|} \quad ; \quad \lambda^2 < 0
\end{cases}$$

fluctuation-dissipation ratio measures distance from equilibrium

$$\chi(t, s) = \frac{TR(t, s)}{\partial_s C(t, s)} \simeq \begin{cases} 
1/2 + O(e^{-6\lambda^2s}) \quad ; \quad \lambda^2 > 0 \\
2/3 \quad ; \quad \lambda^2 = 0 \\
1 + O(e^{-|\lambda^2||t-s|}) \quad ; \quad \lambda^2 < 0
\end{cases}$$

relaxation far from equilibrium, when $\chi \neq 1$, if $\lambda^2 \geq 0 \ (T \leq T_c)$
Consequences:

If $\lambda^2 > 0$ : free random walk, the system never reaches equilibrium!
If $\lambda^2 = 0$ : slow relaxation, because of critical fluctuations

In both situations : observe

1. slow dynamics (non-exponential relaxation)
2. time-translation-invariance broken
3. dynamical scaling behaviour

$\rightarrow$ the conditions for physical ageing are all satisfied if $T \leq T_c$
$\rightarrow$ the system remains out of equilibrium

If $\lambda^2 < 0$ : rapid relaxation, with finite relaxation time
\[ \tau_{\text{rel}} \sim 1/|\lambda^2|, \] towards unique equilibrium state
II. Local scale-invariance for $z \neq 2$

Extend known cases $z = 1, 2 \implies$ **axioms of LSI**:

MH 97/02, Baumann & MH 07

1. Möbius transformations in time (generator $X_n$)

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1$$

require commutator: $[X_n, X_{n'}] = (n - n')X_{n+n'}$

2. Dilatation generator: $X_0 = -t \partial_t - \frac{1}{z} r \cdot \partial_r - \frac{x}{z}$

Implies simple power-law scaling $L(t) \sim t^{1/z}$ (no glasses!).

3. Spatial translation-invariance $\rightarrow 2^e$ family $Y_m$ of generators.

4. $X_n$ contain phase terms from the scaling dimension $x = x_\phi$

5. $X_n, Y_m$ contain further ‘mass terms’ (Galilei!)

6. finite number of independent conditions for $n$-point functions.
Theorem: LSI without ‘masses’

Commutators $[X_n, X_{n'}] = (n - n')X_{n+n'}$, $[X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$ with $n, n' \in \mathbb{Z}$ and $m \in \mathbb{Z} - 1/z$ have only the realisations:

<table>
<thead>
<tr>
<th>$z$</th>
<th>$X_n$ = $-t^{n+1} \partial_t - \frac{n+1}{z} t^n r \partial_r - \frac{(n+1)x}{z} t^n - \frac{n(n+1)}{2} B_{10} t^{n-1} r^z$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Y_{k-1/z} = -t^k \partial_r - \frac{z^2}{2} k B_{10} t^{k-1} r^{-1+z}$</td>
</tr>
<tr>
<td>2</td>
<td>$X_n$ = $-t^{n+1} \partial_t - \frac{1}{2} (n + 1) t^n r \partial_r - \frac{1}{2} (n + 1) xt^n$</td>
</tr>
<tr>
<td></td>
<td>$-\frac{n(n+1)}{2} B_{10} t^{n-1} r^2 - \frac{(n^2-1)n}{6} B_{20} t^{n-2} r^4$</td>
</tr>
<tr>
<td></td>
<td>$Y_{k-1/2} = -t^k \partial_r - 2k B_{10} t^{k-1} r - \frac{4}{3} k (k - 1) B_{20} t^{k-2} r^3$</td>
</tr>
<tr>
<td>1</td>
<td>$X_n$ = $-t^{n+1} \partial_t - A_{10}^{-1} [(t + A_{10} r)^{n+1} - t^{n+1}] \partial_r$</td>
</tr>
<tr>
<td></td>
<td>$-(n + 1) xt^n - \frac{n+1}{2} \frac{B_{10}}{A_{10}} [(t + A_{10} r)^n - t^n]$</td>
</tr>
<tr>
<td></td>
<td>$Y_{k-1} = -(t + A_{10} r)^k \partial_r - \frac{k}{2} B_{10} (t + A_{10} r)^{k-1}$</td>
</tr>
</tbody>
</table>

Free parameters (two in each case): $z, A_{10}, B_{10}, B_{20}$
similar classification from a geometric point of view  

1. **generic** $z$ and $B_{10} = 0 : \implies [Y_m, Y_{m'}] = 0$.

