Unified A Posteriori Error Control for all Nonstandard Finite Elements[1]

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[1] we know of

Model Example

Flux or stress field p in equilibrium equation $g + \operatorname{div} p = 0$ is approximated by piecewise constant p_ℓ and yields equilibrium residual

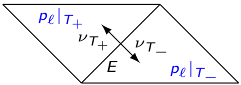
$$\operatorname{Res}(v) := \int_\Omega (g \cdot v - p_\ell : D_\ell v)\, dx \qquad \text{for } v \in V := H^1_0(\Omega; \mathbb{R}^m)$$

Endow V with norm $\|v\|_V := \|Dv\|_{L^2(\Omega)}$ s.t. $V^* \approx H^{-1}(\Omega)$ and

$$\|\operatorname{div}(p - p_\ell)\|_{V^*} = \|\operatorname{Res}\|_{V^*} := \sup_{v \neq 0} \frac{\operatorname{Res}(v)}{\|v\|_V}$$

Estimation by edge residuals $\eta_E := |E|(p_\ell|_{T_+} \cdot \nu_{T_+} + p_\ell|_{T_-} \cdot \nu_{T_-})$ on each interior edge $E = \partial T_+ \cap \partial T_-$ with outer unit normals $\nu_{T_\pm}$

$$\eta_\mathcal{E} := \left(\sum_{E \in \mathcal{E}} |\eta_E|^2 \right)^{1/2}$$



Estimation of Equilibrium Error

Up to data oscillation $\operatorname{osc}(g,\cdot)$, edge estimator $\eta_{\mathcal{E}}$ is reliable & efficient

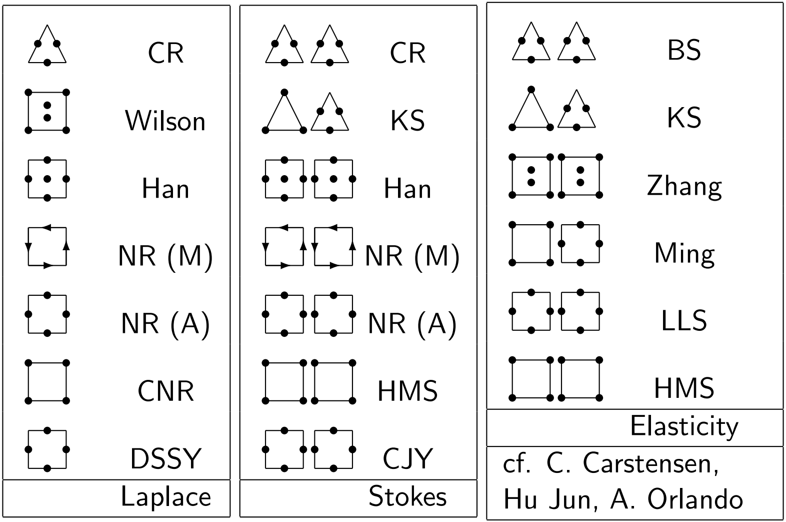
$$\|\,\mathrm{Res}\,\|_{V^*} \approx \eta_{\mathcal{E}} \pm \operatorname{osc}(g, \text{elements} \cup \text{edges})$$

Remark: Edge residual estimator is equivalent to many others (cf. Ainsworth/Oden, Babuška/Strouboulis, Verfürth)

Example: For triangulation of $\Omega \subset \mathbb{R}^2$ and first-order conforming or nonconforming FEM

$$p_\ell := D_\ell u_\ell \quad \text{and} \quad V_\ell^c := P_1(\mathcal{T}_\ell; \mathbb{R}^m) \cap V \subset \ker \mathrm{Res}$$

Schemes & Applications



Unified Analysis of. . .

- **applications:**
  Laplace, Stokes, Navier-Lamé, Maxwell equations. . .

- **schemes:**
  (all?!) conforming, nonconforming, tri/quad, mixed, mortar elements, dG, hanging nodes. . .

