$\mathbf{H}(\mathrm{div})$-CONFORMING p-INTERPOLATION IN TWO DIMENSIONS: ERROR ESTIMATES AND APPLICATIONS

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Problem formulation

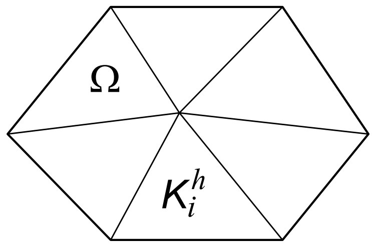
Notation

$\Omega \subset \mathbb{R}^2$ – a polygonal domain; $\bar{\Omega} = \cup_i \bar{K}_i^h$;

$h > 0$ – mesh parameter; $p \geq 1$ – polynomial degree;

$\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2) \in \Omega$;

$\mathbf{H}^r(\mathrm{div}, \Omega) := \{\mathbf{u} \in \mathbf{H}^r(\Omega);\ \mathrm{div}\, \mathbf{u} \in H^r(\Omega)\}$, $r \geq 0$.



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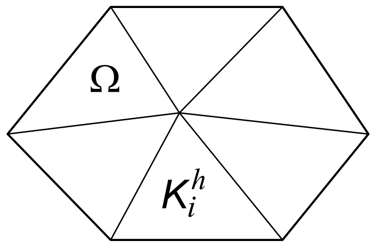

The problem

Given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \Omega)$ with $r > 0$, find $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}))$ such that

- $v_1(\mathbf{x})$, $v_2(\mathbf{x})$ are piecewise polynomials of degree p,
- $\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$,
- $\mathbf{u}(\mathbf{x}) \approx \mathbf{v}(\mathbf{x})$, i.e., $\|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}(\mathrm{div}, \Omega)} \to 0$ as $h \to 0$ and/or $p \to \infty$.

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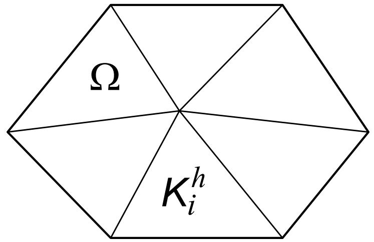

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Applications

* Mixed finite element methods for elliptic problems
* FEM for Maxwell's equations in 2D (due to isomorphism of div and curl)
* $\mathbf{H}$(div)-conforming BEM for Maxwell's equations in 3D

Problem formulation

Reference element K: equilateral triangle T or unit square Q.

Polynomial space on K: the Raviart-Thomas space of order $p \geq 1$,

$$\boldsymbol{P}_p^{\mathrm{RT}}(K) = \begin{cases} (\mathcal{P}_{p-1}(T))^2 \oplus \mathbf{x}\,\mathcal{P}_{p-1}(T) & \text{if} \quad K = T, \\ \mathcal{P}_{p,p-1}(Q) \times \mathcal{P}_{p-1,p}(Q) & \text{if} \quad K = Q. \end{cases}$$

The problem

Given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ with $r > 0$, find $\mathbf{u}_p \in \boldsymbol{P}_p^{\mathrm{RT}}(K)$ and $\delta_p(r)$ such that

i) $\mathbf{u}_p$ is well-defined and stable (with respect to p) for any $r > 0$;

ii) $\mathbf{u}_p$ allows to construct $\mathbf{H}(\mathrm{div})$-conforming approximations on a patch of elements (e.g., $\mathbf{u}_p$ interpolates normal components of $\mathbf{u}$ along ∂K);

iii) $\|\mathbf{u} - \mathbf{u}_p\|_{\mathbf{H}(\mathrm{div},\, K)} \preceq \delta_p(r)\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\, K)}$ and $\delta_p(r) \to 0$ as $p \to \infty$.

Classical $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{RT}

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ the interpolant $\Pi_p^{\mathrm{RT}}\mathbf{u}$ is defined by the conditions

$$\langle \mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall\, \mathbf{v} \in \begin{cases} (\mathcal{P}_{p-2}(T))^2 & \text{if } K = T, \\ \mathcal{P}_{p-2,p-1}(Q) \times \mathcal{P}_{p-1,p-2}(Q) & \text{if } K = Q; \end{cases}$$

$$\langle (\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}) \cdot \mathbf{n}, w \rangle_{0,\ell} = 0 \quad \forall\, w \in \mathcal{P}_{p-1}(\ell) \text{ and } \forall\, \ell \subset \partial K.$$

Error estimation for p-interpolation on the square Q

[Suri '90], [Milner, Suri '92], [Stenberg, Suri '97], [Ainsworth, Pinchedez '02]

$$\|\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},Q)} \preceq p^{-(r-1/2-\varepsilon)} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},Q)}, \quad r > 1/2.$$

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Conclusions:

– lack of stability (with respect to p) for low-regular fields;

– optimal p-estimates can hardly be achieved;

– it is not clear how to deal with triangular elements.

