A fast nonpolynomial FEM for scattering from polygons

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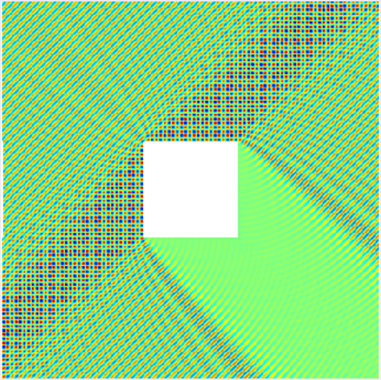
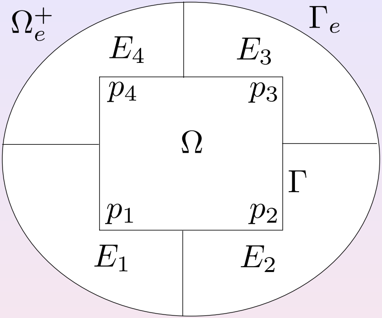
A sound-soft scattering problem

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } \mathbb{C}\backslash\Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ \frac{\partial u_s}{\partial r} - iku_s &= o(r^{-1/2}) \end{aligned}$$

u_i: Incident Wave
u_s: Scattered Field
$u = u_i + u_s$: Full Field

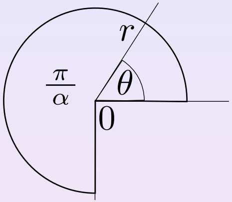


Domain decomposition



- $E_i \cap E_j = \emptyset$ for all $i \neq j$
- $\bigcup_i \overline{E_i} = \overline{\Omega_e}$
- $\Gamma_i \cap \partial\Omega$ consists of two straight lines whose origin is the corner at p_i
- The intersection $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$ is a connected analytic curve

Basis functions in E_i



Close to corner with angle π/α:

$$u(r,\theta) = \sum_{j=1}^{\infty} \gamma_j J_{\alpha j}(kr) \sin \alpha j\theta, \gamma_j \in \mathbb{C}$$

In E_i define local approximation space

$$V_i := \{g : \; g(r,\theta) = \sum_{j=1}^{N_i} c_j J_{\alpha j}(kr) \sin \alpha j\theta, \; c_j \in \mathbb{C}\}$$

In E_i approximate full field u

Approximating the solution towards infinity

On Γ_e choose absorbing boundary conditions.

- Simple approximation: $\frac{\partial u}{\partial n} - iku = 0$
- Hankel function expansion: $u(r,\theta) \approx \sum_{j=0}^{N} c_j H_j^{(1)}(r,\theta) e^{ij\theta}$
- Boundary Integral Equations

Here, use fundamental solutions:

Γ_i: Closed analytic Jordan curve in Ω_e

$$u(x) = \int_{\Gamma_i} H_0^{(1)}(k|x-y|) g(y) dy, \; x \in \Omega_e^+$$

Ansatz: $g(y) = \sum_{j=1}^{N} c_j \delta(y - y_j)$

$$V_e := \{g : \; g(x) = \sum_{j=1}^{N_e} c_j H_0^{(1)}(k|x-y_j|), \; c_j \in \mathbb{C}\}$$

In Ω_e^+ approximate scattered field u_s

A least-squares formulation

Def.: $v \in V$ if $v|_{E_i} \in V_i$ and $v|_{\Omega_e^+} \in V_e$

Define

$$J(v) := \sum_{i<j} \int_{\Gamma_i \cap \Gamma_j} |[\nabla v](\mathbf{x})|^2 ds + k^2 |[v](\mathbf{x})|^2 ds$$
$$+ \sum_{i=1}^{r} \int_{\Gamma_i \cap \Gamma_e} |[\nabla(\hat{u}_{inc} + v)](\mathbf{x})|^2 + k^2 |[\hat{u}_i + v](\mathbf{x})|^2 ds$$

with

$$\hat{u}_{inc}(\mathbf{x}) := \begin{cases} u_{inc}(\mathbf{x}) & \mathbf{x} \in \Omega_e^+ \\ 0 & \mathbf{x} \in \Omega_e \end{cases}$$

Least-Squares FEM [Sto98, MW99]

$$v_{LS} = \arg\min_{v \in V} J(v)$$

Formulating the numerical least-squares problem

Choose quadrature points ξ_j, $j = 1, \ldots, m$ and corresponding weights ω_j.
Define $(A)_{ij} = \phi_j(\xi_i)$, $W = \mathrm{diag}(\omega_1, \ldots, \omega_m)$, $b_j = f(\xi_j)$.

$$\begin{aligned} \int_\Gamma |\sum_{j=1}^{n} \phi_j(\xi) x_j - f(\xi)|^2 d\xi &\approx x^H A^H W A x - 2\mathrm{Re}\{x^H A^H W b\} + b^H W b \\ &= \|W^{1/2}(Ax - b)\|_2^2 \end{aligned}$$

Solving least-squares problem $\|W^{1/2}(Ax - b)\|_2$ directly numerically more stable than solving $A^H W A x = A^H W b$.