2. $z = 2$. Find infinite-dimensional extension of $\mathfrak{sch}_1$:

   \[ Z_n^{(0)} := -2t^n, \quad Z_m^{(1)} := -2t^{m-1/2}r, \quad Z_n^{(2)} := -nt^{n-1}r^2 \]

   \[
   [Y_m, Y_{m'}] = (m - m')(4B_{20}Z_{m+m'}^{(2)} + B_{10}Z_{m+m'}^{(0)}) \\
   [X_n, Z_{n'}^{(0,2)}] = -n'Z_{n+n'}^{(0,2)}, \quad [X_n, Z_m^{(1)}] = -(n/2 - m)Z_{n+n'}^{(1)} \\
   [Y_m, Z_{m'}^{(1)}] = -Z_{m+m'}^{(0)}, \quad [Y_m, Z_n^{(2)}] = -nZ_{m+n}^{(1)}
   \]

   For $B_{20} = 0$ and $B_{10} = \mathcal{M}/2$ one is back to $\mathfrak{sv}_1 \supset \mathfrak{sch}_1$.

3. $z = 1$. Then $[Y_n, Y_{n'}] = A_{10}(n - n')Y_{n+n'}$, in $d = 1$ dimensions.

   If $A_{10} \neq 0$, isomorphic to $\mathfrak{vect}(S^1) \times \mathfrak{vect}(S^1)$.

   In the limit $A_{10} \to 0$, contraction to $\mathfrak{av}_1 \supset \mathfrak{alt}_1 = \text{CGA}(1); (\gamma \in \mathbb{R})$

   \[
   X_n = -t^{n+1}\partial_t - (n + 1)t^n r \partial_r - (n + 1)t^n x - n(n + 1)\gamma t^{n-1}r \\
   Y_n = -t^{n+1}\partial_r - (n + 1)\gamma t^n
   \]

   two Virasoro-like **independent** central charges

For $d = 2$ so-called **exotic** central extension of $\mathfrak{alt}_2$, but incompatible

with $\infty$-dim. extension $\mathfrak{alt}_2 \subset \mathfrak{av}_2$
consider $z$ arbitrary, set $B_{10} = 0$.

For the case $z = 1$, see Havas & Plebanski 78, Negro et al 97, MH 97 & 02; ... 09-10.

Extend to $z \neq 1, 2$ by generators with mass terms, for $d = 1$:

$$ Y_{1-1/z} := -t \partial_r - \mu z r \nabla_r^{2-z} - \gamma z(2 - z) \partial_r \nabla_r^{-z} $$

$$ X_1 := -t^2 \partial_t - \frac{2}{z} tr \partial_r - \frac{2(x + \xi)}{z} t - \mu r^2 \nabla_r^{2-z} $$

$$ -2\gamma(2 - z) r \partial_r \nabla_r^{-z} - \gamma(2 - z)(1 - z) \nabla_r^{-z} $$

- depend on two parameters $\gamma, \mu$ and on two dimensions $x, \xi$
- contains fractional derivative $\nabla_r^{\alpha} f(r) := i^\alpha \int_{\mathbb{R}^d} \frac{dk}{(2\pi)^d} |k|^{\alpha} e^{ir \cdot k} \hat{f}(k)$
- some properties: $\nabla_r^{\alpha} \nabla_r^{\beta} = \nabla_r^{\alpha+\beta}$, $[\nabla_r^{\alpha}, r_i] = \alpha \partial_{r_i} \nabla_r^{\alpha-2}$

$$ \nabla_r^{\alpha} \exp(iq \cdot r) = i^\alpha |q|^{\alpha} \exp(iq \cdot r) $$
**Fact 1**: simple algebraic structure:

\[
[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}
\]

→ Generate \( Y_m \) from \( Y_{-1/z} = -\partial_r \).