Goals of Unified Analysis

- Generalise analysis to cover many different discretisation schemes and applications in one framework
- Reduce repetition of similar mathematical arguments and focus on specific properties/difficulties
- No optimal constants but common point of departure and guiding principles

A Unified Approach

Generic Approach

- For each *Application*: Verify $\|\text{error}\| \approx \|\text{residual}\|_*$
- For each *Scheme*: Determine discrete space $V_\ell \subset \ker(\text{residual})$ and design computable lower/upper bounds of $\|\text{residual}\|_*$

Topics

- *Mixed Setting* for Unified A Posteriori Error Control
- Unified *Equilibrium Estimator*
- Analysis of *Consistency Residual*
- Applications: Poisson, Lamé, Stokes, ...

Mixed Setting for Unified Analysis

Abstract problem formulation:

Given $X_\ell \times Y_\ell \subset X \times Y$, $\mathcal{A} : X \times Y \to (X \times Y)^*$

$\ell := \ell_X + \ell_Y \in (X \times Y)^*$, find $(x, y) \in X \times Y$ s.t.

$$(PM) \qquad \mathcal{A}(x, y)(\xi, \eta) = \ell(\xi, \eta) \qquad \text{for all } (\xi, \eta) \in X \times Y$$

Given $a \in (X \times X)^*$, $\Lambda : Y \to X$, $b \in (X \times Y)^*$ with $b(\cdot, v) := a(\Lambda v, \cdot)$

$c \in (Y \times Y)^*$, $\mathcal{A}(x, y) := a(x, \cdot) - b(\cdot, y) + b(x, \cdot) + c(y, \cdot)$

(PM) then reads

$$a(x, \xi) - b(\xi, y) = \ell_X(\xi) \qquad \text{for all } \xi \in X$$
$$b(x, \eta) + c(y, \eta) = \ell_Y(\eta) \qquad \text{for all } \eta \in Y$$

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(PM) then reads

$$\begin{aligned} a(x,\xi) - b(\xi,y) &= \ell_X(\xi) \qquad \text{for all } \xi \in X \\ b(x,\eta) + c(y,\eta) &= \ell_Y(\eta) \qquad \text{for all } \eta \in Y \end{aligned}$$

Errors & Residuals in Unified Analysis

Given approx. $(x_\ell, \tilde{y}_\ell) \in X_\ell \times Y$ to (x, y), define $\text{Res} := \text{Res}_X + \text{Res}_Y$

(consistency) $$\text{Res}_X := \ell_X - a(x_\ell, \cdot) + b(\cdot, \tilde{y}_\ell) = \ell_X - a(x_\ell - \Lambda \tilde{y}_\ell)$$

(equilibrium) $$\text{Res}_Y := \ell_Y - b(x_\ell, \cdot) - c(\tilde{y}_\ell, \cdot)$$

Remarks:

- $\tilde{y}_\ell \in Y$ close to y_ℓ, not necessarily discrete
- Res_X involves piecewise gradient $D_\ell(y_\ell - \tilde{y}_\ell)$ of $y_\ell - \tilde{y}_\ell \notin Y$

Since $\mathcal{A}$ isomorphism, $\|\text{error}\| \approx \|\text{residual}\|_*$, i.e.

$$\|x - x_\ell\|_X + \|y - \tilde{y}_\ell\|_Y \approx \|\,\text{Res}_X\,\|_{X^*} + \|\,\text{Res}_Y\,\|_{Y^*}$$

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[Preparation] Generic Equilibrium Error Analysis

Two discrete spaces $V_\ell^c \subset V$ and $V_\ell^{nc} \subset H^1(\mathcal{T}_\ell; \mathbb{R}^m)$ with

$$V \xrightarrow{J} V_\ell^c \xrightarrow{\Pi} V_\ell^{nc}$$

(H1) $\exists$ H^1-stable Clément-type operator $J : V \to V_\ell^c$ into (conforming) subspace $V_\ell^c \subseteq V$ with first-order approximation property
(H2) V_ℓ^c and V_ℓ^{nc} piecewise smooth w.r.t. shape-regular $\mathcal{T}_\ell$
(H3) For $p_\ell \in L^2(\Omega; \mathbb{R}^{m\times n})$ $\exists\, \Pi : V_\ell^c \to V_\ell^{nc}$ s.t. $\forall v_\ell \in V_\ell^c$ $\forall T \in \mathcal{T}_\ell$

$$\|D(\Pi v_\ell)\|_{L^2(T)} \lesssim \|Dv_\ell\|_{L^2(\omega_T)}$$
$$\int_T v_\ell \, dx = \int_T \Pi v_\ell \, dx$$
$$\int_\Omega p_\ell : D_\ell v_\ell \, dx = \int_\Omega p_\ell : D_\ell(\Pi v_\ell) \, dx$$