Convergence of $J(v)$

Estimate $J(v)$ by

$$\begin{aligned} J(v) &\leq C_1 \left\{ \|\nabla u_s - \nabla v\|^2_{L^2(\Gamma_e)} + k^2 \|u_s - v\|^2_{L^2(\Gamma_e)} \right\} \\ &+ k^2 C_2 \left\{ \sum_i \|v - u\|^2_{L^\infty(E_i)} + \sum_{i<j} \|\nabla v - \nabla u\|^2_{L^\infty(\Gamma_{ij})} \right\} \end{aligned}$$

- Estimate L^∞ convergence in interior elements
- Estimate L^2 convergence on Γ_e.

Estimates on interior elements

Theorem [Vekua]: Fix $z_0 \in \Omega$. Then there exists a unique function Φ holomorphic in Ω with $\Phi(z_0)$ real such that for u with $Lu = 0$ and L elliptic operator with analytic coefficients

$$u = \mathrm{Re}\{V[\Phi; z_0]\}$$

For $\Delta u = 0$:

$$u(x, y) = \mathrm{Re}\{\Phi(z)\}$$

For $-\Delta u = k^2 u$:

$$u(x, y) = \mathrm{Re}\{\Phi(z) - \int_{z_0}^{z} \Phi(t) \frac{\partial}{\partial t} J_0(k\sqrt{(z - t)(\bar{z} - \bar{z}_0)}) \mathrm{dt}\}$$

Estimates on interior elements...

The fractional degree polynomial

$$p_N(z) := \sum_{j=0}^{N} i\tilde{a}_j z^{\alpha j}, \quad \tilde{a}_j \in \mathbb{R}.$$

is mapped to the particular solution

$$\mathrm{Re}\{V[p_N; 0]\} = \sum_{j=1}^{N} a_j J_{\alpha j}(kr) \sin \alpha j \theta$$

We have

$$\|u - \mathrm{Re}\{V[p_N; \cdot\}]\|_{L^\infty(E_i)} \le \|V\|_{L^\infty(E_i)} \|\Phi - p_N\|_{L^\infty(E_i)}.$$

For full convergence analysis see [Bet07]

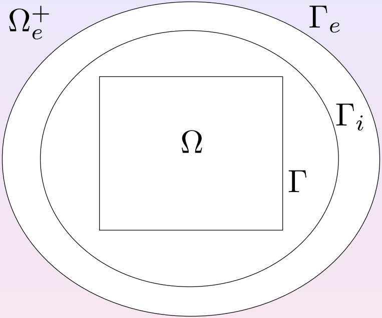
Estimates on interior elements...

Theorem: There exists $\rho_i > 1$ such that for any $1 < \tau < \rho_i$

$$\min_{v \in V_i} \|u - v\|_{L^\infty(E_i)} = O(\tau^{-N_i}),\ N_i \to \infty$$

- Same exponential bounds for derivatives on element boundaries
- Estimate asymptotic for $N_i \to \infty$
- Constants depend on k

Fundamental solutions estimates



Assume $\Gamma_e = \{z \in \mathbb{C} | \ |z| = R_0\}$, $\Gamma_i = \{z \in \mathbb{C} : \ |z| = R\}$

$$v \in V_e \Leftrightarrow v(x) = \sum_{j=1}^{N_e} c_j H_0^{(1)}(k|x - y_j|), \ y_j = Re^{i\frac{2\pi j}{N}}$$

Fundamental solutions estimates...

Define

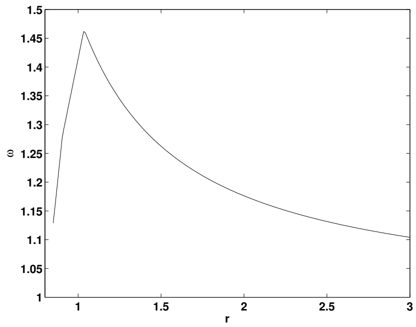
$$t^{(N_e)} := \min_{v \in V_e} \|u - v\|_{L^2(\Gamma_e)}$$

