The constraint of inextensibility imposed on a curve by a Lagrange multiplier

...with an Inf-Sup Puzzle

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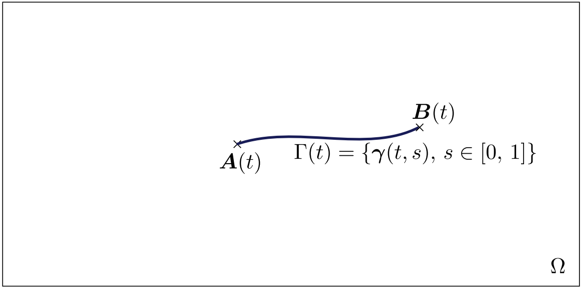
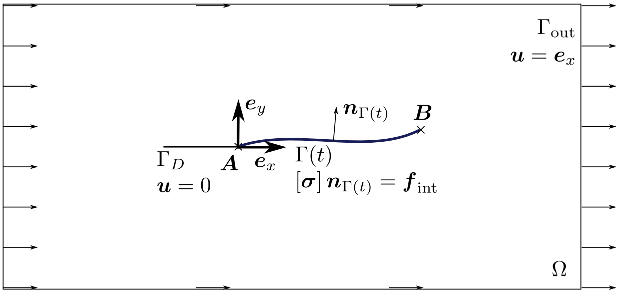
The Setting



$$\begin{cases} \gamma(t,s) = \gamma_0(s) + \int_0^t \mathbf{u}(\tau,\ \gamma(\tau,s))\,\mathrm{d}\tau, \\ J(\mathbf{u}) = \min_{\mathbf{v}\in V^{\mathrm{div}}(t)} J(\mathbf{v}) \end{cases}$$

with $V^{\mathrm{div}}(t) = \left\{\mathbf{v} \in H_0^1(\Omega)^2,\ \mathrm{div}_{\mathrm{s}}\,\mathbf{v} = 0 \text{ in } Z'(\Gamma(t))\right\} \cap V' + \mathbf{u}_D$

The Setting: the flow around a flag



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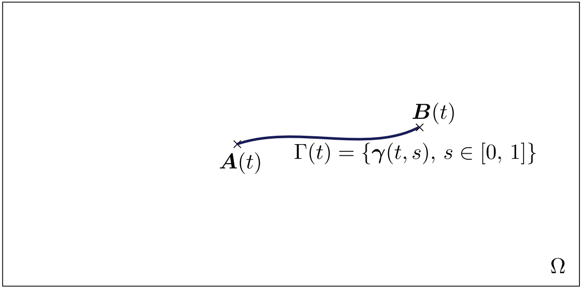
$$\cap \left\{\mathrm{div}\,\mathbf{v} = 0 \text{ in } L^2(\Omega)\right\} + \mathbf{u}_D$$

and $J(\mathbf{v}) = \displaystyle\int_\Omega |\mathbf{D}(\mathbf{v})|^2\, \mathrm{d}x$, where $\mathbf{D}(\mathbf{v}) = \dfrac{1}{2}\left(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathsf{T}}\right)$

The Setting: which functional spaces?



$$\begin{cases} \gamma(t,s) & = & \gamma_0(s) + \displaystyle\int_0^t \mathbf{u}(\tau,\ \gamma(\tau,s))\,\mathrm{d}\tau, \\ J(\mathbf{u}) & = & \displaystyle\min_{\mathbf{v}\in V^{\mathrm{div}}(t)} J(\mathbf{v}) \end{cases}$$

with $V^{\mathrm{div}}(t) = \left\{\mathbf{v} \in H_0^1(\Omega)^2,\ \mathrm{div}_s\,\mathbf{v} = 0 \text{ in } Z'(\Gamma(t))\right\} \cap V' + \mathbf{u}_D$

we have $Z(t) \subset H^{1/2}(\Gamma(t))$, there may be boundary conditions

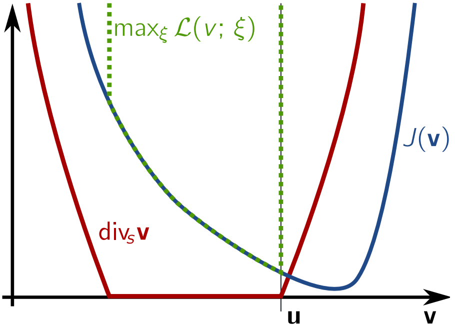
The Variational Setting: saddle-point approach