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, the interpolant $\Pi_p^{\mathrm{div}}\mathbf{u}$ is defined as

$$\Pi_p^{\mathrm{div}}\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathrm{RT}}(K),$$

where

$\mathbf{u}_1 = \sum_{\ell \subset \partial K} \left(\int_\ell \mathbf{u} \cdot \mathbf{n} \right) \boldsymbol{\phi}_\ell$ – the lowest order interpolant ($\boldsymbol{\phi}_\ell \in \boldsymbol{P}_1^{\mathrm{RT}}(K)$),

$\mathbf{u}_2^p$ – the sum of edge interpolants,

$\mathbf{u}_3^p$ – an interior interpolant (vector bubble function) satisfying

$$\langle \mathrm{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p)), \mathrm{div}\, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall\, \mathbf{v} \in \boldsymbol{P}_p^{\mathrm{RT},0}(K),$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p), \mathbf{curl}\, \phi \rangle_{0,K} = 0 \qquad \forall\, \phi \in \mathcal{P}_p^0(K).$$

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

Properties of the operator Π_p^{div}:

– Π_p^{div} is well defined for any $r > 0$;

– it is stable with respect to p for any $r > 0$;

– it preserves polynomial vector fields from $\boldsymbol{P}_p^{\mathrm{RT}}(K)$;

– it works equally well on both triangles and parallelograms;

– it can be easily generalised to allow variation of polynomial degrees;

– it makes de Rham diagram commute.

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

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Interpolation error estimation

If $\mathbf{u} \in \mathbf{H}^{r}(\mathrm{div}, K)$ with $0 < r < 1$, then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\, K)} \le C(\varepsilon)\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^{r}(\mathrm{div},\, K)}, \quad 0 < \varepsilon < r.$$

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Orthogonal (Helmholtz) decomposition of $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{curl}\,\psi, \qquad \langle \mathbf{u}_0, \mathbf{curl}\,\phi\rangle_{0,K} = 0 \quad \forall\, \phi \in H^1(K).$$

Hence, one has limited regularity of $\mathbf{u}_0$ and ψ !

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularized Poincaré integral operators

$R: \ H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K), \ r \geq 0, \qquad \mathrm{div}(R\psi) = \psi \ \ \forall\, \psi \in H^r(K);$

$A: \ \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K), \ r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \ \ \forall\, \mathbf{u} \in \mathbf{H}^r(\mathrm{div}0, K).$

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Lemma 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there exist $\psi \in H^{r+1}(K)$ and $\mathbf{v} \in \mathbf{H}^{r+1}(K)$ such that $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$. Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{r+1}(K)} \preceq \|\mathrm{div}\,\mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \|\psi\|_{H^{r+1}(K)} \preceq \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \qquad (1)$$

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Proof. 1) $\mathrm{div}\,\mathbf{u} \in H^r(K) \ \Rightarrow\ \mathbf{v} := R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^{r+1}(K)$ and

$$\mathbf{u} = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + R(\mathrm{div}\,\mathbf{u}) = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + \mathbf{v}.$$

2) $\mathbf{u} - R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^r(K), \quad \mathrm{div}(\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) = \mathrm{div}\,\mathbf{u} - \mathrm{div}(R(\mathrm{div}\,\mathbf{u})) = 0.$

3) $\psi := A(\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) \in H^{r+1}(K)$ and $\mathbf{curl}\,\psi = \mathbf{u} - R(\mathrm{div}\,\mathbf{u}).$ $\square$

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div}, K)} \preceq p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, K)}.$$

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Immediate (and important) extensions

∗ Brezzi-Douglas-Marini space on the reference triangle

∗ Optimal *hp*-estimates (by the Bramble-Hilbert argument and scaling)

$$\|\mathbf{u} - \Pi_{hp}^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div},\, \Omega)} \preceq h^{\min\{r,p\}} p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\, \Omega)}$$

∗ $\mathbf{H}(\mathrm{curl})$-conforming p-interpolation operator in 2D (due to isomorphism of div and curl). Application: p- and hp-FEM for Maxwell's equations in 2D.

