$H(\mathrm{div})$-conforming Finite Elements for the Brinkman Problem

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Outline of the talk

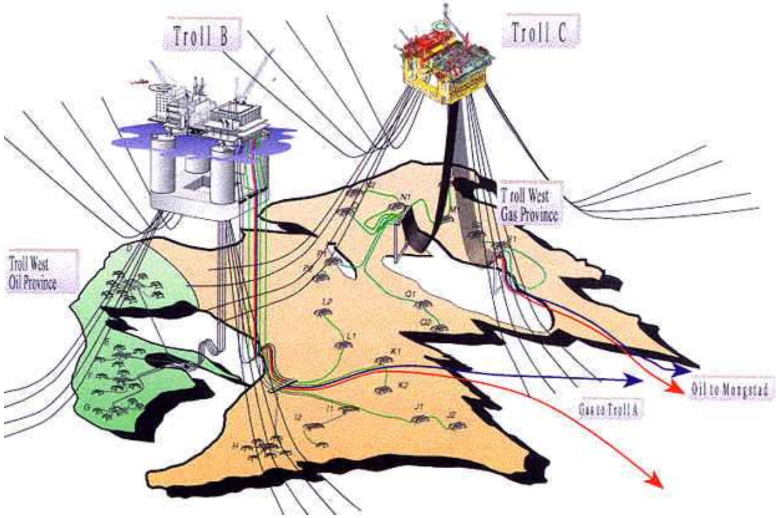
- The Brinkman problem
- Motivation - why use $H(\mathrm{div})$-conforming elements
- Problem setting - the non-conforming framework
- Local postprocessing
- A word on a posteriori
- Hybridization of the system

The Brinkman model

- Describes the flow of a viscous fluid in a porous medium
- Applicable to materials of very high porosity, e.g.
  - Sands, porous stones, petroleum engineering
  - Heat pipes

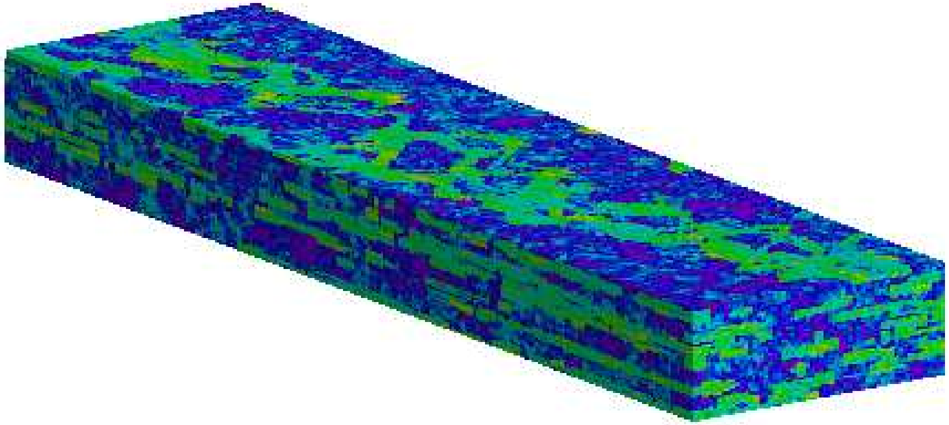
Reservoir modelling

- Oil reservoirs are natural multi-scale problems
  - Field scale - $10 - 100$ kilometres
  - Mesoscale - $10 - 100$ metres
  - Microscale - laboratory sample size
- Multiscale finite element methods or upscaling?



