On the Integral Type Crouzeix-Raviart Nonconforming FE: Lower Bounds for Eigenvalues

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Crouzeix-Raviart Nonconforming FE

$T = \{(t_1, t_2) : \ t_1, t_2 \geq 0, \ t_1 + t_2 \leq 1\}$ - the reference element;

The shape functions of introduced linear element on T are:

$\varphi_1(t_1, t_2) = -1 + 2t_1 + 2t_2; \ \varphi_2(t_1, t_2) = 1 - 2t_1; \ \varphi_3(t_1, t_2) = 1 - 2t_2.$

We define nonconforming piecewise linear finite element space V_h of Crouzeix-Raviart elements with integral type degrees of freedom (Fig. 1) for which $h = \max_{K \in \tau_h}$ is mesh parameter: $V_h = \{v : v_{|_K} \in \mathcal{P}_1$ is integrally continuous on the edges of K, for all $K \in \tau_h$, $\int_{\partial\Omega} v \, dl = 0\}$.

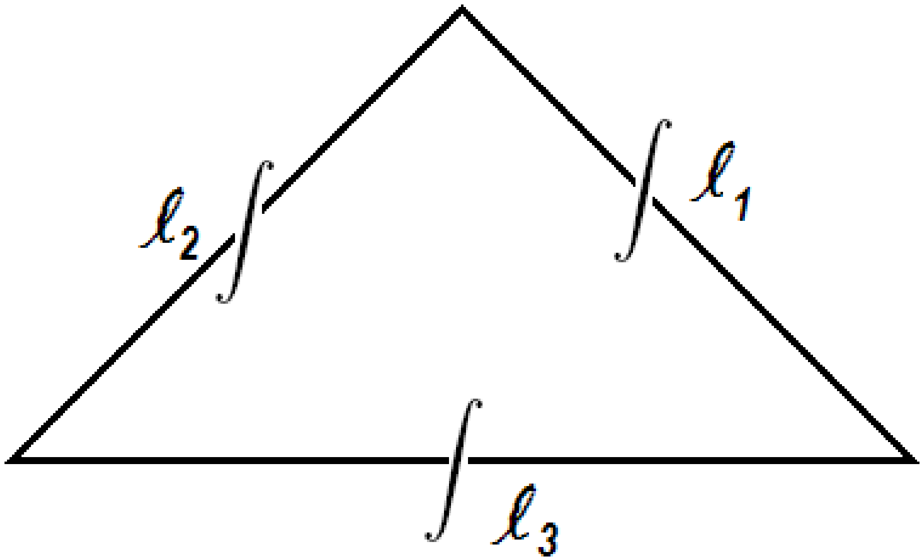

Figure 1:

For any $v \in L_2(\Omega)$ with $v_{|_K} \in H^m(K),\ \forall K \in \tau_h$ we define the mesh-dependent norm and seminorm:

$$\|v\|_{m,h} = \left\{ \sum_{K \in \tau_h} \|v\|_{m,K}^2 \right\}^{1/2}, \quad |v|_{m,h} = \left\{ \sum_{K \in \tau_h} |v|_{m,K}^2 \right\}^{1/2},\ m = 0, 1.$$

We define the following bilinear form on $V_h + H_0^1(\Omega)$:

$$a_h(u, v) = \sum_{K \in \tau_h} \int_K (\nabla u \cdot \nabla v + a_0 uv)\ dx. \tag{1}$$

i_h - the intepolant, associated with the integral type C-R linear FE for any partition τ_h

Then:

$$\forall v \in L_2(\Omega),\ \forall K \in \tau_h,\ \int_{l_j} i_h v\, dl = \int_{l_j} v\, dl,\ j = 1, 2, 3.$$

It is evident that

$$i_h v \in V_h,\ \forall v \in L_2(\Omega);$$
$$i_h v \equiv v,\ \forall v \in V_h.$$

$\mathcal{R}_h : \ V \to V_h$ denotes the elliptic projection operator defined by:

$$a_h(u - \mathcal{R}_h u, v_h) = 0 \quad \forall u \in V, \ \forall v_h \in V_h.$$

Using the interpolation properties of the conforming and nonconforming linear FE triangles we prove the following result:

Theorem 1 *If v belongs to $H^2(\Omega) \cap V$, then*

$$\|v - \mathcal{R}_h v\|_{s,h} \leq C h^{2-s} \|u\|_{2,\Omega}, \ s = 0, 1. \tag{2}$$

A superclose property of the interpolant i_h with respect to the a_h-form:

