A posteriori error bounds in the $L^\infty(L^2)$-norm for the wave problem

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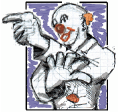
Joint work with O. Lakkis (Sussex), C. Makridakis (Crete)



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Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, open polygonal domain and $T > 0$.
Consider the problem

$$\begin{aligned} \partial_{tt} u - \nabla \cdot (a \nabla u) &= f && \text{in } \Omega \times (0, T] \\ u &= u_0 && \text{in } \Omega \times \{0\} \\ u_t &= u_1 && \text{in } \Omega \times \{0\} \\ u &= 0 && \text{on } \partial\Omega \times (0, T] \end{aligned}$$

for $a = a(x) \in C(\bar{\Omega})$ with $0 < \alpha_{\min} \leq a \leq \alpha_{\max}$ and $f \in L^\infty(0, T; L^2(\Omega))$

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Problem in variational form: Find u as above such that

$$\langle \partial_{tt} u, v \rangle + \langle a \nabla u, \nabla v \rangle = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega),$$

A posteriori error bounds for FEM for wave problems

- Johnson ('93), Süli ('96,'97) duality methods for wave problem as 1st order system.
- Akrivis, Makridakis & Nochetto ('06) time-discrete schemes for 1st order systems.
- Adjerid ('02), Bernardi & Süli ('05) $L^\infty(H^1)$-error a posteriori estimates.

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We are concerned with a posteriori bounds for the $L^\infty(L^2)$-error.

Semi-discrete FEM: Find $U \in V_h$:

$$\langle U_{tt}, V \rangle + a(U, V) = \langle f, V \rangle \quad \forall V \in V_h,$$

where $a(U, V) = \langle a\nabla U, \nabla V \rangle$.

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Fully-discrete FEM: for $n = 1, \ldots, N$, find $U^n \in V_h^n$:

$$\langle \partial^2 U^n, V\rangle + a(U^n, V) = \langle f^n, V\rangle \quad \forall V \in V_h^n,$$

where $f^n := f(t^n, \cdot)$, $k_n = t^{n+1} - t^n$ and

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$$\partial^2 U^n := \frac{\partial U^n - \partial U^{n-1}}{k_n}, \quad \partial U^n := \begin{cases} \dfrac{U^n - U^{n-1}}{k_n}, & \text{for } n = 1, 2, \dots, N, \\ V^0 := \pi^0 u_1 & \text{for } n = 0, \end{cases} \tag{1}$$

with $U^0 := \pi^0 u_0$, and $\pi^0 : L^2(\Omega) \to V_h^0$ suitable projection.

Semi-discrete case We shall make use the ideas of reconstruction, introduced by Makridakis & Nochetto ('03), whereby

$$u - U = \underbrace{u - w}_{\text{"evolution error"}} + \underbrace{w - U}_{\text{"elliptic error"}}$$

with $w \in H_0^1(\Omega)$ the elliptic reconstruction of U.

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Fully-discrete case

Suitable space-time reconstruction (cf. Lakkis & Makridakis ('06)), together with (classical) Baker's techinique Baker ('76).

Definition

We define the elliptic reconstruction $w \in H_0^1(\Omega)$ of U to be the solution of the elliptic problem

$$a(w, v) = \langle AU - \Pi f + f, v \rangle \quad \forall v \in H_0^1(\Omega),$$

where $\Pi : L^2(\Omega) \to V_h$ denotes the orthogonal L^2-projection on the finite element space V_h, and

$$A : V_h \to V_h,$$

denotes the (space-)discrete operator defined by

$$\text{for } Z \in V_h, \quad \langle -AZ, V \rangle = a(Z, V) \quad \forall V \in V_h.$$

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Baker's technique: set $v = v(t, \cdot) = \int_t^\tau \rho(s, \cdot)\mathrm{d}s$

Baker ('76)

Theorem

The following error bound holds:

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^2(\Omega))} \leq & \|\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big) \\ & + 2\int_0^T \|\epsilon_t\| + C_{a,T}\|u_1 - U_t(0)\|,\end{aligned}$$

where $C_{a,T} := \min\{2T, \sqrt{2C_\Omega/\alpha_{\min}}\}$*, where* C_Ω *is the PF constant.*

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Controlling ϵ, ϵ_t: use L^2-norm a posteriori bounds for elliptic problems

$$\|\epsilon\| \leq \mathcal{E}_{L^2}(U, AU - \Pi f + f, \mathcal{T}).$$

$$\|\epsilon_t\| \leq \mathcal{E}_{L^2}(U_t, (AU - \Pi f + f)_t, \mathcal{T}).$$

e.g., Verfürth ('96), Ainsworth & Oden ('01)...

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$$U(t) := \frac{t - t^{n-1}}{k_n} U^n + \frac{t^n - t}{k_n} U^{n-1} - \frac{(t - t^{n-1})(t^n - t)^2}{k_n} \partial^2 U^n,$$

for $t \in (t^{n-1}, t^n]$, $n = 1, \ldots, N$.

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for $t \in (t^{n-1}, t^n]$, $n = 1, \ldots, N$.

Note: $U(t) \in C^1(0, t^N)$ and $U(t^n) = U^n$ and $U_t(t^n) = \partial U^n$ and...