Introduce $\xi \in Z(t)$, and let:

$$\mathcal{L}(\mathbf{v}, \xi) = J(\mathbf{v}) - \int_\Gamma \xi \, \mathrm{div}_\mathbf{s} \, \mathbf{v} \, \mathrm{d}s,$$

the solution is such that:

$$\mathcal{L}(\mathbf{u}, \zeta) = \min_{\mathbf{v}} \max_{\boldsymbol{\xi}} \mathcal{L}(\mathbf{v}, \xi).$$



The Variational Setting: saddle-point

Assume $J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v})$, then it is characterised by:

$$\mathcal{L}(\mathbf{u},\,\zeta) = \min_{\mathbf{v}}\,\max_{\boldsymbol{\xi}}\,\frac{1}{2}a(\mathbf{v},\mathbf{v}) - \int_\Gamma \xi\,\mathrm{div_s}\,\mathbf{v}\,\mathrm{d}s$$

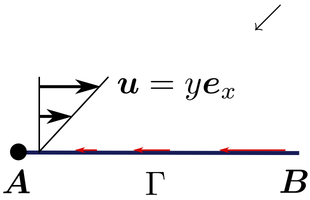
$$\Leftrightarrow \begin{cases} a(\mathbf{u},\mathbf{v}) - \int_\Gamma \zeta\,\mathrm{div_s}\,\mathbf{v}\,\mathrm{d}s & = 0 \quad \forall \mathbf{v}\in V_0 \\ \qquad\qquad\;\; - \int_\Gamma \xi\,\mathrm{div_s}\,\mathbf{u}\,\mathrm{d}s & = 0 \quad \forall \xi \in Z(t) \end{cases}$$

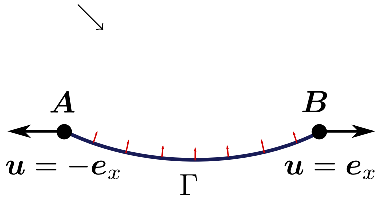
Lagrange multiplier ζ is called the *tension* of the curve (e.g., flag)

The Variational Setting: tension force

By a generalisation of Green's formula on curved domains,

$$\int_\Gamma \xi \, \mathrm{div}_s \, \mathbf{v} \, \mathrm{d}s = \quad -\int_\Gamma \frac{\partial \xi}{\partial s} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}s \quad -\int_\Gamma \kappa \xi \mathbf{n} \cdot \mathbf{v} \, \mathrm{d}s \quad + [\xi \mathbf{t} \cdot \mathbf{v}]_{\mathbf{A}}^{\mathbf{B}}$$

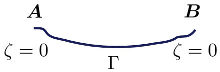




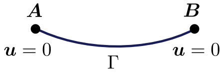
The Variational Setting : space $Z(t)$

If $\xi \in Z(t) \subset H^{1/2}(\Gamma)$, then $\dfrac{\partial \xi}{\partial s} \in (H_{00}^{1/2}(\Gamma))'$ but not $(H_0^{1/2}(\Gamma))'$

$$\int_\Gamma \xi \, \mathrm{div}_s \mathbf{v} \, ds = \quad -\int_\Gamma \frac{\partial \xi}{\partial s} \mathbf{t} \cdot \mathbf{v} \, ds \quad -\int_\Gamma \kappa \xi \mathbf{n} \cdot \mathbf{v} \, ds \quad + [\xi \mathbf{t} \cdot \mathbf{v}]_{\mathbf{A}}^{\mathbf{B}}$$



$\zeta, \xi \in Z(t) = H_{00}^{1/2}(\Gamma) \qquad \mathbf{u}, \mathbf{v} \in H^{3/2}(\Gamma)$



$\zeta, \xi \in Z(t) = H^{1/2}(\Gamma) \qquad \mathbf{u}, \mathbf{v} \in H_0^{3/2} \subset H_{00}^{1/2}(\Gamma)$

Note: $H_{00}^{1/2}(0, +\infty)$ is the space such that $\xi \in H^{1/2}(0, +\infty)$ and $\dfrac{\xi}{\sqrt{x}} \in L^2(0, +\infty)$.