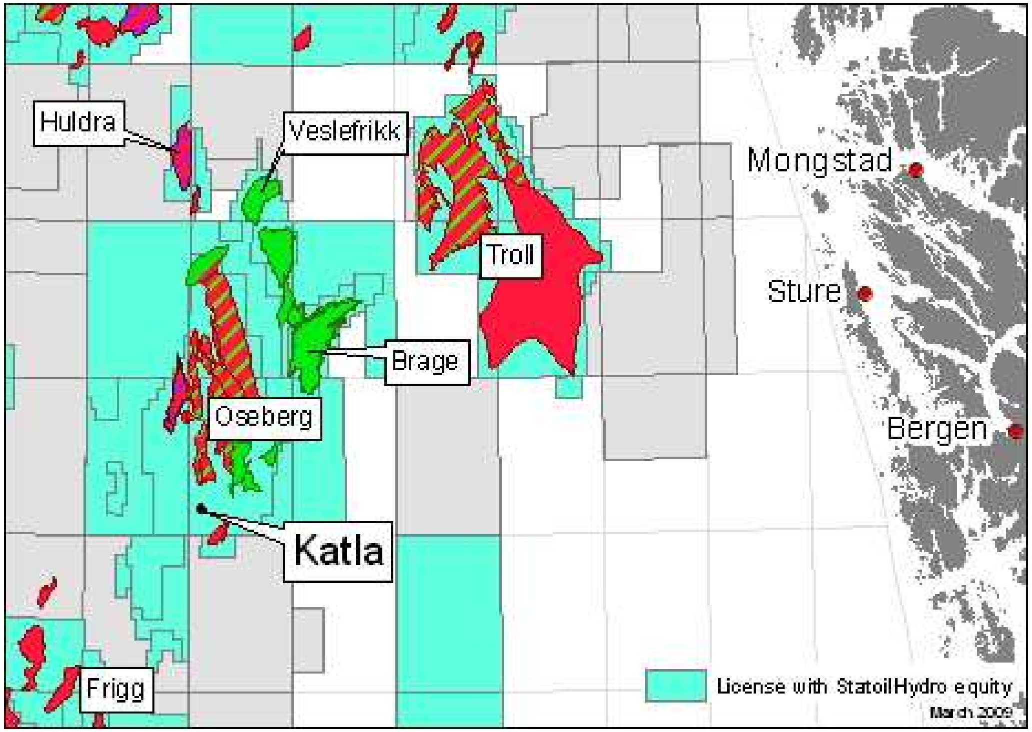
Reservoir modelling

- Typical properties of oil fields:
  - Long cracks, vugs
  - Rock of varying porosity
- Large jumps in parameter values



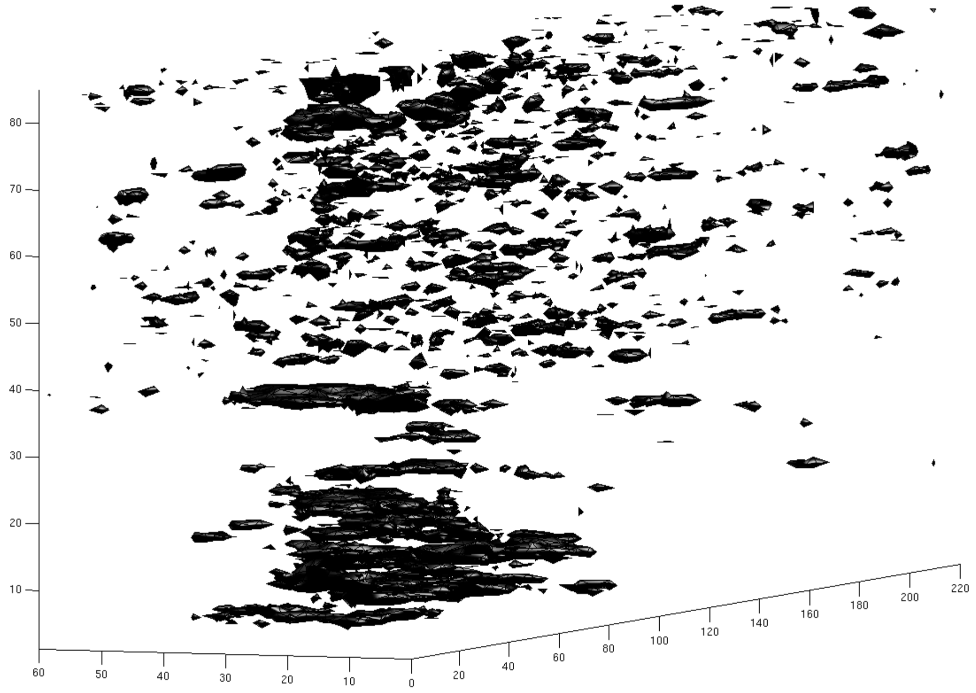
Example: realistic data

SPE10 comparative solution test based on actual data from Tarbert / Upper Ness formations