[Result] Generic Equilibrium Error Analysis

Thm.[CEHL10$^+$]: Suppose $R_{\mathcal{T}} \in L^2(\mathcal{T}_\ell)$, $R_{\mathcal{E}} \in L^2(\bigcup \mathcal{E}_\ell)$ and $\mathrm{Res} : V_\ell^{nc} + V \to \mathbb{R}$ reads

$$\mathrm{Res}(v) := \int_\Omega R_{\mathcal{T}} \cdot v \, dx + \int_{\bigcup \mathcal{E}_\ell} R_{\mathcal{E}} \cdot \langle v \rangle \, ds$$

Suppose (H1)-(H3) and $V_\ell^{nc} \subset \ker \mathrm{Res}$

Then

$$\eta_\ell := \left(\sum_{E \in \mathcal{E}_\ell} h_E \| R_E \|^2_{L^2(E)} \right)^{1/2} = \| h_{\mathcal{E}}^{1/2} R_{\mathcal{E}} \|_{L^2(\bigcup \mathcal{E}_\ell)}$$

is reliable and efficient in the sense

$$\eta_\ell - \mathrm{osc}(R_{\mathcal{T}}, \mathcal{T}_\ell) - \mathrm{osc}(R_{\mathcal{E}}, \mathcal{E}_\ell) \lesssim \| \mathrm{Res} \|_{V^*} \lesssim \eta_\ell + \mathrm{osc}(R_{\mathcal{T}}, \{\omega_z : z \in \mathcal{K}_\ell\})$$

[Result] Generic Equilibrium Error Analysis

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Analysis of Consistency Error

Thm.[CEHL10$^+$]:

Given $X = L^2(\Omega)$, $x_\ell = D_\ell y_\ell \in X_\ell = P_0(\mathcal{T}_\ell; \mathbb{R}^n)$,

$Y = H^1_0(\Omega)$, $y_\ell \in Y_\ell = \mathrm{CR}_1(\mathcal{T}_\ell)$ and

$\mu_\ell := \min_{\eta \in Y} \|x_\ell - D\eta\|_{L^2(\Omega)}$

$$\begin{aligned}
\mu_\ell &\approx \min_{\eta_\ell \in P_1(\mathcal{T}_\ell)\cap Y} \|x_\ell - D_\ell \eta_\ell\|_{L^2(\Omega)} \\
&\approx \min_{\eta_\ell \in P_1(\mathcal{T}_\ell)\cap Y} \|h_{\mathcal{T}}^{-1}(y_\ell - \eta_\ell)\|_{L^2(\Omega)} \\
&\approx \|h_{\mathcal{T}}^{-1}(y_\ell - \mathcal{S} y_\ell)\|_{L^2(\Omega)} \\
&\approx \|h_{\mathcal{E}}^{-1/2}[y_\ell]\|_{L^2(\bigcup \mathcal{E}_\ell)} \\
&\approx \|h_{\mathcal{E}}^{1/2}[D_\ell y_\ell \cdot \tau_{\mathcal{E}}]\|_{L^2(\bigcup \mathcal{E}_\ell)}
\end{aligned}$$

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$$\begin{aligned}
\mu_\ell &\approx \min_{\eta_\ell \in P_1(\mathcal{T}_\ell)\cap Y} \|x_\ell - D_\ell \eta_\ell\|_{L^2(\Omega)} \\
&\approx \min_{\eta_\ell \in P_1(\mathcal{T}_\ell)\cap Y} \|h_\mathcal{T}^{-1}(y_\ell - \eta_\ell)\|_{L^2(\Omega)} \\
&\approx \|h_\mathcal{T}^{-1}(y_\ell - \mathcal{S} y_\ell)\|_{L^2(\Omega)} \\
&\approx \|h_\mathcal{E}^{-1/2}[y_\ell]\|_{L^2(\bigcup \mathcal{E}_\ell)} \\
&\approx \|h_\mathcal{E}^{1/2}[D_\ell y_\ell \cdot \tau_\mathcal{E}]\|_{L^2(\bigcup \mathcal{E}_\ell)}
\end{aligned}$$

