Error Analysis of an Evolution Equation for Microstructure

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Thursday 20/Friday 21 May 2010

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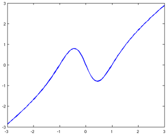
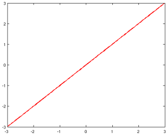
Consider the problem

$$\begin{aligned} \Delta u_t + \operatorname{div}(\sigma(\nabla u)) &= 0 && \text{in } \Omega = (0,1)^2 \\ u &= 0 && \text{on } \partial\Omega \\ u &= u_0 && \text{when } t = 0 \end{aligned}$$

$\sigma = DW$ where W is a double well potential such that $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ globally Lipschitz, and $\sigma(p) \cdot p \geq c|p|^2 - d$ for $c > 0$, $d \geq 0$.

For example, $W(\nabla u) = \frac{1}{2}\frac{1}{u_x^2+1}(u_x^2 - 1)^2 + \frac{1}{2}u_y^2$.



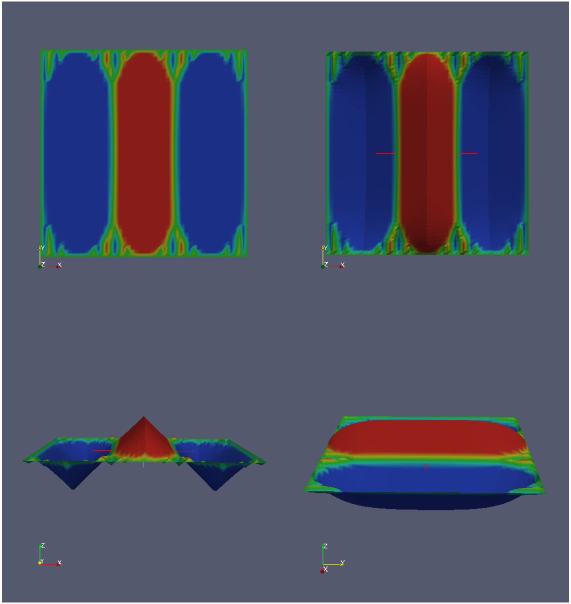


This problem is the $H_0^1(\Omega)$ gradient flow of $I(u) := \int_\Omega W(\nabla u)dx$.
i.e.

$$u_t = -\nabla I(u) \qquad \text{in } H_0^1(\Omega).$$

The direction chosen by the dynamics is the direction of steepest descent.

In our example, the solution would like to satisfy $u_x = \pm 1$, $u_y = 0$.



Questions

- What can we say about the long-time behaviour of solutions and the appearance of microstructure?
- Is microstructure a numerical artifact and can we approximate the solution with FEM?

What can we prove?

Rewrite as

$$u_t = F(u) \qquad \text{in } H_0^1(\Omega)$$

where $F(u) := -\Delta^{-1}\operatorname{div}(\sigma(\nabla u))$. $F : H_0^1(\Omega) \to H_0^1(\Omega)$ is Lipschitz

Standard theory and energy estimates (eg. Henry and Temam) $\Rightarrow$ $\exists!$ solution

$$u(t) = u_0 + \int_0^t F(u(s))ds,$$

$u \in C([0,\infty), H_0^1(\Omega)) \cap L^\infty([0,\infty), H_0^1(\Omega))$,
$u_t \in C((0,\infty), H_0^1(\Omega)) \cap L^\infty((0,\infty), H_0^1(\Omega))$ (both Lipschitz), and

$$\int_\Omega W(\nabla u(t))dx + \int_0^t \|\nabla u_s(s)\|_{L^2(\Omega)}^2 ds = const. \qquad t \geq 0,$$

$\Rightarrow u_t \in L^2((0,\infty), H_0^1(\Omega))$ and $u_t \to 0$ in $H_0^1(\Omega)$ as $t \to 0$.

What can't we prove?

- No additional regularity or compactness (only $H^1_0(\Omega)$) . This limits what we can say about the long-time behaviour of solutions. $\|u(t)\|_{H^1(\Omega)} \leq C$ for all $t \geq 0$ only implies weak convergence.
- Question remains: Are all solutions minimising sequences for $\int_\Omega W(\nabla u(x))dx$, or are some solutions attracted to rest points, for which $\int_\Omega W(\nabla u(x))dx > 0$?

Numerics?

Can we approximate the solution with FEM?

