Zero-range condensation at criticality

Stefan Grosskinsky

Warwick

in collaboration with Ines Armendariz, Michalis Loulakis and Paul Chleboun

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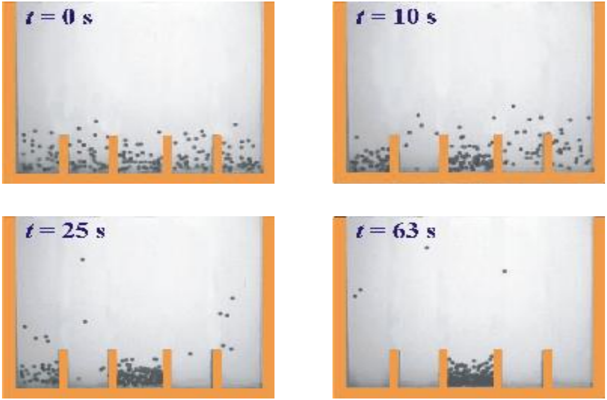
Clustering in granular gases



[van der Meer, van der Weele, Lohse, Mikkelsen, Versluis (2001-02)]
stilton.tnw.utwente.nl/people/rene/clustering.html

The zero-range process

Lattice/vertex set: Λ_L of size L

Jump probabilities: $p(x,y) \in [0,1]$
$\sum_y p(x,y) = 1$, irreducible

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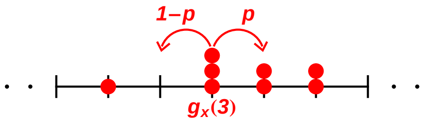

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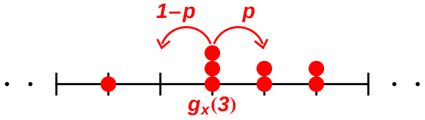

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Generator: $f \in C_0(X_L)$

$$\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} g_x(\eta_x)p(x,y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$$

[Spitzer (1970), Andjel (1982)]

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Condensation phenomena

- spatial heterogeneity

  $\Rightarrow$ condensation on the 'slowest' site

  [Evans (1996), Krug, Ferrari (1996), Benjamini, Ferrari, Landim (1996), Ferrari, Sisko (2007),...]

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- effective attraction of particles due to $g(k) \searrow$
  $\Rightarrow$ condensation on a random site
  a generic class [Evans (2000)]

$$g(n) \simeq 1 + \frac{b}{n^\gamma}\,, \quad \gamma \in (0,1)\ ; \quad \gamma = 1,\ b > 2$$

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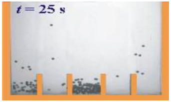

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[Evans, Hanney (2005), Evans (2000)]

- bus route model, ant trails
  epitaxial growth (step flow model)
  network dynamics: rewiring (directed) networks

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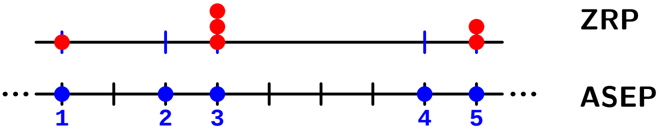

- Phase separation in one-dimensional exclusion models
  [Kafri, Levine, Mukamel, Schütz, Török (2002)]
- connection to traffic flow modeling
  [Kaupuzs, Mahnke, Harris (2006)]

Stationary measures

Factorized stationary weights

$$w^L(\boldsymbol{\eta}) := \prod_{x\in\Lambda_L} w(\eta_x) \quad \text{with} \quad w(n) \sim \prod_{k=1}^{n} g(k)^{-1}$$

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Canonical measures fix $\Sigma_L(\boldsymbol{\eta}) = N$

$$\pi_{L,N}(\boldsymbol{\eta}) := \frac{1}{Z_{L,N}}\, w^L(\boldsymbol{\eta})\, \delta\Big(\sum_x \eta_x, N\Big)$$

Stationary measures

Grand-canonical measures fugacity $\phi \quad \to \quad \phi^{\sum_x \eta_x}$

$$\nu_\phi^L(\boldsymbol{\eta}) = \frac{1}{z(\phi)^L} \prod_{x \in \Lambda_L} w(\eta_x)\, \phi^{\eta_x} \quad \text{where} \quad z(\phi) = \sum_{n=0}^{\infty} w(n)\, \phi^n$$

defined for $\phi < \phi_c$

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defined for $\phi < \phi_c = 1$

$$w(n) \sim \begin{cases} n^{-b} & , \ \gamma = 1 \\ \exp(-\frac{b}{1-\gamma}\, n^{1-\gamma}) & , \ \gamma \in (0,1) \end{cases}$$