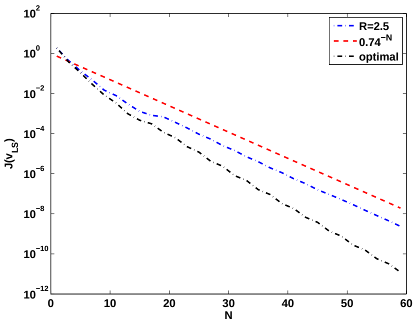
Theorem: Let $\rho := \max_i \frac{R_0}{|p_i|}$. For any $\epsilon > 0$ it holds that

$$t^{(N_e)} = \begin{cases} O\left(\left(\frac{R_0}{R} - \epsilon\right)^{-N_e}\right), & \frac{R_0}{R} < \rho^{\frac{1}{2}} \\ O\left((\rho - \epsilon)^{-\frac{N_e}{2}}\right), & \frac{R_0}{R} > \rho^{\frac{1}{2}} \end{cases}$$

- Estimates asymptotic for fixed k and $N_e \to \infty$
- Large radius R_0 leads to faster exponential convergence of MFS

Convergence for the square scatterer



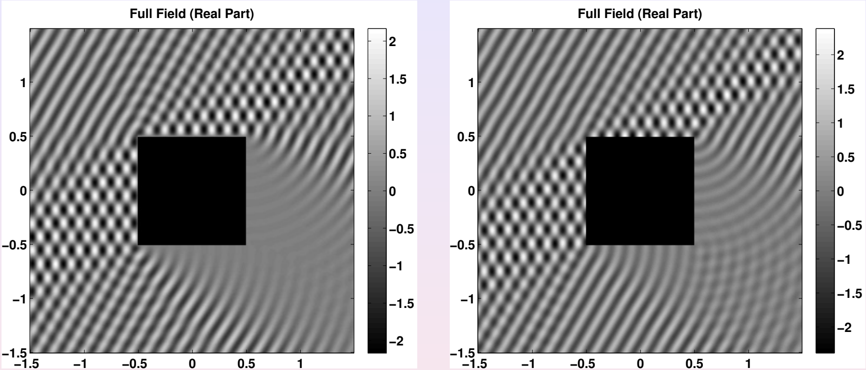


r: radius of outer circle

ω: Overall exponential rate of convergence

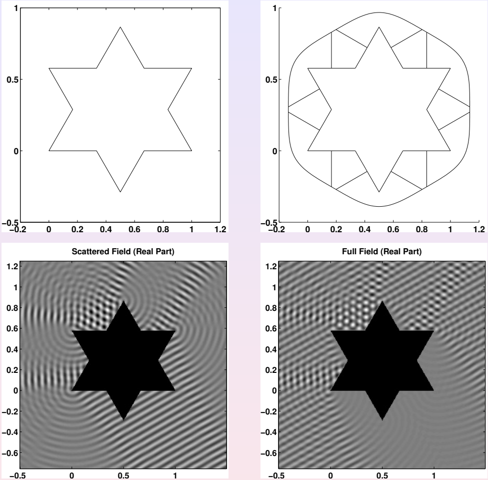
$k = 1$, N Bessel fct. in E_i, 2N fund. sol. in Ω_e^+.

From sound-soft to sound-hard scattering



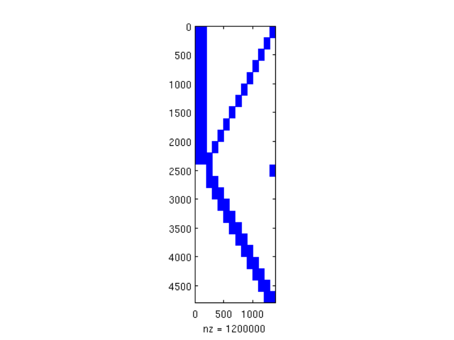
$$\sum_{j=1}^{N_i} c_j J_{\alpha_j}(kr) \sin \alpha j\theta \rightarrow \sum_{j=0}^{N_i} c_j J_{\alpha_j}(kr) \cos \alpha j\theta$$

A snowflake domain



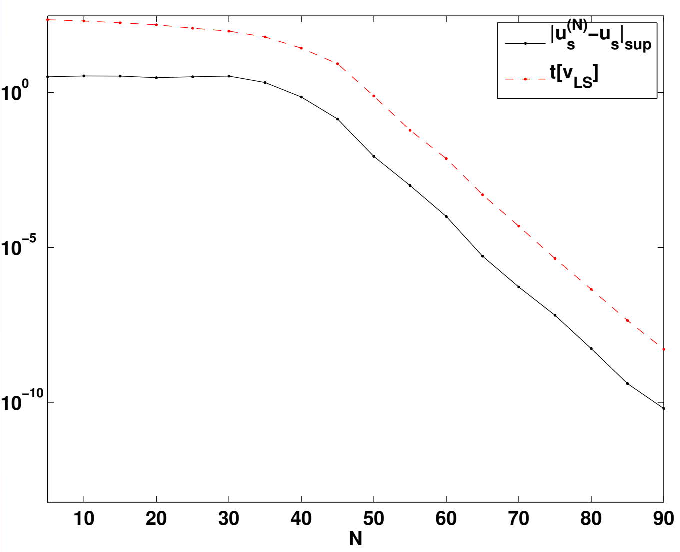
k	N	m	N_V	$t[v_{LS}]$	time
50	70	1980	1071	$3 \cdot 10^{-8}$	8 s
100	90	2460	1377	$4 \cdot 10^{-9}$	15 s
200	130	3660	1989	$5 \cdot 10^{-9}$	44 s
500	260	8700	3978	$2 \cdot 10^{-7}$	7 m