**Fact 2**: LSI-invariant Schrödinger operator:

\[
S := -\mu \partial_t + z^{-2} \nabla^2_r
\]

Let \( x_0 + \xi = 1 - 2/z + (2 - z)\gamma/\mu \). Then \([S, Y_m] = 0 \) and

\[
[S, X_0] = -S, \quad [S, X_1] = -2tS + \frac{2\mu}{z} (x - x_0)
\]

\[\Rightarrow \quad S\phi = 0 \text{ is LSI-invariant equation, if } x_\phi = x_0.\]

**Physical assumption** (hidden & approximate): equations of motion remain of first order in \( \partial_t \), even after renormalisation.
Fact 3: non-trivial conservation laws:
iterated commutator with \( G := Y_{1-1/z}, \ \text{ad} \ G \cdot = [, , G] \)

\[
M_\ell := (\text{ad}_G)^{2\ell+1} Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_r (2\ell+1)(1-z)+1
\]

For \( z = 2 \), \( a_\ell = 0 \) if \( \ell \geq 1 \). For a \( n \)-point function
\( F^{(n)} = \langle \phi_1 \ldots \phi_n \rangle, \ M_\ell F^{(n)} = 0 \) gives in momentum space

\[
\left( \sum_{i=1}^{n} \mu_i^{2\ell-1} |k_i|^{2\ell-(2\ell-1)z} \right) \hat{F}^{(n)}(\{t_i, k_i\}) = 0
\]

\[
\left( \sum_{i=1}^{n} k_i \right) \hat{F}^{(n)}(\{t_i, k_i\}) = 0
\]

\( \implies \) momentum conservation & conservation of \(|k|^{\alpha}\)!

analogous to relativistic factorisable scattering

equil. analogy : 2D Ising model at \( T = T_c \) in magnetic field

\text{Zamolodchikov}^2 79, 89
Consequence: a $L^i$-covariant $2n$-point function $F^{(2n)}$ is only non-zero, if the ‘masses’ $\mu_i$ can be arranged in pairs $(\mu_i, \mu_{\sigma(i)})$ with $i = 1, \ldots, n$ such that $\mu_i = -\mu_{\sigma(i)}$.

generalised Galilei-invariance with $z \neq 2 \implies$ integrability

Corollary 1: Bargman rule: $\langle \phi_1 \ldots \phi_n \bar{\phi}_1 \ldots \bar{\phi}_m \rangle_0 \sim \delta_{n,m}$

Corollary 2: derive reduction formulæ for averages: go to stochastic field-theory, action

$$J[\phi, \bar{\phi}] = J_0[\phi, \bar{\phi}] - T \int \tilde{\phi}^2 - \int_{\phi_t=0} C_{\text{init}} \tilde{\phi}_{t=0}$$

$$+ J_b[\tilde{\phi}]: \text{noise}$$

$\tilde{\phi}$: response field; $C(t, s) = \langle \phi(t)\phi(s) \rangle$, $R(t, s) = \langle \phi(t)\bar{\phi}(s) \rangle$

averages: $\langle A \rangle_0 := \int \mathcal{D}\phi \mathcal{D}\bar{\phi} A[\phi, \bar{\phi}] \exp(-J_0[\phi, \bar{\phi}])$

identify masses (generalised Bargman rule):

$$\mu_{\phi} = -\mu_{\bar{\phi}}$$

Janssen 92, de Dominicis, …
application to the response

\[ R(t, s) = \left\langle \phi(t)\bar{\phi}(s) \right\rangle = \left\langle \phi(t)\bar{\phi}(s)e^{-\mathcal{J}_b[\bar{\phi}]} \right\rangle_0 \]

\[ = \left\langle \phi(t)\bar{\phi}(s) \right\rangle_0 = R_0(t, s) \]

Bargman rule \( \implies \) response function independent of noise!

left side : computed in stochastic models
right side : local scale-symmetry of deterministic equation

**Corollary 3** : response function noise-independent

\[ R(t, s; r) = R(t, s)\mathcal{F}^{(\mu_1, \gamma_1)}(|r|(t - s)^{-1/z}) \]

\[ R(t, s) = r_0 s^{-a} \left( \frac{t}{s} \right)^{1 + a' - \lambda R/z} \left( \frac{t}{s} - 1 \right)^{-1 - a'} \]

\[ \mathcal{F}^{(\mu, \gamma)}(u) = \int_{\mathbb{R}^d} \frac{dk}{(2\pi)^d} |k|^\gamma \exp (iu \cdot k - \mu |k|^z) \]
choice of the (quasi-)primary operators ?