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density $R(\phi) = \big\langle \eta_x \big\rangle_{\nu_\phi} = \phi\,\partial_\phi \log z(\phi) \quad \uparrow \quad$ in ϕ

critical density $\rho_c = \lim_{\phi\to\phi_c} R(\phi) \in (0, \infty]$

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stationary current $j = \langle g(\eta_x)\rangle_{\nu_\phi} = \phi$

Equivalence of ensembles

Previous results

In the thermodynamic limit $L, N \to \infty \;,\quad N/L \to \rho$

$$\pi_{L,N} \xrightarrow{w} \nu_\phi \quad \text{if} \quad \begin{cases} R(\phi) = \rho \;, & \rho < \rho_c \\ \phi = \phi_c \;, & \rho \geq \rho_c \end{cases}$$

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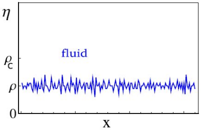
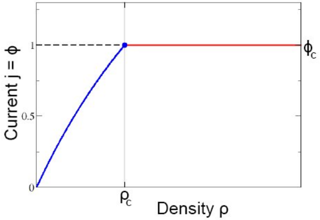
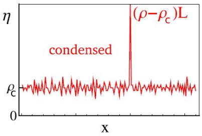


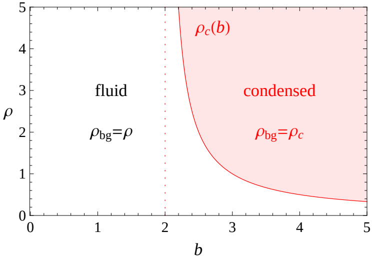




Phase diagram

order parameter $M_L = \max_{x\in\Lambda} \eta_x$

$$\rho_{bg} := \lim_{L\to\infty} \frac{N - M_L}{L-1} = \begin{cases} \rho \ , & \rho \le \rho_c \\ \rho_c \ , & \rho \ge \rho_c \end{cases}$$



$\gamma = 1$

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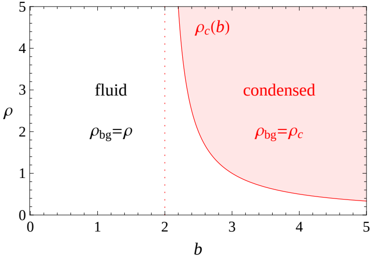

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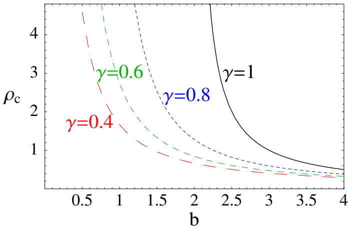

$\gamma \in (0,1)$

Results at criticality

Joint work with Ines Armendariz and Michalis Loulakis

[arXiv:0912.1793]

thermodynamic limit $L, N \to \infty\ ,\quad N/L \to \rho_c$

excess mass $N - \rho_c L = o(L) \nearrow \infty$

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Question

Under the conditional distribution $\pi_{L,N}$, what is the limit

$$\frac{M_L}{N - \rho_c L} \to ? \ \in [0,1]$$

Power law case

$$g(n) \simeq 1 + \frac{b}{n} \quad \Rightarrow \quad \nu_{\phi_c}(\eta_x = n) \sim w(n) \sim n^{-b}$$

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Theorem 1

Assume $b > 3$ ($\sigma^2 < \infty$) and let

$$\Delta_L = \sigma\sqrt{(b-3)L\log L}\ .$$

Then

$$\frac{M_L}{N - \rho_c L} \xrightarrow{\pi_{L,N}} \begin{cases} 0 & , \\ Be(p) & , \\ 1 & , \end{cases} \quad \text{if} \quad \lim \frac{N - \rho_c L}{\Delta_L} \begin{cases} < 1 \\ 1 \\ > 1 \end{cases}$$

with $p \in (0,1)$ explicit, depending on $\lim N - \rho_c L - \Delta_L$.

