On optimality of adaptive finite element methods with Dörfler marking

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Joint work with

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Outline

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Problem and Discretization

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open polygonal domain.

Problem:

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Finite Elements

Let $\mathcal{T}_0$ be an initial, conforming triangulation of Ω.

- We define $\mathbb{T}$ as the set of all conforming refinements of $\mathcal{T}_0$, that can be generated from $\mathcal{T}_0$ using iterative or recursive bisection.
- For $\mathcal{T} \in \mathbb{T}$ we define $\mathbb{V}(\mathcal{T})$ as the space of continuous piecewise affine finite elements over $\mathcal{T}$.

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Reliable and efficient estimators: Ritz Galerkin Solution $U \in \mathbb{V}(\mathcal{T})$

$$|\!|\!| u - U |\!|\!|_\Omega^2 \leq C_1\, \mathcal{E}_\mathcal{T}^2(\mathcal{I}_\mathcal{T}) \quad \text{and} \quad C_2\, \mathcal{E}_\mathcal{T}^2(\mathcal{I}_\mathcal{T}) \leq |\!|\!| u - U |\!|\!|_\Omega^2 + \operatorname{osc}_\mathcal{T}^2(\mathcal{I}_\mathcal{T}, f).$$

Adaptive Finite Element Method (AFEM)

Let $\mathcal{T}_0$ be an initial triangulation of Ω, $k = 0$.

SOLVE

$$U_k = \text{SOLVE}(\mathcal{T}_k)$$

Computes the Ritz approximation in $\mathbb{V}_k = \mathbb{V}(\mathcal{T}_k)$.

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$$\{\mathcal{E}_k(I)\}_{I\in\mathcal{I}_k} = \text{ESTIMATE}(U_k, \mathcal{T}_k)$$

Computes error indicators on $\mathcal{I}_k$.

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$$\mathcal{M}_k = \text{MARK}(\{\mathcal{E}_k(I)\}_{I\in\mathcal{I}_k}) \subset \mathcal{I}_k$$

Choose minimal $\mathcal{M}_k \subset \mathcal{I}_k$ s.t. $\mathcal{E}_k(\mathcal{M}_k) \geq \theta\, \mathcal{E}_k(\mathcal{I}_k)$.

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$$\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$$

Refine elements associated with marked indices $\mathcal{M}_k$

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Increment k and go to step SOLVE.

Approximation Rate

Quantifies convergence speed in Degrees Of Freedom

Total Error

$$|||u - U|||_{\Omega}^2 + \mathcal{E}_{\mathcal{T}}^2(\mathcal{T}) \approx |||u - U|||_{\Omega}^2 + \operatorname{osc}_{\mathcal{T}}^2(\mathcal{T}, f) =: \operatorname{Err}(u - U, f, \mathcal{T})^2$$

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Hence, a good AFEM should yield

$$\operatorname{Err}_k := \operatorname{Err}(u - U_k, f, \mathcal{T}_k) \preccurlyeq \left(\#\mathcal{T}_k - \#\mathcal{T}_0\right)^{-s}$$

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Remark

Generically oscillation is of higher order, hence for fine meshes $\mathcal{T}$:

$$\operatorname{Err}(u-U,f,\mathcal{T})^2 \approx \|u-U\|_\Omega^2$$

Outline

Residual based Estimator

Which indicators for optimal rates?

$$\left.\begin{array}{c}\text{[Stevenson:07]}\\ \text{[CKNS:08]}\\ \text{[Diening and Kreuzer:08]}\end{array}\right\} \Rightarrow \text{Choose residual based estimators!}$$

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Organized element wise:

$$\widehat{\mathcal{E}}^2_{\mathcal{T}}(U,T) := \|h_T f\|^2_{L^2(T)} + \|h_T^{1/2} J(U)\|^2_{L^2(\partial T)} \qquad \forall T \in \mathcal{T},$$

$$\widehat{\operatorname{osc}}^2_{\mathcal{T}}(f,T) := \|h_T(f - f_T)\|^2_{L^2(T)} \qquad \forall T \in \mathcal{T}.$$

Main Auxiliary Results: Contraction and Local Discrete Upper Bound

Theorem ([CKNS:08], [Diening and Kreuzer:08])

There exists $\widehat{C}_0 > 0$, $\alpha \in (0,1)$ *such that*

$$\widehat{\mathrm{Err}}_k \le \widehat{C}_0\, \alpha^{k-\ell}\, \widehat{\mathrm{Err}}_\ell.$$