Theorem 2 *Let $u \in H^2(\Omega)$. Then for any $v_h \in V_h$ the following inequality holds:*

$$a_h(i_h u - u, v_h) \leq Ch^2 \|u\|_{2,\Omega} \|v_h\|_{1,h}. \tag{3}$$

In particular, if $a_0(x) = 0$, then i_h related to the linear C-R nonconforming triangular element coincides with the Ritz projection operator $\mathcal{R}_h$ of the corresponding second-order elliptic problem, i.e.

$$a_h(i_h u - u, v_h) = 0 \ \ \forall u \in V, \ \forall v_h \in V_h.$$

Eigenvalue Problem

Consider the variational elliptic EVP: find $(\lambda, u) \in \mathbf{R} \times H_0^1(\Omega),\ u \neq 0$ such that

$$a(u, v) = \lambda(u, v), \quad \forall\, v \in V. \tag{4}$$

The approximation of EVP (4) by nonconforming FEM is: find $\lambda_h \in \mathbf{R}$ and $u_h \in V_h,\ u_h \neq 0$ such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall\, v_h \in V_h, \tag{5}$$

Patch-recovery Technique

Let us construct macro-elements, unifying four adjacent congruent right-angled isosceles triangles belonging to τ_h. The degrees of freedom of any macro-element $K = \cup_{i=1}^{4} K_i$ from $\widetilde{\tau}_{2h}$ we choose to be the degrees of freedom of $K_i \in \tau_h,\ i = 1, 2, 3, 4$, i.e. these are the integral values of any function $v \in V$ on the edges $l_{i,j},\ j = 1, 2, 3$ of $K_i,\ i = 1, 2, 3, 4$.

Let $\widetilde{V}_{2h}$ be finite element spaces associated with $\widetilde{\tau}_{2h}$. One possible choice for $\widetilde{V}_{2h}$ is to consist of polynomials from $\mathcal{P}_K$, where on any $K \in \widetilde{\tau}_{2h}$

$$\mathcal{P}_K = \mathcal{P}_2 + \text{span}\left\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2,\ i, j = 1, 2, 3;\ i < j\right\}.$$

$$(\lambda_s,\ s = 1, 2, 3 \text{ are baricentric coordinates of } K)$$

Obviously $\mathcal{P}_2 \subset \mathcal{P}_K \subset \mathcal{P}_3$.

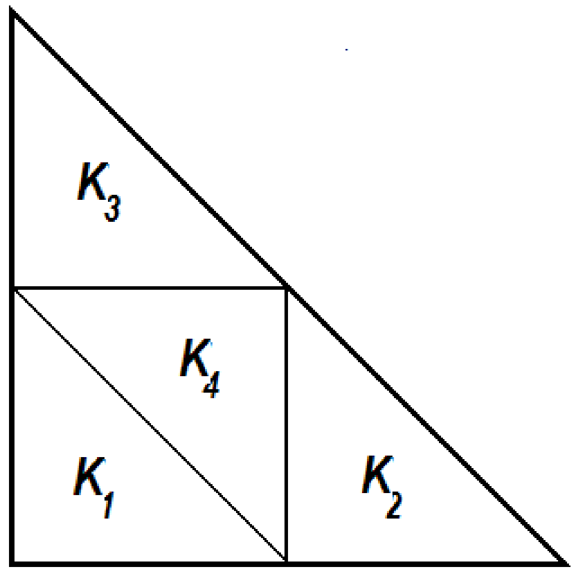

Figure 2:

The interpolation operator $I_{2h} : V_h \rightarrow \widetilde{V}_{2h}$ corresponding to $\widetilde{\tau}_{2h}$ is characterized by edge conditions determined by the degrees of freedom of any $K \in \widetilde{\tau}_{2h}$ It is constructed in such a way that:

$$I_{2h} \circ i_h = I_{2h}, \tag{6}$$

$$\|I_{2h} v_h\|_{r,h} \leq C \|v_h\|_{r,h}, \quad \forall v_h \in V_h,\ r = 0, 1, \tag{7}$$

because the mapping $I_{2h} :\ V_h \rightarrow \widetilde{V}_{2h}$ is bounded.