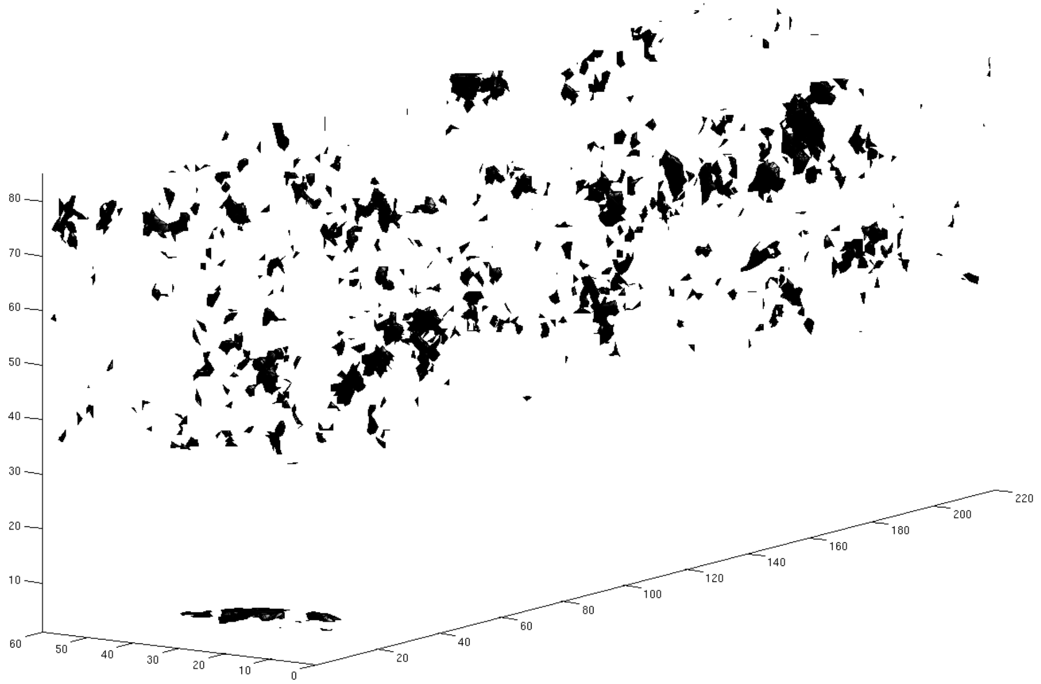
Example: realistic data

Permeability over 5000 millidarcy (void space)



Example: realistic data

Permeability under 0.01 millidarcy (no-flow zone)



The equations

- The strong form

$$
\begin{aligned}
-t^2 \Delta \boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega \\
\text{div } \boldsymbol{u} = g, &\quad \text{in } \Omega
\end{aligned}
$$

- The related weak formulation is

$$
\underbrace{t^2(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v})}_{a(\boldsymbol{u},\boldsymbol{v})} - (\text{div } \boldsymbol{v}, p) - \langle \frac{\partial \boldsymbol{v}}{\partial n}, p \rangle_{\partial \Omega} = (\boldsymbol{f}, \boldsymbol{v})
$$

$$
-(\text{div } \boldsymbol{u}, q) = (g, q)
$$

The problem setting

- The Brinkman problem lies between the Stokes and the Darcy problems
- For the Darcy case, we have the pairing $H(\mathrm{div}, \Omega) \times L^2(\Omega)$
- A non-conforming approximation for the Stokes part
- Solution: Nitsche's method to enforce tangential continuity

Motivation

- $H(\mathrm{div})$-conforming approximation gives
  - An elementwise mass preserving method
  - Useful tools for the error analysis
  - Optimal convergence rate

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- $H(\mathrm{div})$-conforming approximation gives
  - An elementwise mass preserving method
  - Useful tools for the error analysis
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- Properties of the related pressure approximation
  - Low-order approximation $\rightarrow$ very few DOFs
  - Superconvergence $\rightarrow$ local postprocessing
  - Optimal convergence rate

The FE spaces

- We use the BDM spaces of order k

$$\boldsymbol{V}_h^{BDM} = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega) \mid \boldsymbol{v}|_K \in [P_k(K)]^n \; \forall K \in \mathcal{K}_h\},$$
$$Q_h = \{q \in L^2(\Omega) \mid q|_K \in P_{k-1}(K) \; \forall K \in \mathcal{K}_h\}.$$

- This pairing satisfies the equilibrium property $\mathrm{div}\, \boldsymbol{V}_h \subset Q_h$
- Only the normal component of the flux is continuous