Up to finite time it is possible to prove that $u_h \longrightarrow u$ in $H_0^1(\Omega)$ as $h \longrightarrow 0$ but we do not get a rate of convergence (due to lack of additional regularity).

Modify problem to create additional regularity...

Regularized Problem

Consider the problem

$$\begin{aligned} \Delta u_t - \epsilon \Delta^2 u + \operatorname{div}(\sigma(\nabla u)) = 0 \qquad & \text{in } \Omega = (0,1)^2 \\ \Delta u = u = 0 \qquad & \text{on } \partial\Omega \\ u = u_0 \in H_0^1(\Omega) \qquad & \text{at } t = 0. \end{aligned}$$

Rewrite as

$$u_t - \epsilon \Delta u = F(u) \qquad \text{in } H_0^1(\Omega)$$

$F(u) := -\Delta^{-1} \operatorname{div}(\sigma(\nabla u)).$

In the gradient flow representation the additional term is bending energy

$$I(u) = \int_\Omega W(\nabla u) + \frac{\epsilon}{2}(\Delta u)^2.$$

Using same techniques as before we can prove similar results, e.g.
$\exists!$ solution $u \in C([0,\infty), H_0^1(\Omega)) \cap C((0,\infty), V)$

$$u(t) = \mathrm{e}^{\epsilon \Delta t}\, u_0 + \int_0^t \mathrm{e}^{\epsilon \Delta (t-s)}\, F(u(s))ds$$

and . . .

$(V := \{v \in H_0^1(\Omega) : \Delta v \in H_0^1(\Omega)\})$.

. . . it is also possible to prove that

$$\|u\|_{H^1} \leq C \qquad t \in [0, \infty)$$

$$\|u\|_{H^2} \leq \begin{cases} C\epsilon^{-1/2} t^{-1/2} & t \in (0, T] \\ C\epsilon^{-1/2} & t \in (T, \infty) \end{cases}$$

$$\|u_t\|_{H^1} \leq \begin{cases} Ct^{-1} & t \in (0, T] \\ C & t \in (T, \infty) \end{cases}$$

... and other results eg. higher regularity, $\exists$ Lyapunov function, $u_t \to 0$ in H^1, $\exists$ compact attractor of finite dimension...

Semi-discrete Problem

$V_h \subset H_0^1(\Omega)$. Define $\Delta_h : V_h \to V_h$

$$(\Delta_h u_h, \phi_h)_{L^2} = -(\nabla u_h, \nabla \phi_h)_{L^2} \qquad \forall u_h, \phi_h \in V_h.$$

Also define elliptic projection operator $R = R(h)$ and L^2 projection operator $P = P(h)$ by

$$\begin{aligned} (\nabla(Ru - u), \nabla \phi_h)_{L^2} &= 0 \qquad && \forall \phi_h \in V_h, u \in H_0^1(\Omega) \\ (Pu - u, \phi_h)_{L^2} &= 0 \qquad && \forall \phi_h \in V_h, u \in H_0^1(\Omega). \end{aligned}$$

We have $\Delta_h R = P\Delta$. Assume

$$\begin{aligned} &\|u - Ru\|_{L^2} + h\|u - Ru\|_{H^1} \lesssim h^s \|u\|_{H^s} \\ &\|u - Pu\|_{L^2} + h\|u - Pu\|_{H^1} \lesssim h^s \|u\|_{H^s} \qquad s = 1, 2. \end{aligned}$$

Apply the Galerkin method: Find $u_h \in C([0,\infty), V_h)$ such that $u = u_{0h} := Ru_0$ at $t = 0$ and

$$u_{h,t} - \epsilon \Delta_h u_h = F_h(u_h) \qquad \text{in } V_h \text{ for } t > 0$$

where $F_h(u_h) := RF(u_h) = -\Delta_h^{-1} \operatorname{div}(\sigma(\nabla u_h))$.
Same theory $\Rightarrow$ $\exists!$ solution u_h such that

$$u_h(t) = e^{\epsilon \Delta_h t}\, u_{0h} + \int_0^t e^{\epsilon \Delta_h (t-s)}\, F_h(u_h(s)) ds$$

Same regularity results as before (except H^2 norm replaced with $\|\Delta_h u_h\|_{L^2}$).

