$H(\text{div})$-conforming Finite Elements for the Brinkman Problem

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Outline of the talk

- The Brinkman problem
- Motivation - why use $H(\text{div})$-conforming elements
- Problem setting - the non-conforming framework
- Local postprocessing
- A word on a posteriori
- Hybridization of the system
The Brinkman model

- Describes the flow of a viscous fluid in a porous medium
- Applicable to materials of very high porosity, e.g.
  - Sands, porous stones, petroleum engineering
  - Heat pipes
Reservoir modelling

- Oil reservoirs are natural multi-scale problems
  - Field scale - $10^{-1} - 100$ kilometres
  - Mesoscale - $10^{-1} - 100$ metres
  - Microscale - laboratory sample size
- Multiscale finite element methods or upscaling?
Reservoir modelling

- Typical properties of oil fields:
  - Long cracks, vugs
  - Rock of varying porosity
- Large jumps in parameter values
Example: realistic data

SPE10 comparative solution test based on actual data from Tarbert / Upper Ness formations
Example: realistic data

Permeability over 5000 millidarcy (void space)
Example: realistic data

Permeability under 0.01 millidarcy (no-flow zone)
The strong form

\[-t^2 \Delta u + u + \nabla p = f, \quad \text{in } \Omega\]
\[\text{div } u = g, \quad \text{in } \Omega\]

The related weak formulation is

\[\underbrace{t^2 (\nabla u, \nabla v) + (u, v) - \text{(div } v, p) - \langle \frac{\partial v}{\partial n}, p \rangle_{\partial \Omega}}_{a(u,v)} = (f, v)\]

\[-(\text{div } u, q) = (g, q)\]
The problem setting

- The Brinkman problem lies between the Stokes and the Darcy problems.
- For the Darcy case, we have the pairing $H(\text{div}, \Omega) \times L^2(\Omega)$.
- A non-conforming approximation for the Stokes part.
- Solution: Nitsche’s method to enforce tangential continuity.
Motivation

$H(\text{div})$-conforming approximation gives

- An elementwise mass preserving method
- Useful tools for the error analysis
- Optimal convergence rate
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Properties of the related pressure approximation
- Low-order approximation $\rightarrow$ very few DOFs
- Superconvergence $\rightarrow$ local postprocessing
- Optimal convergence rate
The FE spaces

- We use the BDM spaces of order $k$

$$V_h^{BDM} = \{ \mathbf{v} \in H(\text{div}, \Omega) \mid \mathbf{v}|_K \in [P_k(K)]^n \ \forall K \in \mathcal{K}_h \},$$

$$Q_h = \{ q \in L^2(\Omega) \mid q|_K \in P_{k-1}(K) \ \forall K \in \mathcal{K}_h \}.$$

- This pairing satisfies the equilibrium property

$$\text{div } V_h \subset Q_h$$

- Only the normal component of the flux is continuous
Nitsche’s method

To get a stable formulation, a modified bilinear form is introduced

$$a_h(u, v) = (u, v) + t^2 \sum_{K \in \mathcal{K}_h} (\nabla u, \nabla v)_K$$

$$+ t^2 \sum_{E \in \mathcal{E}_h} \left\{ \frac{\alpha}{h_K} \langle [u], [v] \rangle_E - \langle \{ \frac{\partial u}{\partial n} \}, [v] \rangle_E - \langle \{ \frac{\partial v}{\partial n} \}, [u] \rangle_E \right\}.$$

Original idea due to Nitsche in the 70s
The mesh dependent norms

For the flux $u$ we use

$$\|u\|_{t,h}^2 = \|u\|^2 + t^2 \sum_{K \in \mathcal{K}_h} \|\nabla u\|_{0,K}^2 + t^2 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[u \cdot \tau]\|_{0,E}^2.$$

For the pressure $p$

$$\|p\|_{t,h}^2 = \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla p\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} \frac{h_E}{h_E^2 + t^2} \|[p]\|_{0,E}^2.$$

Idea: use the norms from primal mixed formulation for the dual mixed formulation!
A priori results

We have the following quasioptimal result

\[ \| u - u_h \|_{t,h} + \| P_h p - p_h \|_{t,h} \leq C \| u - R_h u \|_{t,h}. \]

The constant \( C \) is independent of the parameter \( t \).

Assuming sufficient regularity, this gives optimal convergence rates for all parameter values.

Noteworthy: a superconvergence result for the pressure.
The postprocessing method

- Optimal order convergence for
  \[ \|u - u_h\|_{t,h} + \|p - p_h^\ast\|_{t,h} : \]
  - \( h^{k+1} \) rate in the pure Darcy case \( t = 0 \)
  - \( h^k \) rate in the case \( t > 0 \) → optimal rate for Stokes

- Allows the use of residual-based a posteriori error estimates

- Performed elementwise, thus computationally cheap
Convergence test

The problem changes numerically at $t = h$!
SPE10: layer 67
A word on a posteriori

- We have developed a sharp and reliable residual-based estimator
- Analysis relies on
  - The saturation assumption
  - Interpolation properties of $R_h$
  - The equilibrium property $\text{div } V_h \subset Q_h$
  - Definition of the postprocessing method
Hybridization

- Darcy: enforce normal continuity via Lagrange multipliers
  - Symmetric, positive definite system
- Nitsche adds connections for the flux variable
  - Add another Lagrange multiplier for the jump
- Makes domain decomposition easy
- Adaptive skeleton mesh?
Matrix after hybridization
Conclusions

- $H(\text{div})$-conforming elements can be extended to cover the case of viscous flow in the Brinkman model
- Numerically light postprocessing scheme
- Optimal a priori results
- Reliable and sharp a posteriori indicator
- Applications to multiscale FEM?