Error analysis of the inverse Poisson problem

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May 20th–21st, 2010

Outline

About the inverse Poisson problem

- Occures in indirect measurement problems, where the underlying physics can be modeled using the Poisson equation
- Finite number of direct measurements
- Ill-conditioned
- Some examples of applications are EEG, inverse ECG and inverse heat flow problems

The forward problem

- The forward problem models how the measured quantities depend on the quantity of interest
- The physical model

$$\begin{aligned} -\Delta u &= f \quad x \in \Omega \\ u &= 0 \quad x \in \partial\Omega \end{aligned}$$

- The measurements are modeled as linear functionals on the function u
- The measurement vector

$$m = \begin{pmatrix} h_1(u) \\ h_2(u) \\ \vdots \\ h_n(u) \end{pmatrix} = Hu = HK^{-1}f$$

The inverse problem

- The problem: given a measurement m, find a reconstruction of the loading function f.
- With suitable additional information of f in the form of regularization and boundary conditions, the problem can be reduced to a quadratic minimization problem

$$f^r = \underset{f \in F}{\operatorname{argmin}} \|HK^{-1}f - m\|^2 + b(f - \bar{f}, f - \bar{f})$$

- A very common type of regularization is a smoothness prior, for which it holds that

$$|b(f, g)| \leq \gamma \|f\|_1 \|g\|_1 \qquad \forall f, g \in F$$

and

$$b(f, f) \geq \alpha \|f\|_1^2 \qquad \forall f \in F$$

Variational form

- The minimization problem is written as a variational problem: find $f^r \in F$ such that

$$a(f^r, g) = l(g) \qquad \forall g \in F,$$

where

$$a(f, g) = (HK^{-1}f)^T(HK^{-1}g) + b(f, g)$$

and

$$l(g) = m^T(HK^{-1}g) + b(\bar{f}, g)$$

Properties of the bilinear form

- Continuity

$$
\begin{aligned}
|a(f,g)| &\leq |(HK^{-1}f)^T(HK^{-1}g)| + |b(f,g)| \\
&\leq C\|H\|^2_{-2}\|K^{-1}f\|_2\|K^{-1}g\|_2 + \gamma\|f\|_1\|g\|_1 \\
&\leq C\|H\|^2_{-2}\|f\|_0\|g\|_0 + \gamma\|f\|_1\|g\|_1 \\
&\leq C\|H\|^2_{-2}\|f\|_1\|g\|_1
\end{aligned}
$$

- Coercivity

$$\alpha\|f\|_1^2 \leq b(f,f) \leq a(f,f)$$

- Unfortunately $a(f,g)$ and $l(g)$ cannot be evaluated directly

Modified problem

- Replace $K^{-1}f$ with the FE solution $K_h^{-1}f$
- Modified problem: find $\hat{f}^r \in F$ such that

$$\hat{a}(\hat{f}^r, g) = \hat{l}(g) \qquad \forall g \in F,$$

where

$$\hat{a}(f, g) = (HK_h^{-1}f)^T(HK_h^{-1}g) + b(f, g)$$

and

$$\hat{l}(g) = m^T(HK_h^{-1}g) + b(\bar{f}, g)$$

Modified problem

- The differences in the bilinear and linear forms are

$$a(f,g) = \hat{a}(f,g) + E_a(f,g)$$

and

$$l(g) = \hat{l}(g) + E_l(g),$$

where

$$\begin{aligned} E_a(f,g) = &(H(K^{-1} - K_h^{-1})f)^T (HK_h^{-1} g) + \\ &(HK_h^{-1} f)^T (H(K^{-1} - K_h^{-1})g) + \\ &(H(K^{-1} - K_h^{-1})f)^T (H(K^{-1} - K_h^{-1})g) \end{aligned}$$

and

$$E_l(g) = m^T (H(K^{-1} - K_h^{-1})g)$$

Consistency error

- Using the definitions of f^r and $\hat{f}^r$ and the operators $E_a(f,g)$ and $E_l(g)$, one gets

$$a(f^r - \hat{f}^r, g) = E_l(g) - E_a(\hat{f}^r, g) \quad \forall g \in F$$

- Now

$$\begin{aligned} |E_l(g)| &\le \|m\| \|H(K^{-1} - K_h^{-1})g\| \\ &\le C\|m\| \|H\|_0 \|K^{-1}g - K_h^{-1}g\|_0 \\ &\le Ch^2 \|m\| \|H\|_0 \|g\|_0 \end{aligned}$$

and

$$|E_a(\hat{f}^r, g)| \le Ch^2 \|H\|_0^2 \|\hat{f}^r\|_0 \|g\|_0$$

Consistency error

- The consistency error can now be estimated

$$\begin{aligned}\alpha\|f^r-\hat{f}^r\|_1^2 &\le a(f^r-\hat{f}^r, f^r-\hat{f}^r)\\ &\le |E_l(f^r-\hat{f}^r)|+|E_a(\hat{f}^r, f^r-\hat{f}^r)|\\ &\le Ch^2(\|m\|\|H\|_0+\|H\|_0^2\|\hat{f}^r\|_0)\|f^r-\hat{f}^r\|_0\\ &\le Ch^2(\|m\|\|H\|_0+\|H\|_0^2\|\hat{f}^r\|_0)\|f^r-\hat{f}^r\|_1\end{aligned}$$