2D Navier-Stokes flow around 1D inextensible membrane

Let $a(\mathbf{u},\mathbf{v}) = \int_\Omega 2\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\,\mathrm{d}x + \mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x + \mathrm{Re}_\Gamma\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}s$,
we have

$$a(\mathbf{u},\mathbf{v}) - \int_\Omega p\,\mathrm{div}\,\mathbf{v}\,\mathrm{d}x + \int_\Gamma \left(\frac{\partial\zeta}{\partial s}\mathbf{t} + \kappa\zeta\mathbf{n}\right)\mathbf{v}\,\mathrm{d}s = 0 \quad \forall \mathbf{v}\in V_0$$

$$-\int_\Omega q\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x = 0 \quad \forall q\in L^2_0(\Omega)$$

$$\int_\Gamma \left(\frac{\partial\xi}{\partial s}\mathbf{t} + \kappa\xi\mathbf{n}\right)\cdot\mathbf{u}\,\mathrm{d}s = 0 \quad \forall \xi\in Z(t)$$

Equivalent strong formulation (under regularity assumptions)

$$\mathrm{Re}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \mathbf{div}\ 2\mathbf{D}(\mathbf{u}) + \boldsymbol{\nabla} p = \mathbf{0} \qquad \text{in } \Omega\setminus\Gamma(t)$$

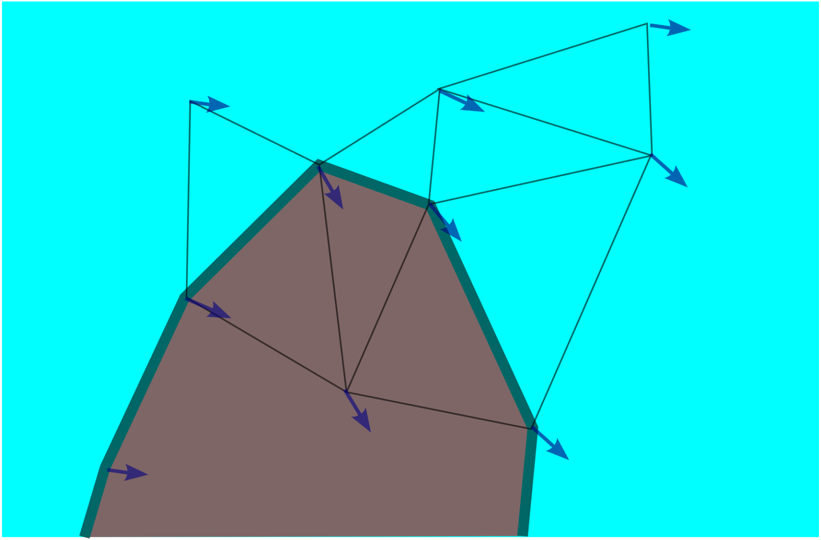
$$\mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega$$

$$\mathrm{div}_s\,\mathbf{u} = 0 \qquad \text{on } \Gamma(t)$$

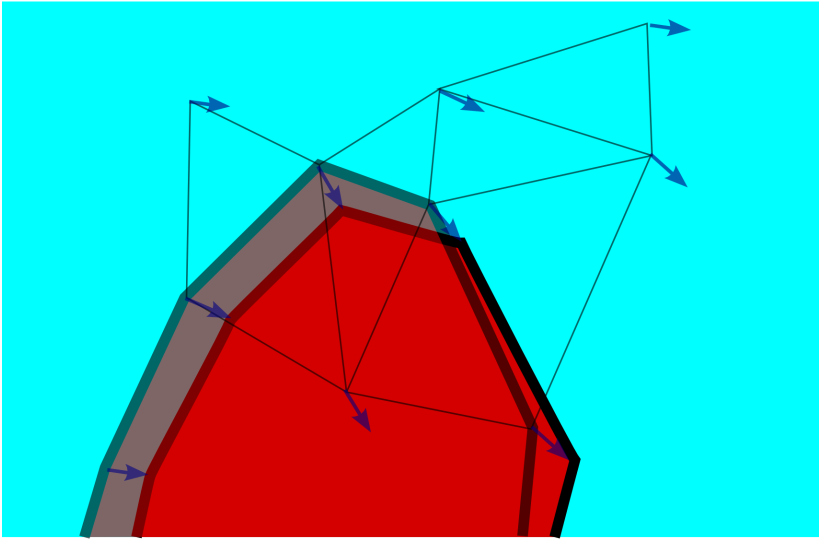
with a boundary condition on $\Gamma(t)$

$$-[p]\mathbf{n} + [2\mathbf{D}(\mathbf{u})]\mathbf{n} = \mathrm{Re}_\Gamma\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{\partial\zeta}{\partial s}\mathbf{t} + \kappa\zeta\mathbf{n}$$