Not so immediate (but also important) application: a priori error analysis of the *hp*-BEM with quasi-uniform meshes for the EFIE

$\Gamma \subset \mathbb{R}^3$ is a Lipschitz polyhedral surface,
$\mathbf{u} \in \mathbf{X} = \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ solves the EFIE,
$\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ is a discrete solution by Galerkin BEM (based on RT-elements).

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$*$ *The regularity of* $\mathbf{u}$ *is stated in terms of Sobolev spaces*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \preceq \|\mathbf{u} - P_{hp}\,\mathbf{u}\|_{\mathbf{X}} \overset{!}{\preceq} h^{1/2}\, p^{-1/2}\, \|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)}$$

$$\preceq h^{1/2+\min\{r,p\}}\, p^{-(r+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}_{-}^{r}(\mathrm{div}_\Gamma, \Gamma)}, \quad r > 0,$$

where $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ is the orthogonal projection w.r.t. the norm in $\mathbf{X}$.

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$$\preceq h^{1/2+\min\{r,p\}}\,p^{-(r+1/2)}\,\|\mathbf{u}\|_{\mathbf{H}_-^r(\mathrm{div}_\Gamma,\,\Gamma)}, \qquad r > 0,$$

where $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ is the orthogonal projection w.r.t. the norm in $\mathbf{X}$.

$*$ *Singular behaviour of* $\mathbf{u}$ *is explicitly specified*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \preceq h^{\alpha}\,p^{-2\alpha}\left(1 + \log\frac{p}{h}\right)^{\beta},$$

where α corresponds to the strongest singularity, $\beta \in \mathbb{N}_0$, p is large enough.

Three-dimensional case

[Demkowicz, Buffa '05]:

H^1-, $\mathbf{H}(\mathbf{curl})$-, and $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operators in 3D

Main properties of these operators:

– well defined and stable with respect to p;

– make de Rham diagram commute;

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Main difficulty: polynomial extensions from an edge ℓ to triangular face T

Given $f \in \mathcal{P}_p^0(\ell)$, find $F \in \mathcal{P}_p(T)$ such that:

i) $F = f$ on ℓ, $F = 0$ on $\partial T \backslash \ell$;

ii) $\|F\|_{\tilde{H}^{1/2}(T, \partial T \backslash \ell)} \le C(p)\, \|f\|_{L^2(\ell)}$.

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ii) $\|F\|_{\tilde{H}^{1/2}(T, \partial T \backslash \ell)} \le C(p) \, \|f\|_{L^2(\ell)}$.

[Heuer, Leydecker '08]:

$C(p) = O\left(\log^{1/2} p\right)$ is the best available result to date, but it is not optimal!

Proof of Theorem 1

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}, \quad \psi \in H^{r+1}(K),\ \mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\mathbf{v}$.

3) $\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u} = \mathbf{curl}(\psi - \Pi_p^1\psi) + (\mathbf{v} - \Pi_p^{\mathrm{div}}\mathbf{v})$.

4) $\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{H}(\mathrm{div},\,K)} = \|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{L}^2(K)} = |\psi - \Pi_p^1\psi|_{H^1(K)}$

$$\preceq p^{-r}\,\|\psi\|_{H^{1+r}(K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

5) $\|\mathbf{v} - \Pi_p^{\mathrm{div}}\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\,K)}$

$$\leq \inf_{\mathbf{v}_p}\Big(\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}(\mathrm{div},\,K)} + \|\Pi_p^{\mathrm{div}}(\mathbf{v} - \mathbf{v}_p)\|_{\mathbf{H}(\mathrm{div},\,K)}\Big)$$

$$\overset{\varepsilon\in(0,1)}{\preceq} \inf_{\mathbf{v}_p}\Big(\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^\varepsilon(K)} + \|\mathrm{div}(\mathbf{v} - \mathbf{v}_p)\|_{L^2(K)}\Big) \preceq \inf_{\mathbf{v}_p}\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^1(K)}.$$

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4) $$\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{H}(\mathrm{div},\,K)} = \|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{L}^2(K)} = |\psi - \Pi_p^1\psi|_{H^1(K)}$$
$$\preceq p^{-r}\,\|\psi\|_{H^{1+r}(K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

5) $$\|\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq \inf_{\mathrm{v}_p} \|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^1(K)}$$
$$\preceq p^{-r}\,\|\mathbf{v}\|_{\mathbf{H}^{1+r}(K)} \preceq p^{-r}\,\|\mathrm{div}\,\mathbf{u}\|_{H^r(K)}.$$

6) Combine 4) and 5), then use 3). $\square$

Thank you for attention!