Nitsche's method

- To get a stable formulation, a modified bilinear form is introduced

$$a_h(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v}) + t^2 \sum_{K\in\mathcal{K}_h} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_K$$

$$+\, t^2 \sum_{E\in\mathcal{E}_h} \Big\{ \underbrace{\frac{\alpha}{h_K}\langle [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]\rangle_E}_{\text{jump penalty}} - \underbrace{\langle \{\frac{\partial \boldsymbol{u}}{\partial n}\}, [\![\boldsymbol{v}]\!]\rangle_E}_{\text{symmetry}} - \underbrace{\langle \{\frac{\partial \boldsymbol{v}}{\partial n}\}, [\![\boldsymbol{u}]\!]\rangle_E}_{\text{partial integration}} \Big\}\,.$$

- Original idea due to Nitsche in the 70s

The mesh dependent norms

- For the flux $\boldsymbol{u}$ we use

$$|||\boldsymbol{u}|||_{t,h}^2 = \|\boldsymbol{u}\|^2 + t^2 \sum_{K\in\mathcal{K}_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + t^2 \sum_{E\in\mathcal{E}_h} \frac{1}{h_E} \|[\![\boldsymbol{u}\cdot\boldsymbol{\tau}]\!]\|_{0,E}^2.$$

- For the pressure p

$$|||p|||_{t,h}^2 = \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2+t^2} \|\nabla p\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} \frac{h_E}{h_E^2+t^2} \|[\![p]\!]\|_{0,E}^2$$

- Idea: use the norms from primal mixed formulation for the dual mixed formulation!

A priori results

- We have the following quasioptimal result

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{t,h} + |||P_h p - p_h|||_{t,h} \leq C|||\boldsymbol{u} - \boldsymbol{R}_h \boldsymbol{u}|||_{t,h}.$$

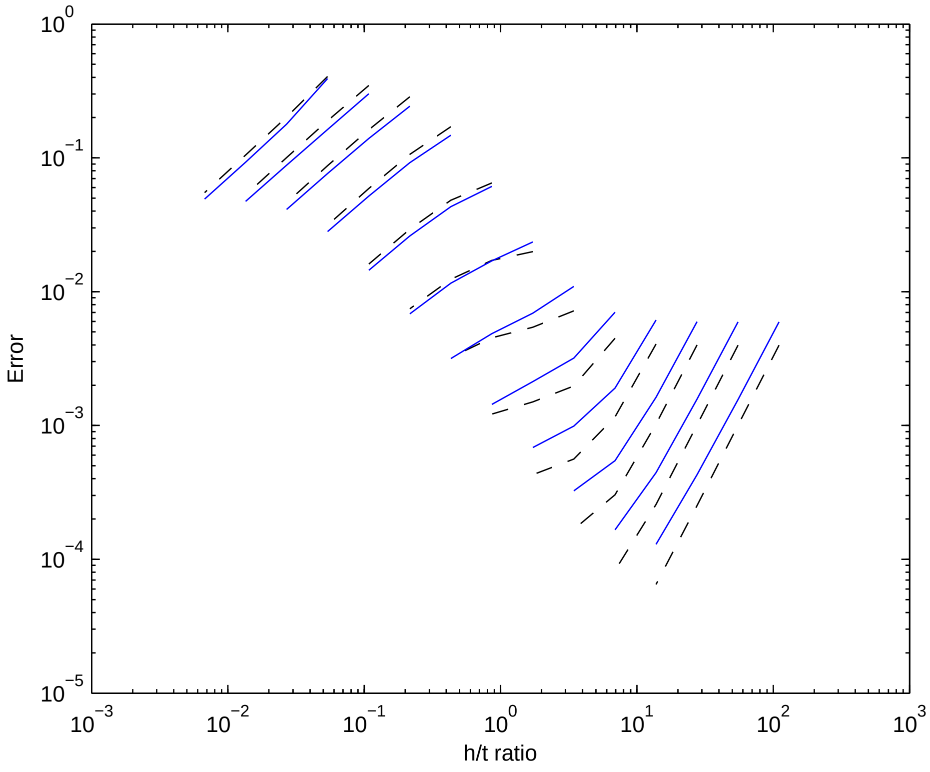
- The constant C is independent of the parameter t
- Assuming sufficient regularity, this gives optimal convergence rates for all parameter values
- Noteworthy: a superconvergence result for the pressure

The postprocessing method

- Optimal order convergence for $|||\boldsymbol{u} - \boldsymbol{u}_h|||_{t,h} + |||p - p_h^*|||_{t,h}$:
  - h^{k+1} rate in the pure Darcy case $t = 0$
  - h^k rate in the case $t > 0 \to$ optimal rate for Stokes
- Allows the use of residual-based a posteriori error estimates
- Performed elementwise, thus computationally cheap

Convergence test

The problem changes numerically at $t = h$!