Discretisation : 1. interface tracking



Discretisation : 1. interface tracking



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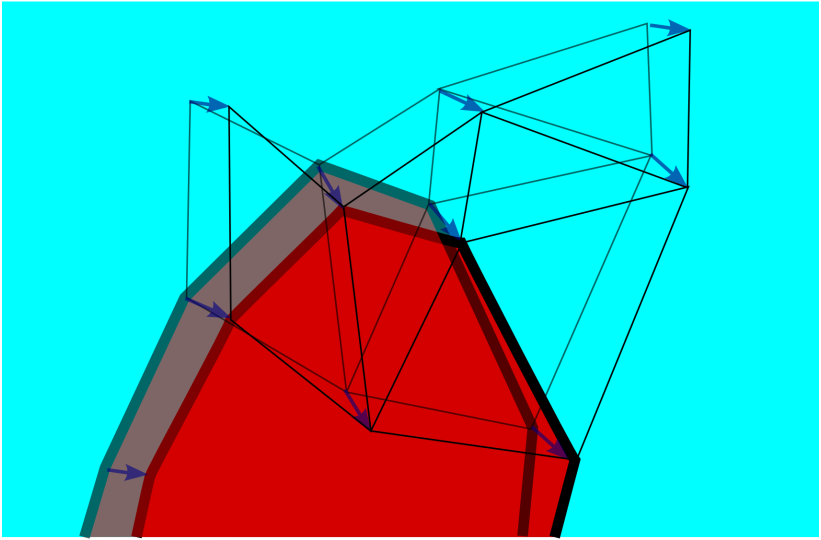
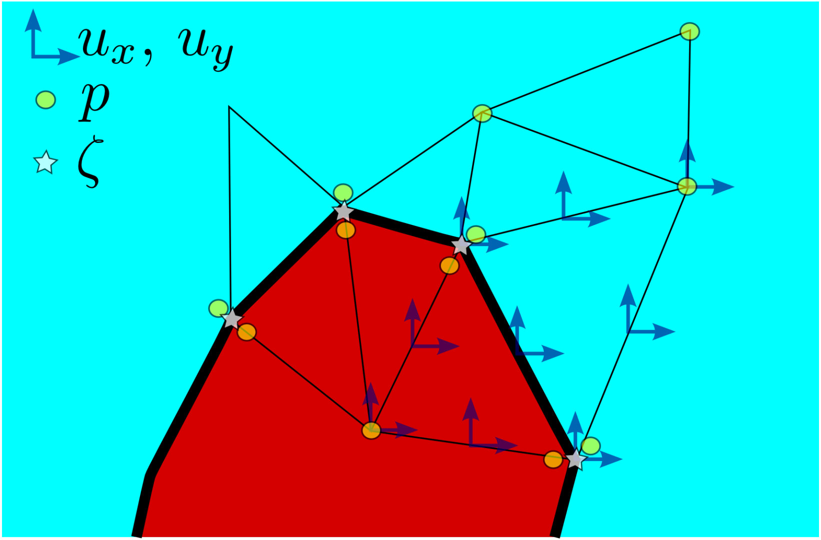


Discretisation : 2. finite elements



$$u_x,\ u_y \in P_2(\Omega) \cap C^0(\Omega), \qquad p \in P_1(\Omega) \cap C^0(\Omega \setminus \Gamma(t)),$$

$$\zeta \in P_1(\Gamma(t)) \cap C^0(\Gamma(t))$$

Resolution : augmented Lagrangian method

- Structure of the linear system :

$$\begin{pmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} U_h \\ P_h \\ Z_h \end{pmatrix} = \begin{pmatrix} F_h \\ 0 \\ 0 \end{pmatrix}$$

momentum
incompressibility
inextensibility

- In search of the saddle point: augmented Lagrangian $(\mathbf{u}_h, \{ p_h, \zeta_h \})$

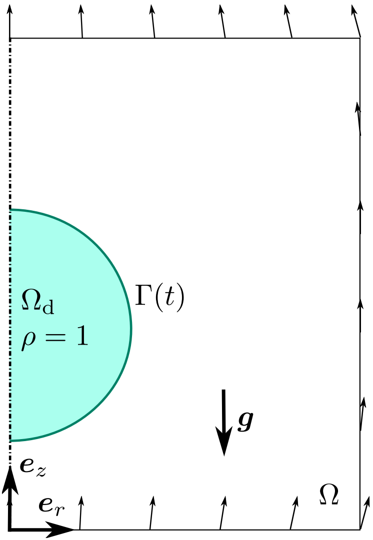
- Uzawa algorithm:

$$\begin{aligned} \left[A + r(B^T B + C^T C)\right] \; U_h^{n+1} &= F_h - B^T P_h^n - C^T Z_h^n \\ P_h^{n+1} &= P_h^n + rBU_h^{n+1} \\ Z_h^{n+1} &= Z_h^n + rCU_h^{n+1} \end{aligned}$$

- Residuals at convergence of order 10^{-8} (momentum eq.) and 10^{-12} (incompressibility, inextensibility eqs.)