Finite transformation (spatial part here for $z = 2$): 
\[ t = \beta(t'), \quad r = r' \sqrt{\frac{d\beta(t')}{dt'}} \quad \text{and} \quad \beta(0) = 0 \]
\[
\phi(t, r) = \dot{\beta}(t')^{-x/2} \left( \frac{d \ln \beta(t')}{d \ln t'} \right)^{-\xi} \exp \left[ -\frac{Mr'^2}{4} \frac{d \ln \dot{\beta}(t')}{dt'} \right] \phi'(t', r')
\]

reduce to usual lsi-primary operator $\Phi(t, r) := t^{-2\xi/z} \phi(t, r)$.

Then $\Phi(t) = \dot{\beta}(t')^{-(x+2\xi)/z} \Phi'(t')$, transforms as a primary.

**a) mean-field equation** $\partial_t m = \Delta m + 3(\lambda^2 - \nu(t))m$ reduces to diffusion equation $\partial_t \Phi = \Delta \Phi$ via

\[
m(t, r) = \Phi(t, r) \exp \int_0^t d\tau \ 3(\lambda^2 - \nu(\tau))
\]

**two cases**:
\[
\begin{cases} 
\text{if} \quad T = T_c \Leftrightarrow \lambda^2 = 0 : & \Phi(t) \sim t^{1/2} m(t) \\
\text{if} \quad T < T_c \Leftrightarrow \lambda^2 > 0 : & \Phi(t) \sim 1 \cdot m(t)
\end{cases}
\]
magnetisation $m(t)$ and primary operator $\Phi(t)$ distinct

b) **kinetic spherical model** equation

$$\partial_t \phi(t) = \Delta \phi(t) - \nu(t) \phi(t) + \text{noise} , \quad \nu(t) \sim t^{-1}$$

gauge transformation $\Phi(t) = \phi(t) \exp[-\int_0^t d\tau \nu(\tau)]$, gives diffusion eq. for $\Phi$

c) **kinetic Glauber-Ising model** $T = T_c$

1D $a' - a = -\frac{1}{2}$
2D $a' - a \simeq -0.17(2)$
3D $a' - a \simeq -0.022(5)$

* $2^{\text{nd}}$-order $\varepsilon$-expansion disagrees with lattice data \cite{Pleimling & Gambassi 05}
* $a' - a < 0$ required to match LSI with lattice data, but still disagrees with FT
  $\Rightarrow$ resum $\varepsilon$-expansion to be able to compare with lattice data?
Some known values of $a$, $a'$ and $\lambda_R/z$ at $T = T_c$.

<table>
<thead>
<tr>
<th>model</th>
<th>$d$</th>
<th>$a$</th>
<th>$a' - a$</th>
<th>$\lambda_R/z$</th>
<th>Réf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ising</td>
<td>1</td>
<td>0</td>
<td>$-1/2$</td>
<td>$1/2$</td>
<td>Godrèche &amp; Luck 00</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.115</td>
<td>$-0.17(2)$</td>
<td>$0.732(5)$</td>
<td>H &amp; P 03</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.506</td>
<td>$-0.022(5)$</td>
<td>$1.36(2)$</td>
<td>H &amp; P 03</td>
</tr>
<tr>
<td>EA spin glass</td>
<td>3</td>
<td>0.060(4)</td>
<td>$-0.76(3)$</td>
<td>$0.38(2)$</td>
<td>H &amp; P 05</td>
</tr>
<tr>
<td>FA</td>
<td>1</td>
<td>1</td>
<td>$-3/2$</td>
<td>$2$</td>
<td>Mayer et al 06</td>
</tr>
<tr>
<td></td>
<td>$&gt;2$</td>
<td>$1 + d/2$</td>
<td>$-2$</td>
<td>$2 + d/2$</td>
<td>Mayer et al 06</td>
</tr>
<tr>
<td>contact proc.</td>
<td>1</td>
<td>$-0.681$</td>
<td>$0.270(10)$</td>
<td>$1.76(5)$</td>
<td>H, E &amp; P 06</td>
</tr>
<tr>
<td>NEKIM</td>
<td>1</td>
<td>$-0.430(2)$</td>
<td>$0.00(1)$</td>
<td>$1.9(2)$</td>
<td>Odor 06</td>
</tr>
<tr>
<td>OJK model</td>
<td>$\geq 2$</td>
<td>$(d - 1)/2$</td>
<td>$-1/2$</td>
<td>$d/4$</td>
<td>Mazenko 04</td>
</tr>
</tbody>
</table>