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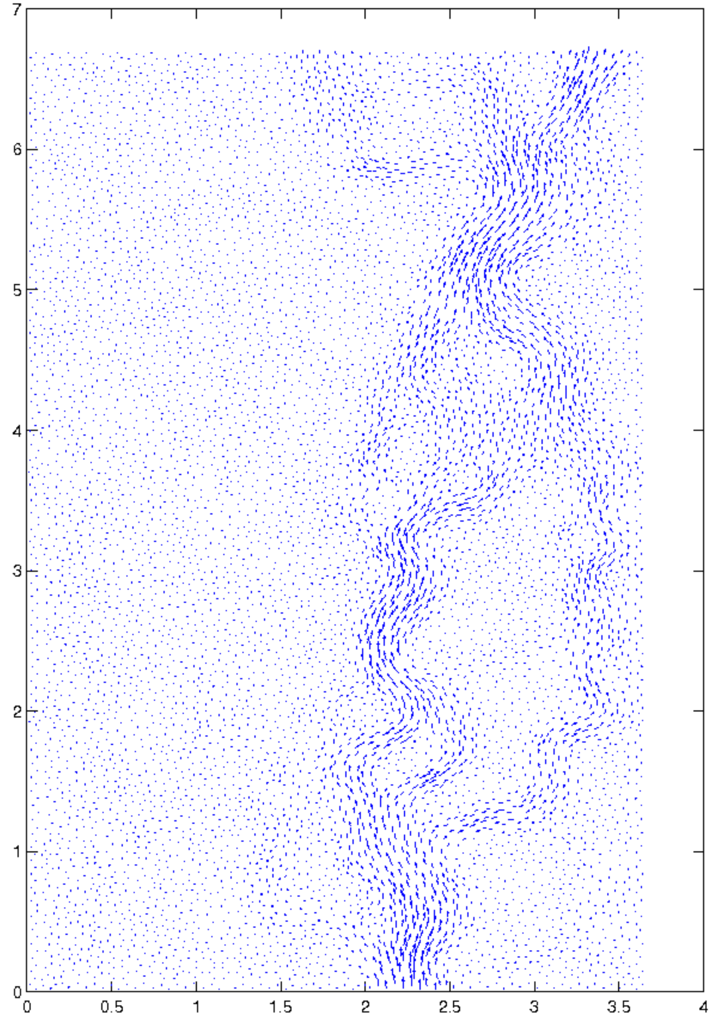
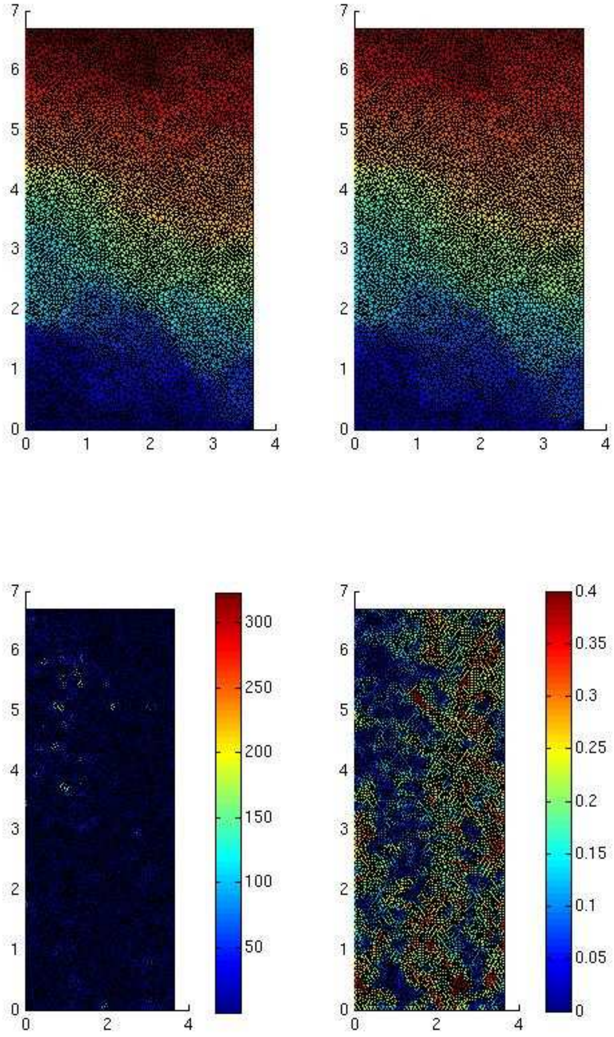