Closed membrane: object rendered undeformable

by incompressibility and inextensibility constraints



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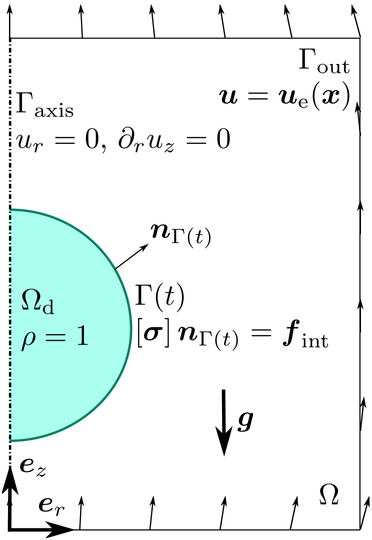
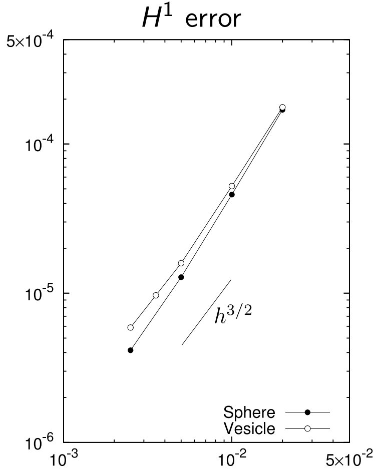
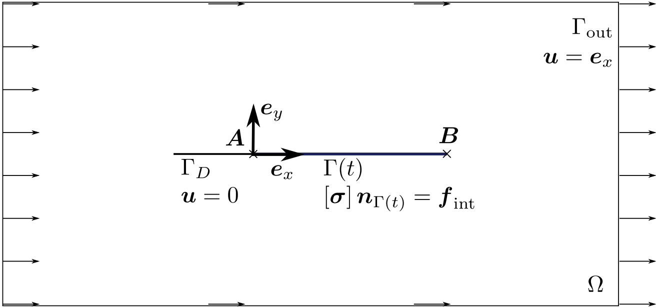
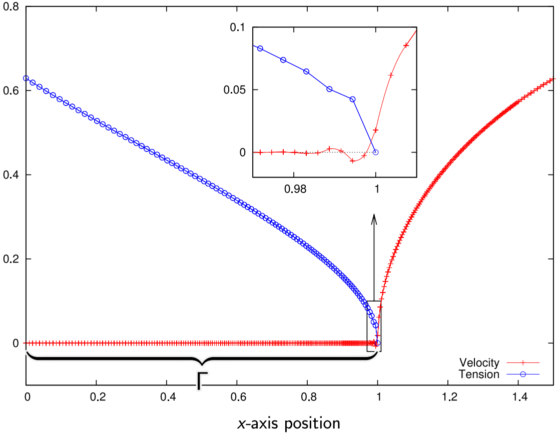




‘Open’ membrane...

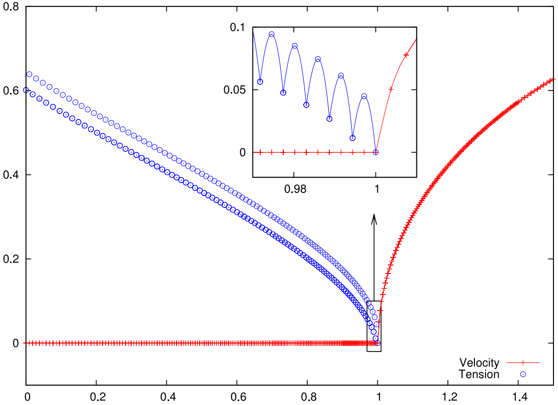


‘Open’ membrane... and open problem: inf-sup condition



$$\text{With } \zeta \in Z_h^1(t) = \left\{ \xi \in C^0(\Gamma),\ \xi_{|_e} \in P_1(e), \forall e \in \Gamma_h \right\}$$

'Open' membrane... and open problem: inf-sup condition



With $\zeta \in Z_h^1(t) = \{\xi \in C^0(\Gamma),\ \xi_{|_e} \in P_1(e), \forall e \in \Gamma_h\}$

With $\zeta \in Z_h^2(t) = \{\xi \in C^0(\Gamma),\ \xi_{|_e} \in P_2(e), \forall e \in \Gamma_h\}$

Stationary inertia-less flag aligned in a laminar flow field

- Solution is identical to the flow past a plate
- Comparison with the asymptotic solution at lee point