- Thus

$$\|f^r-\hat{f}^r\|_1\le\frac{C}{\alpha}h^2(\|m\|\|H\|_0+\|H\|_0^2\|\hat{f}^r\|_0)$$

Discretization error

- The modified variational problem is solved in a finite dimensional subspace: find $\hat{f}_h^r \in F_h$ such that

$$\hat{a}(\hat{f}_h^r, g) = \hat{l}(g) \quad \forall g \in F_h$$

- Standard estimate

$$\|\hat{f}^r - \hat{f}_h^r\|_1 \leq \frac{C}{\alpha} h \|\hat{f}^r\|_2$$

- Nitsche's trick

$$\|\hat{f}^r - \hat{f}_h^r\|_0 \leq \frac{C}{\alpha} h^2 \|\hat{f}^r\|_2$$

Total error

- Combining the estimates

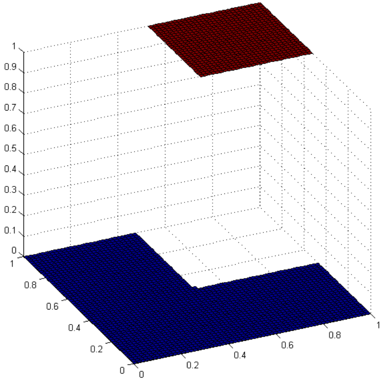
$$\begin{aligned}\|f^r - \hat{f}_h^r\|_1 &= \|f^r - \hat{f}^r + \hat{f}^r - \hat{f}_h^r\|_1 \\ &\leq \|f^r - \hat{f}^r\|_1 + \|\hat{f}^r - \hat{f}_h^r\|_1 \\ &\leq \frac{C}{\alpha}(h^2(\|m\|\|H\|_0 + \|H\|_0^2\|\hat{f}^r\|_0) + h\|\hat{f}^r\|_2) \\ &\leq \frac{C}{\alpha} h\|\hat{f}^r\|_2\end{aligned}$$

- For L^2-norm

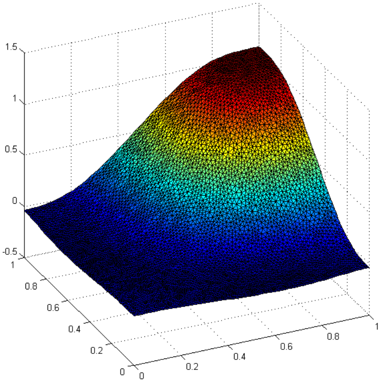
$$\begin{aligned}\|f^r - \hat{f}_h^r\|_0 &\leq \|f^r - \hat{f}^r\|_1 + \|\hat{f}^r - \hat{f}_h^r\|_0 \\ &\leq \frac{C}{\alpha} h^2(\|m\|\|H\|_0 + \|H\|_0^2\|\hat{f}^r\|_0 + \|\hat{f}^r\|_2)\end{aligned}$$

Test setting

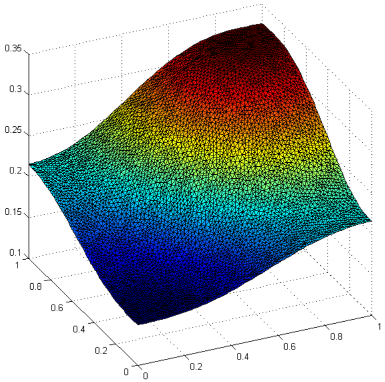
- $b(f, g) = \alpha^2 \|f\|_1 \|g\|_1$
- Original load as shown



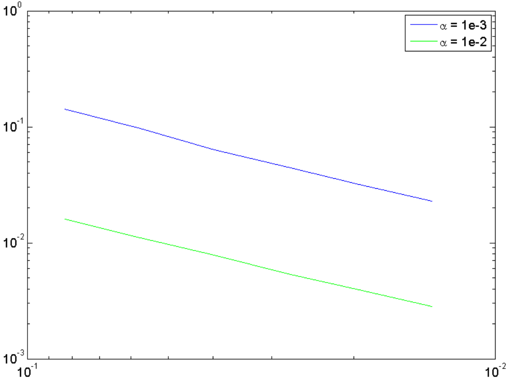
Reconstruction with $\alpha = 10^{-3}$



Reconstruction with $\alpha = 10^{-2}$

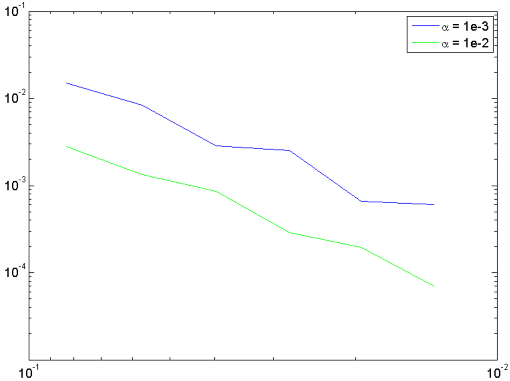


Convergence in H^1-norm



blue: $h^{1.01}$, green: $h^{0.96}$

Convergence in L^2-norm



blue: $h^{1.89}$, green: $h^{2.01}$

Conclusions & future work

- Regularized inverse Poisson problem is very similar to the forward Poisson problem
- Consistency error is at most the same order of magnitude as the discretization error
- Error is inversely proportional to the amount of regularization

- How about problems with limited regularity?
    - A posteriori error estimation?
- What happens when measurement functionals are not in L^2?