Stretched exponential case

$$g(n) \simeq 1 + \frac{b}{n^\gamma}\ , \quad \gamma \in (0,1) \quad \Rightarrow \quad \nu_{\phi_c}(\eta_x = n) \sim w(n) \sim e^{-\frac{b}{1-\gamma} n^{1-\gamma}}$$

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Theorem 2

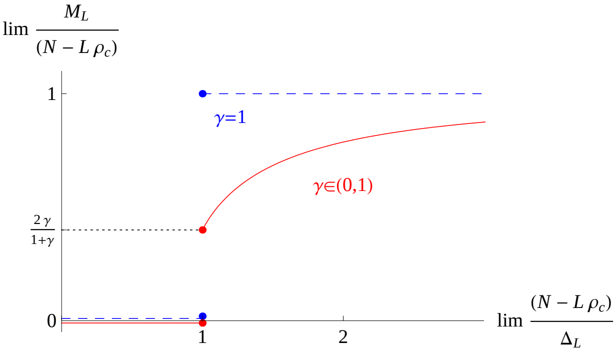
Let $\Delta_L = c_\gamma (\sigma^2 L)^{1/(1+\gamma)}$.

$$\frac{M_L}{N - \rho_c L} \xrightarrow{\pi_{L,N}} \begin{cases} 0 & , \\ \frac{2\gamma}{1+\gamma} Be(p) & , \\ a(t) & , \end{cases} \quad \text{if} \quad t = \lim \frac{N - \rho_c L}{\Delta_L} \begin{cases} < 1 \\ 1 \\ > 1 \end{cases}$$

with $p \in (0,1)$ explicit, depending on $\lim N - \rho_c L - \Delta_L$,

$a(t)$ implicit, $a(1) = \frac{2\gamma}{1+\gamma}$, $a(t) \nearrow 1$ as $t \to \infty$.

Law of large numbers



Fluctuations

Power law case

Assume $b > 3$ ($\sigma^2 < \infty$). Then

$$\pi_{L,N}\Big(\frac{M_L}{L^{1/(b-1)}} \leq x\Big) \to e^{-ux^{1-b}} \quad \text{if} \quad \frac{M_L}{N-\rho_c L} \to 0\ ,$$

i.e. the fluctuations are Fréchet, and

$$\frac{M_L - (N-\rho_c L)}{\sqrt{L\sigma^2}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{if} \quad \frac{M_L}{N-\rho_c L} \to 1\ ,$$

i.e. the fluctuations are Gaussian .

Fluctuations

Stretched exponential case

If $\frac{M_L}{N-\rho_c L} \to 0$ there exist

$$y_L \lim \Big(\frac{1-\gamma}{b}\log L\Big)^{1/(1-\lambda)} \quad \text{and} \quad b_L \sim y_L^\gamma / b$$

such that

$$\pi_{L,N}\Big(\frac{M_L - y_L}{b_L} \leq x\Big) \to e^{-e^{-x}} \quad \text{(Gumbel)}\ .$$

If $\frac{M_L}{N-\rho_c L} \to a(t)$ there exists $a_L \to a(t)$ such that

$$\frac{M_L - (N-\rho_c L)\, a_L}{\sqrt{L}} \xrightarrow{d} \mathcal{N}\Big(0, \frac{\sigma^2}{1-\gamma(1-a(t))/a(t)}\Big)\ .$$