The structure of A



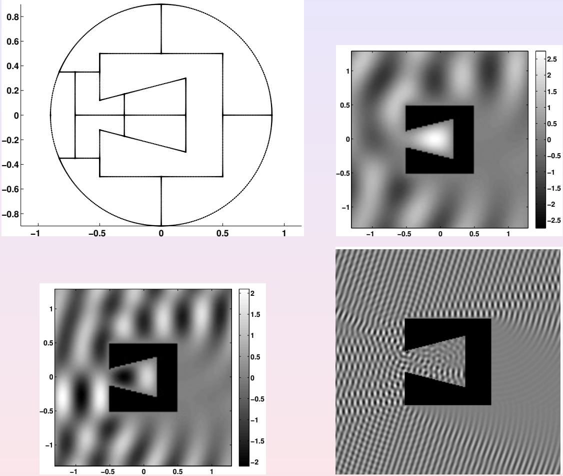
- A numerically singular
- Use backward stable least-squares solver [BB10]

Rate of Convergence



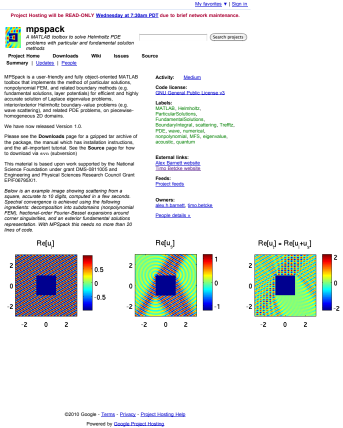
Convergence for $k = 100$

A cavity



k=100: 14 seconds for setup and solution ($t[v_{LS}] \approx 3 \cdot 10^{-8}$). 46 seconds for plotting on $6 \cdot 10^4$ grid points.

MPSPACK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mpspack

A MATLAB toolbox to solve Helmholtz PDE problems with particular and fundamental solution methods

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MPSpack is a user-friendly and fully object-oriented MATLAB toolbox that implements the method of particular solutions, nonpolynomial FEM, and related boundary methods (e.g. fundamental solutions, layer potentials) for efficient and highly accurate solution of Laplace eigenvalue problems, interior/exterior Helmholtz boundary-value problems (e.g. wave scattering), and related PDE problems, on piecewise-homogeneous 2D domains.

We have now released Version 1.0.

Please see the **Downloads** page for a gzipped tar archive of the package, the manual which has installation instructions, and the all-important tutorial. See the **Source** page for how to download via `svn` (subversion)

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Below is an example image showing scattering from a square, accurate to 10 digits, computed in a few seconds. Spectral convergence is achieved using the following ingredients: decomposition into subdomains (nonpolynomial FEM), fractional-order Fourier-Bessel expansions around corner singularities, and an exterior fundamental solutions representation. With MPSpack this needs no more than 20 lines of code.

Activity: Medium

Code license:
GNU General Public License v3

Labels:
MATLAB, Helmholtz, ParticularSolutions, FundamentalSolutions, BoundaryIntegral, scattering, Trefftz, PDE, wave, numerical, nonpolynomial, MFS, eigenvalue, acoustic, quantum

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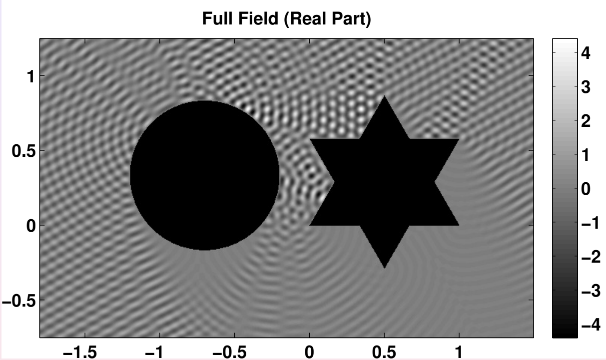
- Object-Oriented Matlab Toolbox
- Simple and fast solution of many interior and exterior Helmholtz and Laplace problems
- Extensive tutorial available
- All examples in this talk implemented in MPSPACK
- Manual mesh generation (to be changed in the future)

Multiply connected domains



k=100: Setup and solve around 7.5 min, $t[v_{LS}] \approx 4 \cdot 10^{-7}$

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Thanks!