$\implies: a \neq a'$ should be the generic case.

$\implies: \text{order-parameter } m(t) \text{ does in general not transform in the most simple way!}$
Corollary 4:
Correlators obtained from factorised 4-point responses:

\[ C(t, s) = \langle \phi(t) \phi(s) \rangle = \langle \phi(t) \phi(s) e^{-J_b[\phi]} \rangle_0 \]

example: contribution of ‘initial’ noise at time \( u \):

\[ C_{\text{init}}(t, s; r) = \int_{\mathbb{R}^{2d}} dR dR' \ F^{(4)}(t, s, u; r, R, R') \ C(u, R - R') \]

4-pt function ‘initial’ correlator

\[ = c_0 (ts)^{2\xi/z + F} s^{4\tilde{x}/z - 2F} (t - s)^{-2(2\xi + x)/z} \]

\[ \times \int_{\mathbb{R}^d} dk \ |k|^{2\beta} \ exp[ir \cdot k - \alpha |k|^z (t - s)] \ \hat{C}(s, k) \]

where we have also sent \( u \rightarrow s \).

Relevant, e.g. for phase-ordering kinetics \( \rightarrow z = 2 \) Bray & Rutenberg 94

Ising model, more precise ‘initial’ correlator:

\[ C(t; r) = \frac{2}{\pi} \ \arcsin \left( \exp \left[ -\frac{r^2}{L(t)^2} \right] \right) \]

Ohta, Jasnow, Kawasaki ’82
III. How to test the foundations of LSI

theory is built on:

a) simple scaling – domain sizes \( L(t) \sim t^{1/z} \)
b) invariance under Möbius transformation \( t \mapsto t/(\gamma t + \delta) \)
c) Galilei-invariance generalised to \( z \neq 2 \)

together with spatial translation-invariance

\[ \implies \text{extended Bargman rules} \]
\[ \implies \text{factorisation of } 2n\text{-point functions} \]

<table>
<thead>
<tr>
<th>Möbius transformation</th>
<th>autoresponse ( R(t, s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>generalised Galilei-invariance</td>
<td>space-time response ( R(t, s; r) )</td>
</tr>
<tr>
<td>factorisation</td>
<td>two-time correlation function</td>
</tr>
</tbody>
</table>
Example: Ising model, space-time behaviour (parameter-free!):

\[ \rho_r^{(2)}(y, \mu) = s^{d/2-a} \rho^{(2)}(t/s, \mu) \]

(a,b) 2D; \( \mu = 1, 2, 4 \)
(b) \( s=100 \), \( s=121 \), \( s=144 \), \( s=169 \), \( s=196 \), \( s=225 \)

(c,d) 3D; \( \mu = 1, 2, 3 \)
(c) \( s=25 \), \( s=36 \), \( s=64 \), \( s=81 \), \( s=100 \)
(d) \( s=25 \), \( s=36 \), \( s=64 \), \( s=81 \), \( s=100 \)


2D Ising model, $T < T_c$: autocorrelator in the scaling limit

$$C(ys, s) = C_0 y^\rho (y - 1)^{-\rho - \lambda c/z} \int_0^\infty dx \, e^{-x} f_\nu \left( \frac{x}{y - 1} \right)$$

$$f_\nu(\sqrt{u}) = \int_0^\infty dv \, \arcsin(e^{-\nu v}) \, J_0(\sqrt{uv})$$

parameters to be fitted: $\rho, \nu$.