Finite size effects

Joint work with Paul Chleboun

Recursion relation

$$Z_{L,N} = \sum_{k=0}^{N} w(k)\, Z_{L-1,N-k}$$

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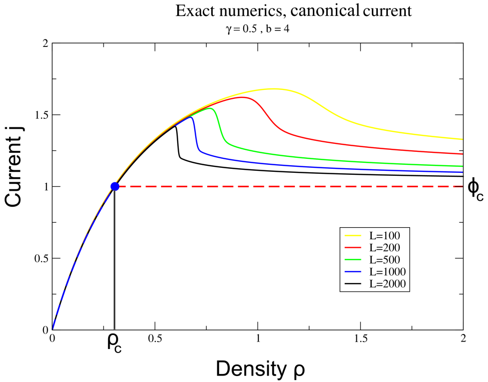
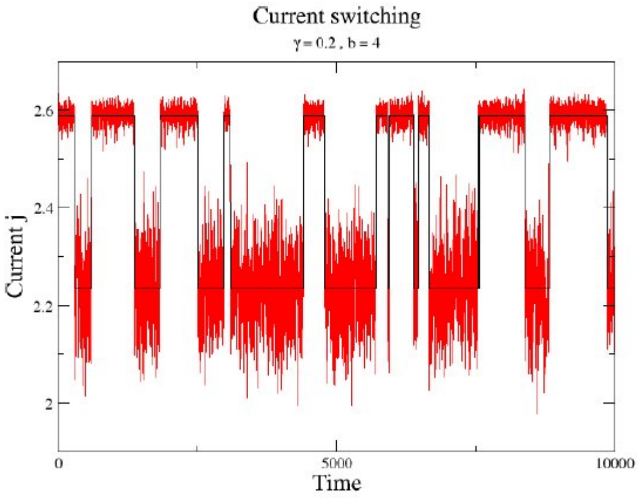
canonical current

$$j = Z_{L,N-1}/Z_{L,N}$$



[Angel, Evans, Mukamel (2004)]

Monte Carlo simulations



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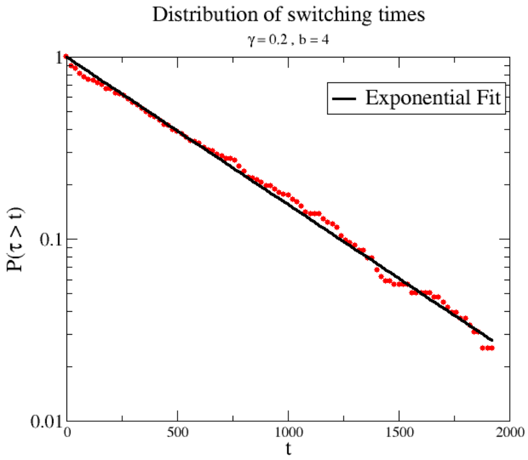
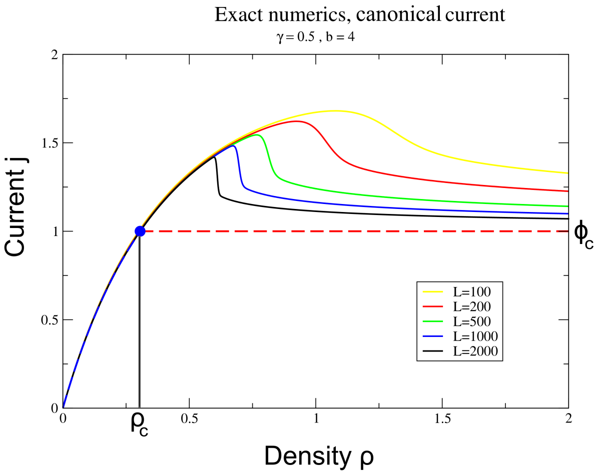


Approximations



Approximations

Cut-off grand canonical distribution

$$z_N(\phi) = \sum_{n=0}^{N} w(n)\,\phi^n\ , \quad \rho_N(\phi) = \phi\,\partial_\phi \log z_N(\phi)$$

$\longrightarrow$ Continuation of 'fluid phase' to higher currents $\phi_N(\rho) > 1$

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$$\phi_N(\rho_{bg}) = g(M_L) = 1 + \frac{b}{M_L^\gamma}$$

$\rightarrow$ describes 'condensed phase'

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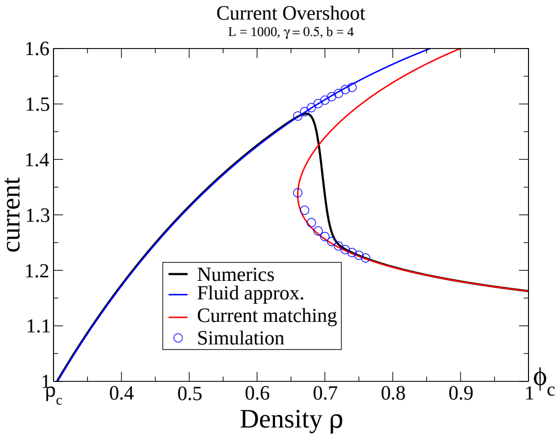

Conclusion

- rigorous results on the behaviour at the critical point
- significant finite size effects
  relevant e.g. in granular clustering and traffic models
- counter-intuitive behaviour for continuous PT
  metastable states and switching