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Remark (vanishing moment property)

The remainder $\mu^n(t)$ satisfies

$$\int_{t^{n-1}}^{t^n} \mu^n(t)\,\mathrm{d}t = 0,$$

Definition

Let U^n, $n = 0, \ldots, N$, be the fully-discrete solution computed by the method, $\Pi^n : L^2(\Omega) \to V_h^n$ be the orthogonal L^2-projection, and $A^n : V_h^n \to V_h^n$ to be the discrete operator defined by

$$\text{for } q \in V_h^n, \quad \langle A^n q, \chi \rangle = a(q, \chi) \quad \forall \chi \in V_h^n.$$

We define the *elliptic reconstruction* $w^n \in H_0^1(\Omega)$ such that

$$a(w^n, v) = \langle g^n, v \rangle \quad \forall v \in H_0^1(\Omega),$$

with

$$g^n := A^n U^n - \Pi^n f^n + \bar{f}^n,$$

where $\bar{f}^0(\cdot) := f(0, \cdot)$ and $\bar{f}^n(\cdot) := k_n^{-1} \int_{t^{n-1}}^{t^n} f(t, \cdot)\mathrm{d}t$ for $n = 1, \ldots, N$. (Special care for $n = 1$, also)

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$$w(t) := \frac{t - t^{n-1}}{k_n} w^n + \frac{t^n - t}{k_n} w^{n-1} - \frac{(t - t^{n-1})(t^n - t)^2}{k_n} \partial^2 w^n,$$

Theorem

$$\begin{aligned}\|e\|_{L^\infty(0,t^N;L^2(\Omega))} \leq & \|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big) \\ & + 2\Big(\int_0^{t^N} \|\epsilon_t\| + \sum_{i=1}^{4} \eta_i(t^N)\Big) + C_{a,N}\|u_1 - V^0\|,\end{aligned} \tag{2}$$

where $C_{a,N} := \min\{2t^N, \sqrt{2C_\Omega/\alpha_{\min}}\}$.

We have made use of the following notation:
mesh change indicator

$$\eta_1(\tau) := \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \|(I - \Pi^j)U_t\| + \int_{t^{m-1}}^{\tau} \|(I - \Pi^m)U_t\| + \sum_{j=1}^{m-1} (\tau - t^j)\|(\Pi^{j+1} - \Pi^j)\partial U^j\| + \tau\|(I - \Pi^0)V^0(0)\|$$

evolution error indicator

$$\eta_2(\tau) := \int_0^\tau \|\mathcal{G}\|,$$

where $\mathcal{G} : (0, T] \to \mathbb{R}$ with $\mathcal{G}|_{(t^{j-1},t^j]} := \mathcal{G}^j$, $j = 1, \ldots, N$ and

$$\mathcal{G}^j(t) := \frac{(t^j - t)^2}{2}\partial g^j - \Big(\frac{(t^j - t)^4}{4k_j} - \frac{(t^j - t)^3}{3}\Big)\partial^2 g^j - \gamma_j,$$

with $\gamma_j := \gamma_{j-1} + \frac{k_j^2}{2}\partial g^j + \frac{k_j^3}{12}\partial^2 g^j$, $j = 1, \ldots, N$, $\gamma_0 = 0$;

data error indicator

$$\eta_3(\tau) := \frac{1}{2\pi} \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} k_j^3 \|\bar{f}^j - f\|^2 \Big)^{1/2} + \Big(\int_{t^{m-1}}^{\tau} k_m^3 \|\bar{f}^m - f\|^2 \Big)^{1/2};$$

time reconstruction error indicator

$$\eta_4(\tau) := \frac{1}{2\pi} \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} k_j^3 \|\mu^j \partial^2 U^j\|^2 \Big)^{1/2} + \Big(\int_{t^{m-1}}^{\tau} k_m^3 \|\mu^m \partial^2 U^m\|^2 \Big)^{1/2}.$$

Finally, we bound the ϵ and ϵ_t terms using

Lemma

We have

$$\|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\|\epsilon(0)\| \leq \delta_1(t^N) + \sqrt{2}C_{\mathrm{el}}\mathcal{E}^0,$$

where

$$\delta_1(t^N) := \max\Big\{\frac{8k_1}{27}C_{\mathrm{el}}\mathcal{E}(V^0, \partial g^0, \mathcal{T}^0),$$
$$\Big(\frac{35}{27} + \frac{31}{27}\max_{1\leq j\leq N}\frac{k_j}{k_{j-1}}\Big)\max_{0\leq j\leq N}\big(C_{\mathrm{el}}\mathcal{E}^j + C_\Omega\alpha_{\min}^{-1}\|\bar{f}^j - f^j\|\big)\Big\},$$

with $\mathcal{E}^j := \mathcal{E}(U^j, A^jU^j - \Pi^jf^j + f^j, \mathcal{T}^j)$, $j = 0, 1, \ldots, N$.

Lemma

We have

$$\int_0^{t^N} \|\epsilon_t\| \le \delta_2(t^N),$$

where

$$\delta_2(t^N) := \frac{2}{3}\sum_{j=0}^{N}(2k_j + k_{j+1})\Big(C_{\mathrm{el}}\mathcal{E}_\partial^j + C_\Omega \alpha_{\min}^{-1}\|\partial f^j - \partial \bar{f}^j\|\Big),$$

with

$$\mathcal{E}_\partial^j := \mathcal{E}(\partial U^j, \partial(A^j U^j) - \partial(\Pi^j f^j) + \partial f^j, \hat{\mathcal{T}}^j), \quad j = 0, 1, \ldots, N.$$

Current work: (G., Lakkis & Makridakis, in preparation)

- explicit methods for wave problem
- high order time-stepping for wave problem