Remark: $|\!|\!| u - U_k |\!|\!|_\Omega^2 + \gamma\, \widehat{\mathcal{E}}_k^2(\mathcal{T}_k) \approx \widehat{\mathrm{Err}}_k^2$

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Theorem ([Stevenson:07], [CKNS:08])

Let $\mathcal{T}_* \in \mathbb{T}$ *being a refinement of* $\mathcal{T}_k$ *with corresponding Ritz approximation* U_*

$$\|U_k - U_*\|_\Omega^2 \le \widehat{D}_1 \widehat{\mathcal{E}}_{\mathcal{T}_k}^2(\mathcal{T}_k \setminus \mathcal{T}_*).$$

$\mathcal{T}_k \setminus \mathcal{T}_*$ *is the set of elements that have to be refined in order to obtain* $\mathcal{T}_*$.

Remark: Residual based estimator: $\widehat{D}_1 = \widehat{C}_1$.

[CKNS]: Optimal Rates for Residual based Estimator

Let (u, f) can be approximated with rate $s > 0$. Let $\mathcal{T}_0$ satisfy some initial marking conditions [Binev, Dahmen, DeVore:04,Stevenson:08] and assume

$$\theta \in (0, \theta_*) \qquad \text{with} \qquad \theta_*^2 := \frac{\widehat{C}_2}{1 + \widehat{D}_1}.$$

Theorem ([Stevenson:07], [CKNS:08])

AFEM produces optimal rates, i.e.,

$$\left(\|u - U_k\|_\Omega^2 + \widehat{\operatorname{osc}}_k^2 \right)^{1/2} \le C \left(1 - \frac{\theta^2}{\theta_*^2}\right)^{-1/2} \left(\#\mathcal{T}_k - \#\mathcal{T}_0\right)^{-s} \qquad \forall k \in \mathbb{N}.$$

The constant C depends on α and on C_0.

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BUT: WHAT'S ABOUT OTHER ESTIMATORS?

Other Estimators: Hierarchical Estimator

With local side and element hat functions ϕ_σ and ϕ_T:

$$\mathcal{E}_{\mathcal{T}}^2(U,\sigma) := \left\langle \operatorname{Res}(U), \tfrac{\phi_\sigma}{|\!|\!|\phi_\sigma|\!|\!|_\Omega} \right\rangle^2 + \sum_{T\subset\omega_\sigma} \left(\left\langle \operatorname{Res}(U), \tfrac{\phi_T}{|\!|\!|\phi_T|\!|\!|_\Omega} \right\rangle^2 + \|h_T(f-f_T)\|_{L^2(T)}^2 \right)$$

$$\operatorname{osc}_{\mathcal{T}}^2(f,\sigma) := \sum_{T\subset\omega_\sigma} \|h_T(f-f_T)\|_{L^2(T)}^2, \qquad \forall \sigma \in \mathcal{S}.$$

Other Estimators: Estimator Based on Local Problems [Morin, Nochetto, and Siebert:03]

Let ϕ_z, $z \in \mathcal{N}$, be the Lagrange basis functions.

$\mathbb{W}_z$: Continuous quadratic finite elements in ω_z with vanishing trace on $\partial\omega_z$. If $z \in \mathcal{N} \cap \Omega$, then additionally $\int_{\omega_z} \psi\phi_z = 0$ for all $\psi \in \mathbb{W}_z$.

For each vertex $z \in \mathcal{N}$ solve the linear problem

$$\eta_z \in \mathbb{W}_z : \quad \int_{\omega_z} \nabla\eta_z \cdot \nabla\psi \, \phi_z \, dx = \left\langle \operatorname{Res}(U), \psi\phi_z \right\rangle \quad \forall\, \psi \in \mathbb{W}_z.$$

Organized by nodes

$$\mathcal{E}_{\mathcal{T}}^2(U,z) := \|\nabla\eta_z\phi_z^{\frac{1}{2}}\|_{L^2(\omega_z)}^2 + \|h_z(f - f_z)\phi_z^{\frac{1}{2}}\|_{L^2(\omega_z)}^2,$$

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Other Estimators: Recovery Based Estimator [Bartels and Carstensen:02] and [Zienkiewicz and Zhu:87]

Define the averaging operator $\mathcal{G} : \mathbb{V} \to \mathbb{V}(\mathcal{T})^d$ by the nodal values

$$(\mathcal{G}V)(z) = \frac{1}{|\omega_z|} \int_{\omega_z} \nabla V \, dx, \quad z \in \mathcal{N}.$$

Local error indicators

$$\mathcal{E}^2_{\mathcal{T}}(U, z) := \Big\{ \|\nabla U - \mathcal{G}U\|^2_{L^2(\omega_z)} + h_z^2 \|f - f_z\|^2_{L^2(\omega_z)} \Big\}, \quad z \in \mathcal{N},$$

$$\operatorname{osc}^2_{\mathcal{T}}(f, z) := \|h_z(f - f_z)\|^2_{L^2(\omega_z)}, \qquad z \in \mathcal{N}.$$

Other Estimators: Equilibrated Residual Estimator [Braes et. al.]