[Application] Laplace Equation

Poisson model problem:

$$\Delta u = g \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega$$

leads to primal mixed formulation with

$$a(p,q) := \int_\Omega p \cdot q \, dx, \quad \Lambda u := D_\ell u, \quad b(q,u) = \int_\Omega D_\ell u \cdot q \, dx$$

Given $g \in L^2(\Omega)$, find $(p,u) \in X \times Y := L^2(\Omega;\mathbb{R}^n) \times H_0^1(\Omega)$ s.t.

$$\mathcal{A}(p,u)(q,v) = (g,v)_{L^2(\Omega)} \qquad \text{for all } (q,v) \in X \times Y$$

Since $\mathcal{A}$ isomorphism

$$\|p - p_\ell\|_{L^2} + \|u - \tilde{u}_\ell\|_{H^1} \approx \|\operatorname{Res}_X\|_{X^*} + \|\operatorname{Res}_Y\|_{Y^*}$$

[Application (cont.)] Laplace Equation

Treatment of different schemes in the unified framework:

Conforming FEM Discrete solution $u_\ell = \tilde{u}_\ell \in Y$ defines discrete flux $p_\ell := \Lambda u_\ell = D u_\ell$. Consistency error vanishes, equilibrium residual treated as suggested previously.

Nonconforming FEM Discrete solution u_ℓ defines discrete flux $p_\ell := D_\ell u_\ell$. Same analysis for equilibrium residual while consistency residual $\|p_\ell - D\tilde{u}_\ell\|_{L^2(\Omega)}$ can be bounded as before.

Mixed FEM Discrete solution is discrete flux p_ℓ and $u_\ell \notin Y$ is Lagrange multiplier. Equilibrium residual reduces to $\operatorname{osc}(g, \text{elements})$, consistency residual $\min_{\tilde{u}_\ell \in H^1_0(\Omega)} \|p_\ell - D\tilde{u}_\ell\|_{L^2(\Omega)}$ can be bounded as in [Carstensen, Math. Comp. 1997].

Stokes Equations

Let $a(u,v) := -\int_\Omega uv\ dx$, $\Lambda u := \operatorname{div} u$, $c(u,v) := \int_\Omega 2\mu\varepsilon(u) : \varepsilon(v)\ dx$

$\ell_X(p,q) := -\int_\Omega pq\ dx$ and $X := L^2_0(\Omega;\mathbb{R}^n)$, $Y := H^1_0(\Omega;\mathbb{R}^n)$

With $\varepsilon(v) := \operatorname{sym} Dv$ and deviatoric operator $\operatorname{dev}\sigma := \sigma - \frac{1}{n}(\operatorname{tr}(\sigma))\mathbf{1}$
the linear operator $\mathcal{A} : X \times Y \to (X \times Y)^*$ reads

$$\mathcal{A}(\sigma,u)(\tau,v) := \frac{1}{2\mu}(\operatorname{dev}\sigma, \operatorname{dev}\tau)_{L^2(\Omega)} - (\sigma,\varepsilon(v))_{L^2(\Omega)} - (\tau,\varepsilon(u))_{L^2(\Omega)}$$

$\mathcal{A}$ isomorphism...

Stokes Equations

Stokes problem: Given $g \in L^2(\Omega; \mathbb{R}^n)$, find $(\sigma, u) \in X \times Y$ s.t.

$$\mathcal{A}(\sigma, u)(\tau, v) = (g, v)_{L^2(\Omega)} \qquad \text{for all } (\tau, v) \in X \times Y$$

Conforming or nonconforming FEM yield u_ℓ and p_ℓ with
$\sigma = 2\mu\varepsilon(u) - p\mathbf{1}$ and $\sigma_\ell = 2\mu\varepsilon_\ell(u_\ell) - p_\ell\mathbf{1}$