Error Analysis

To analyse the error we follow standard theory (eg. Larsson for short time error), but pay particular attention to the dependence on ϵ,

to show that

$$\|u_h - u\|_{H^1} \lesssim h\epsilon^{-1/2} t^{-1/2} \qquad t \in (0, T].$$

Split the error into 2 parts

$$e = u_h - u = \underbrace{u_h - Ru}_{\theta(t)\in V_h} + \underbrace{Ru - u}_{\rho(t)}.$$

$\rho(t)$ is just the elliptic projection error,

$$\|\rho(t)\|_{H^1} \lesssim h\|u(t)\|_{H^2} \lesssim h\epsilon^{-1/2}t^{-1/2} \qquad \text{for } t \in (0, T].$$

$\theta(t)$ satisfies the following equation,

$$\theta_t - \epsilon\Delta_h\theta = F_h(u_h) - F_h(u) + (P - R)(u_t - F(u)).$$

$$\theta_h(t) = e^{\epsilon\Delta_h t}\,\theta_{0h} + \int_0^t e^{\epsilon\Delta_h(t-s)}\Big[F_h(u_h) - F_h(u) + (P - R)(u_s - F(u))\Big]\,ds$$

$$
\begin{aligned}
\theta(t) = {} & \mathrm{e}^{\epsilon\Delta_h t}\,\theta_0 \\
& + \int_0^t \mathrm{e}^{\epsilon\Delta_h(t-s)}\left[F_h(u_h(s)) - F_h(u(s))\right] ds && \text{(T1)} \\
& - \int_0^t \mathrm{e}^{\epsilon\Delta_h(t-s)}(P-R)F(u(s))ds && \text{(T2)} \\
& + \mathrm{e}^{\epsilon\Delta_h t/2}(P-R)u(\tfrac{t}{2}) - \mathrm{e}^{\epsilon\Delta_h t}(P-R)u_0 && \text{(T3-4)} \\
& + \epsilon\int_0^{\frac{t}{2}} \Delta_h\,\mathrm{e}^{\epsilon\Delta_h(t-s)}(P-R)u(s)ds && \text{(T5)} \\
& + \int_{\frac{t}{2}}^t \mathrm{e}^{\epsilon\Delta_h(t-s)}(P-R)u_s(s)ds && \text{(T6)}
\end{aligned}
$$

Now take $\|\cdot\|_{H^1}$ of each term separately and use our regularity estimates to get the result.

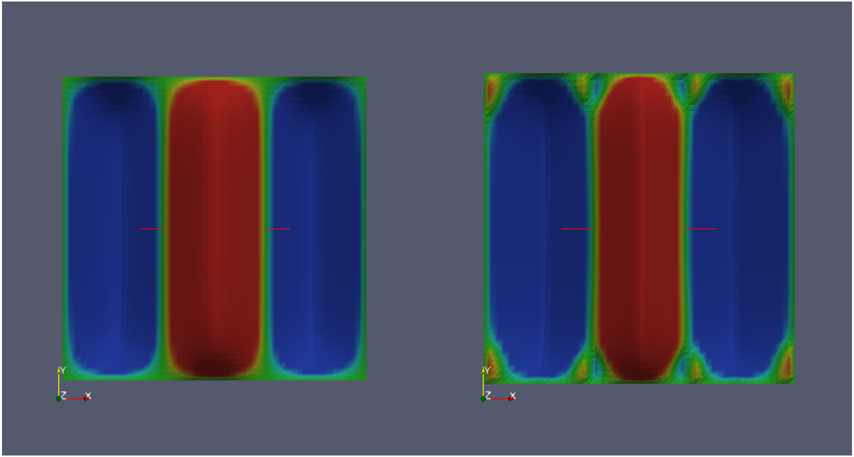
$$\|u_{\epsilon h} - u_\epsilon\|_{H^1} \lesssim h\epsilon^{-1/2} t^{-1/2} \qquad t \in (0, T].$$

If we choose $h^{2-2\delta} < \epsilon$ for some $\delta > 0$ then

$$\|u_{\epsilon h} - u_\epsilon\|_{H^1} \lesssim h^\delta t^{-1/2}$$

independent of ϵ.

Unfortunately we can only prove that up to finite time $u_\epsilon \longrightarrow u$ in $H_0^1(\Omega)$ as $\epsilon \longrightarrow 0$. We do not have a rate of convergence for the regularization error.



left: $\epsilon = 0.001$, right: $\epsilon = 0.0001$. $h = 0.02$.

Further Work

- Long-time convergence result . . . convergence of attractors.
- Error in L^2.
- Regularization error? How does regularization change the long-time behaviour of the PDE?
- Existence of rest points for the original PDE.