Given $z \in \mathcal{N}$ find $\xi_z \in \mathbb{RT}^{-1}(\mathcal{T}, z)$ with minimal L^2-norm such that

$$
\begin{aligned}
\operatorname{div} \xi_{z\,|T} &= -\frac{1}{|T|}\int_T f\phi_z\,dx =: f_{z\,|T}, && \text{in each } T \subset \omega_z,\\
[\![\xi_z]\!]_{|\sigma} &= \int_\sigma J(U)\phi_z\,do = \frac{1}{2}J(U)_{|\sigma}, && \text{on each } \sigma \subset \Sigma_z,\\
\xi_z \cdot \boldsymbol{\nu} &= 0, && \text{on } \partial\omega_z,
\end{aligned}
\tag{1}
$$

where $\mathbb{RT}^{-1}(\mathcal{T}, z)$ denotes the local broken Raviart-Thomas space

$$\mathbb{RT}^{-1}(\mathcal{T}, z) := \left\{\vec{g} \in L^2(\omega_z)^2 \mid \vec{g}_{|T}(x) = \vec{a} + bx, \vec{a} \in \mathbb{R}^2, b \in \mathbb{R}\ \forall T \subset \omega_z\right\}.$$

Local error indicators

$$\mathcal{E}_{\mathcal{T}}^2(U, z) := \left\{\|\xi_z\|^2_{L^2(\omega_z)} + h_z^2\|f - f_z\|^2_{L^2(\omega_z)}\right\}, \quad z \in \mathcal{N},$$

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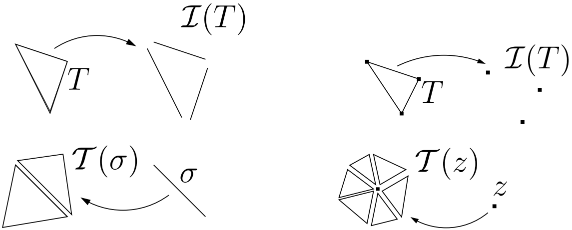
Notation: Index-Set $\mathcal{I} = \mathcal{T}$, $\mathcal{I} = \mathcal{S}$, or $\mathcal{I} = \mathcal{N}$

Relations between triangulation $\mathcal{T}$ and corresponding index set $\mathcal{I}$:

$$\mathcal{T}(\mathcal{I}') := \{T \mid I \subset T \text{ for some } I \in \mathcal{I}'\} \qquad \mathcal{I}' \subset \mathcal{I},$$

$$\mathcal{I}(\mathcal{T}') := \{I \mid I \subset T \text{ for some } T \in \mathcal{T}'\} \qquad \mathcal{T}' \subset \mathcal{T}.$$



Local Equivalence

All indicators are locally equivalent to the residual based indicators: Partially in [Verfürth:1996].

$$\hat{c}\,\mathcal{E}^2_{\mathcal{T}}(\mathcal{I}') \le \widehat{\mathcal{E}}^2_{\mathcal{T}}\big(\mathcal{T}(\mathcal{I}')\big) \qquad \forall \mathcal{I}' \subset \mathcal{I}$$
$$\tilde{c}\,\widehat{\mathcal{E}}^2_{\mathcal{T}}(\mathcal{T}') \le \mathcal{E}^2_{\mathcal{T}}\big(\mathcal{I}(\mathcal{T}')\big) \qquad \forall \mathcal{T}' \subset \mathcal{T};$$

This yields a Dörfler property for the residual estimator with $\hat{\theta} = \hat{c}\,\tilde{c}\,\theta$:

$$\widehat{\mathcal{E}}^2_k\big(\mathcal{T}(\mathcal{M}_k)\big) \ge \hat{c}\,\mathcal{E}^2_k(\mathcal{M}_k) \ge \hat{c}\,\theta^2\mathcal{E}^2_k(\mathcal{I}_k) \ge \hat{c}\,\tilde{c}\,\theta^2\widehat{\mathcal{E}}^2_k(\mathcal{T}_k) = \hat{\theta}^2\,\widehat{\mathcal{E}}^2_k(\mathcal{T}_k)$$