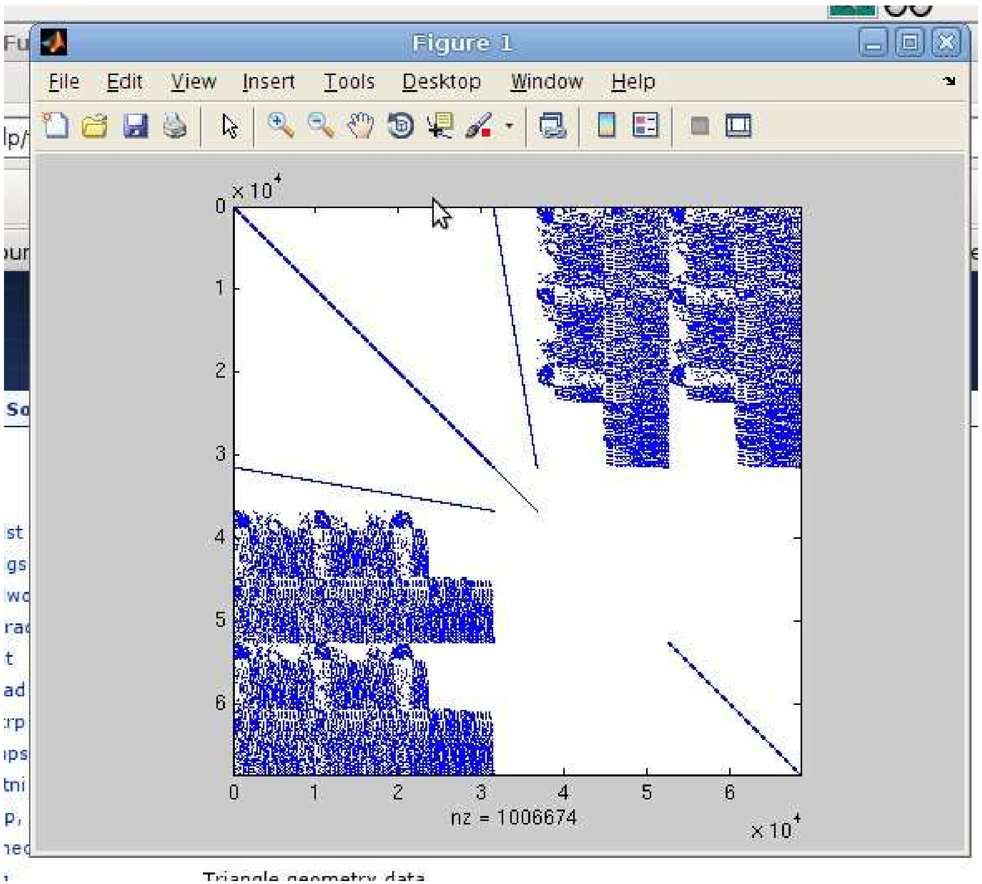
A word on a posteriori

- We have developed a sharp and reliable residual-based estimator
- Analysis relies on
  - The saturation assumption
  - Interpolation properties of $\boldsymbol{R}_h$
  - The equilibrium property $\operatorname{div} \boldsymbol{V}_h \subset Q_h$
  - Definition of the postprocessing method

Hybridization

- Darcy: enforce normal continuity via Lagrange multipliers
  - Symmetric, positive definite system
- Nitsche adds connections for the flux variable
  - Add another Langrange multiplier for the jump
- Makes domain decomposition easy
- Adaptive skeleton mesh?

Matrix after hybridization



Conclusions

- $H(\mathrm{div})$-conforming elements can be extended to cover the case of viscous flow in the Brinkman model
- Numerically light postprocessing scheme
- Optimal a priori results
- Reliable and sharp a posteriori indicator
- Applications to multiscale FEM?