![Graphs showing autocorrelator behavior](image)

of practical importance:
'good' choice of 'initial' correlations $C_{\text{ini}}(r) = c_0 \delta(r)$ not sufficient

Baumann & MH 10

$\implies$ for the first time, a theoretical calculation for $C(t, s)$ reproduces the simulations for all $t/s$!
Tests of LSI for $z \neq 2$:

- **spherical model with conserved order-parameter, $T = T_c$, $z = 4$**
  
  
  **Baumann & MH 06**

- **Mullins-Herring model for surface growth, $z = 4$**
  
  
  **Röthlein, Baumann, Pleimling 06**

- **spherical model with long-ranged interactions, $T \leq T_c$, $0 < z = \sigma < 2$**
  
  
  **Cannas et al. 01; Baumann, Dutta, MH 07; Dutta 08**

- **ferromagnets at their critical point (Ising, XY), $z \approx 2.0 - 2.2$**
  
  
  **MH, Enss, Pleimling 06; Abriet & Karevski 04**

- **critical particle-reaction models (DP ?, NEKIM), $z \approx 1.6 - 2$**
  
  
  **Ódor 06**

- **particle-reaction models with Lévy-flight transport, $0 < z = \eta < 2$**
  
  
  **Durang & MH 09**

important: consideration of invariant differential equation

**NB**: all of the exactly solved models in this list are **markovian**!
What tests of LSI have been achieved?

1. $R(t, s)$:
   - $T < T_c, d = 2$: Ising, Potts, spherical (A&B), disord. Ising
   - $T < T_c, d = 3$: Ising, XY, spherical (A&B)
   - $T = T_c, d \leq 2$: Ising, spherical (A&B), HvL, DP ?, NEKIM
   - $T = T_c, d = 3$: Ising, spherical (A&B), BCPD/L, BPCPD
   - growth: Edwards-Wilkinson, Mullins-Herring

2. $R(t, s; r)$:
   - $T < T_c$: Ising, Potts-3 & 8, spherical (A&B)
   - $T = T_c$: Ising 1D, spherical (A&B), BCPD/L, BPCDP
   - growth: Edwards-Wilkinson, Mullins-Herring

**Difficulty**: oscillating dependence on $|r|$

3. $C(t, s)$:
   - $T < T_c$: Ising 2D, Potts 2D, spherical (A&B)
   - $T = T_c$: Ising 1D, spherical (A&B), BCPD/L, BPCPD

**Required**: precise single-time correlator $C(t, r)$
look for extensions of dynamical scaling in ageing systems
recently, scaling derived for phase-ordering Arenzon et al. 07

here: **hypothesis** of generalised Galilei-invariance
leads to Bargman rule if $z = 2$
and further to ‘integrability’ if $z \neq 1, 2$.

**hidden** dynamical symmetry of deterministic part of (linear &
first-order !) Langevin equations

Tests: derive two-time response and correlation functions

LSI exactly proven for linear Langevin equations
very good numerical evidence for non-linear systems

Some questions (the list could/should be extended):
- how to physically justify Galilei-invariance?
- how to extend to non-linear equations?
- **non**-markovian effects? choice of fractional derivative?
- what is the algebraic (non-Lie !) structure of LSI?
- treatment of master equations with LSI?
This book is Volume I of a two-volume set describing the two main classes of non-equilibrium phase transitions. It covers the statics and dynamics of transitions into an absorbing state. Volume II will cover dynamical scaling in far-from-equilibrium relaxation behaviour and ageing.

The first volume begins with an introductory chapter which recalls the main concepts of phase transitions, set for the convenience of the reader in an equilibrium context. The extension to non-equilibrium systems is made by using directed percolation as the main paradigm of absorbing phase transitions and, in view of the richness of the known results, an entire chapter is devoted to it, including a discussion of recent experimental results. Scaling theories and a large set of both numerical and analytical methods for the study of non-equilibrium phase transitions are thoroughly discussed.

The techniques used for directed percolation are then extended to other universality classes and many important results of model parameters are provided for easy reference.

Vol. 2 – co-author M. Pleimling – will treat ageing phenomena in simple magnets and LSI (to appear still in 2010)