Error control: Given FE solution (σ_ℓ, u_ℓ) to (σ, u), then

$$\|\sigma - \sigma_\ell\|_{L^2(\Omega)} \lesssim \min_{\tilde{u}_\ell \in Y} 2\mu\|\varepsilon_\ell(u_\ell - \tilde{u}_\ell)\|_{L^2(\Omega)} + \|\operatorname{Res}_Y\|_{Y^*} + \|\operatorname{div}_\ell u_\ell\|_{L^2(\Omega)}$$

Examples: all known conforming and nonconforming FEM

[Application] Linear Elasticity

Fourth-order elasticity tensor for $\lambda, \mu > 0$,

$$\mathbb{C}\tau := \lambda\,\mathrm{tr}(\tau)\mathbf{1} + 2\mu\tau \qquad \text{for all } \tau \in \mathbb{R}^{n\times n}$$

Let $a(\sigma,\tau) := \int_\Omega (\mathbb{C}^{-1}\sigma) : \tau \, dx$, $\Lambda u := \mathbb{C}\varepsilon(u)$ and

$X := L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$, $Y := H^1_0(\Omega; \mathbb{R}^n)$

The linear operator $\mathcal{A} : X \times Y \to (X \times Y)^*$ then reads

$$\mathcal{A}(\sigma,u)(\tau,v) := (\mathbb{C}^{-1}\sigma,\tau)_{L^2(\Omega)} - (\sigma,\varepsilon(v))_{L^2(\Omega)} - (\tau,\varepsilon(u))_{L^2(\Omega)}$$

$\mathcal{A}$ isomorphism, λ-independent operator norms of $\mathcal{A}$ and $\mathcal{A}^{-1}$

Linear Elasticity

Navier-Lamé problem: Given $g \in L^2(\Omega; \mathbb{R}^n)$, find $(\sigma, u) \in X \times Y$ s.t.

$$\mathcal{A}(\sigma, u)(\tau, v) = (g, v)_{L^2(\Omega)} \qquad \text{for } (\tau, v) \in X \times Y$$

For conforming or nonconforming FEM, $\sigma = \mathbb{C}\varepsilon(u)$ and $\sigma_\ell = \mathbb{C}\varepsilon_\ell(u_\ell)$

Error control: Given FE solution (σ_ℓ, u_ℓ) to (σ, u),

$$\|\sigma - \sigma_\ell\|_{L^2(\Omega)} \lesssim \min_{\tilde{u}_\ell \in Y} \|\varepsilon_\ell(u_\ell - \tilde{u}_\ell)\|_{L^2(\Omega)} + \|\operatorname{Res}_Y\|_{Y^*}$$

is robust for $\lambda \to \infty$

Examples: Conforming FEM, Kouhia&Stenberg, PEERS

Conclusions for Unified Analysis

- Framework for unified a posteriori analysis covers large class of applications & discretisation schemes
- Similarities are exposed and encountered obstacles/challenges guide development of specific error estimators
- Approach doesn't strive for most accurate/efficient estimators but rather provides an initial line of attack for as many problems as possible

Wrap-Up of Unified A Posteriori Analysis

- C. Carstensen: *A unifying theory of a posteriori finite element error control (Numer. Math. 2005)*
- C. Carstensen, Jun Hu, A. Orlando: *Framework for the a posteriori error analysis of nonconforming finite elements (SINUM 2007)*
- C. Carstensen, Jun Hu: *A Unifying Theory of A Posteriori Error Control for Nonconforming FEM (Numer. Math. 2007)*
- C. Carstensen, ME, R.H.W. Hoppe, C. Löbhard: *A Unifying Theory of A Posteriori Error Control (2010^{+})*

Extensions of Unified Analysis

Unified Analysis has also been applied/extended to

- Maxwell Equations [C. Carstensen, R.H.W. Hoppe 2009]
- Mortar FEM and higher order FEM
- dG FEM [C. Carstensen, T. Gudi, M. Jensen 2009]
- Hanging nodes [C. Carstensen, Jun Hu 2008]

References

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- C. Carstensen, T. Gudi, M. Jensen: *A unifying theory of a posteriori error control for discontinuous Galerkin FEM*
- C. Carstensen, R.H.W. Hoppe: *Unified framework for an a posteriori analysis of non-standard finite element approximations of H(curl)-elliptic problems*