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Implies error reduction for the 'residual' total error; [CKNS:08], [Diening and Kreuzer:08]:

$$\widehat{\mathrm{Err}}_k \le C_0\,\alpha^{k-\ell}\widehat{\mathrm{Err}}_\ell(\mathcal{T}_\ell)$$

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Let $\mathcal{T} \in \mathbb{T}$ and $\mathcal{T}_*$ be a refinement of $\mathcal{T}$, then

$$|\!|\!|U - U_*|\!|\!|^2_\Omega \le \widehat{C}_1 \widehat{\mathcal{E}}^2_{\mathcal{T}}(\mathcal{T} \setminus \mathcal{T}_*) \le \tilde{c}^{-1}\,\widehat{D}_1 \mathcal{E}^2_{\mathcal{T}}\big(\mathcal{I}(\mathcal{T} \setminus \mathcal{T}_*)\big) = D_1 \mathcal{E}^2_{\mathcal{T}}\big(\mathcal{I}(\mathcal{T} \setminus \mathcal{T}_*)\big)$$

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Drawback: bad constant D_1 in definition of $\theta_* = \dfrac{C_2}{1 + D_1}$.

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Discrete local upper bound can be achieved via equivalence of estimators as well:

Let $\mathcal{T} \in \mathbb{T}$ and $\mathcal{T}_*$ be a refinement of $\mathcal{T}$, then

$$\|U - U_*\|^2_\Omega \leq \widehat{C}_1 \widehat{\mathcal{E}}^2_{\mathcal{T}}(\mathcal{T} \setminus \mathcal{T}_*) \leq \tilde{c}^{-1}\, \widehat{D}_1 \mathcal{E}^2_{\mathcal{T}}\big(\mathcal{I}(\mathcal{T} \setminus \mathcal{T}_*)\big) = D_1 \mathcal{E}^2_{\mathcal{T}}\big(\mathcal{I}(\mathcal{T} \setminus \mathcal{T}_*)\big)$$

Drawback: bad constant D_1 in definition of $\theta_* = \dfrac{C_2}{1 + D_1}$.

Better: Try to directly Calculate discrete local upper bound

Other Estimators: Locally Conditioned Robust Estimators

- **Estimator for Equations with discontinuous diffusion coefficients [Petzold:02]**

$$-\operatorname{div} k(x)\nabla u = f \quad \text{in } \Omega$$

Coefficients k positive, pw constant with quasi-monotone jumps on $\mathcal{T}_0$.

$$\mathcal{E}^2_{\mathcal{T}}(U,T) := \frac{h_T^2}{k|_T}\|f\|^2_{L^2(T)} + \frac{h_T}{k|_T}\|J(U)\|^2_{L^2(\partial T)} \qquad \forall T \in \mathcal{T},$$

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- **Estimator for singularly perturbed reaction-diffusion equations [Verfürth:1998]**

$$-\Delta u + \kappa u = f \quad \text{in } \Omega \qquad \kappa \gg 1.$$

$$\mathcal{E}_{\mathcal{T}}^2(U,T) := \|\alpha_T f\|_{L^2(T)}^2 + \|\alpha_T^{1/2} J(U)\|_{L^2(\partial T)}^2, \quad \alpha_T := \min\{h_T, \kappa^{-1}\}.$$

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Advantage of robust estimators:

$$\hat{\theta}_* = \frac{\widehat{C}_2}{1+\widehat{D}_1} \ll \frac{C_2}{1+D_1} = \theta_*.$$

If calculated without equivalence of indicators!

Optimal Rate

Let (u, f) can be approximated with rate $s > 0$. Let $\mathcal{T}_0$ satisfy some initial marking conditions [Binev, Dahmen, DeVore:04, Stevenson:08] and assume

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Theorem ([Stevenson:07], [CKNS:08])

AFEM produces optimal rates, i.e.,

$$\left(\|u - U_k\|_\Omega^2 + \mathrm{osc}_k^2 \right)^{1/2} \leq C \left(1 - \frac{\theta^2}{\theta_*^2}\right)^{-1/2} \left(\#\mathcal{T}_k - \#\mathcal{T}_0\right)^{-s} \qquad \forall k \in \mathbb{N}.$$

Constant C depends on s, θ and the equivalence of estimators.

Thank You For Your Attention!

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