At that, having in mind that the interpolation polynomial $I_{2h} v_{|_K}$ belongs to the set $\mathcal{P}_K$, for any $v \in H^3(\Omega) \cap V$ it follows that

$$\|I_{2h} v - v\|_{1,\Omega} \leq C h^2 \|v\|_{3,\Omega}. \tag{8}$$

The next theorem contains the main superconvergent estimation:

Theorem 3 *Let* $u \in H^3(\Omega) \cap V$*. Then the following estimate holds:*

$$\|I_{2h} \circ \mathcal{R}_h u - u\|_{1,h} \leq Ch^2 \|u\|_{3,\Omega} \tag{9}$$

The main result concerning patch-recovery technique applied to the second-order EVP is given in the following theorem:

Theorem 4 *Let* (λ, u) *be any exact eigenpair and* (λ_h, u_h) *be its FE approximation using triangular nonconforming C-R linear elements. Assume also that* u *satisfies the conditions of Theorem 3 are fulfilled. Then:*

$$\|I_{2h} u_h - u\|_{1,h} \leq Ch^2 \|u\|_{3,\Omega}, \tag{10}$$

$$\left| \frac{a_h(I_{2h}u_h, I_{2h}u_h)}{(I_{2h}u_h, I_{2h}u_h)} - \lambda \right| \leq Ch^4 \|u\|^2_{3,\Omega}. \tag{11}$$

Patch-recovery Technique - Numerical Results

Let Ω be a square domain:

$$\Omega: \; 0 < x_i < \pi, \quad i = 1, 2.$$

Consider the following model problem:

$$-\Delta u = \lambda u \quad \text{in} \;\; \Omega,$$

$$u = 0 \quad \text{on} \;\; \partial\Omega.$$

The exact eigenvalues are equal to $k_1^2 + k_2^2,\; k_j = 1, 2, \ldots,\; j = 1, 2$ $(2, 5, 5, 8, 10, 10, \ldots)$

Table 1: Eigenvalues computed by means of C-R integral type nonconforming FEs (NC) and after applying of patch-recovery technique (PR)

h		λ_1	λ_2	λ_3	λ_4
$\pi/4$	NC	1.965475477	4.546032933	4.546036508	7.430949878
$\pi/4$	PR	2.048733065	5.377641910	5.379034337	8.858183829
$\pi/8$	NC	1.991417651	4.888133308	4.888134617	7.868940522
$\pi/8$	PR	2.001716041	5.030155947	5.030153808	8.039386123
$\pi/16$	NC	1.997857237	4.972126030	4.972127107	7.971004421
$\pi/16$	PR	2.000447081	5.008219681	5.008225792	8.007441874

Eigenvalue Problem (Nonconvex Domain)

Theorem 5 *Let* (λ_k, u_k) *and* $(\lambda_{h,k}, u_{h,k})$ *be the solutions of (4) and (5), respectively and* a_h *is determined by (1) with* $a_0 = 0$.

Assume that Ω *is* *<u>not convex</u>* *and the eigenfunctions being normalized* $\|u_k\|_{0,\Omega} = \|u_{h,k}\|_{0,\Omega} = 1$. *Then*

$$\lambda_{h,k} \leq \lambda_k. \tag{12}$$

usual C-R element: Armentano & Duran 2004
integral-type C-R element: Andreev & Racheva ?

Eigenvalue Problem (Convex Domain)

The next lemma proves supercloseness between any approximate eigenfunction and the integral type interpolant of the corresponding exact eigenfunction.

Lemma 1 *Let* (λ, u) *and* (λ_h, u_h) *be any corresponding eigenpairs obtained by (4) and (5), respectively. If* $i_h u$ *is the C-R linear interpolant of the exact eigenfunction and supposing that the partition is quasiuniform and* $u \in H^2(\Omega) \cap V$*, then the following estimate holds:*

$$\|u_h - i_h u\|_{1,h} \leq Ch^2 \|u\|_{2,\Omega}. \tag{13}$$

conforming case: Andreev 1990
nonconforming case: Andreev $\&$ Racheva ?

Eigenvalue Problem (Convex Domain)

The approximation by integral type nonconforming linear element gives asymptotic lower bounds of the exact eigenvalues:

Theorem 6 *Let* (λ_k, u_k) *and* $(\lambda_{h,k}, u_{h,k})$ *be the solutions of (4) and (5), respectively and let also the conditions of Lemma 1 be fulfilled.*

Assume that Ω *is* <u>*convex*</u> *and eigenfunctions being normalized* $\|u_k\|_{0,\Omega} = \|u_{h,k}\|_{0,\Omega} = 1$. *If the mesh parameter* h *is small enough, then:*

$$\lambda_{h,k} \leq \lambda_k. \tag{14}$$

Eigenvalue Problem (Nonconvex Domain)

$$\lambda_k - \lambda_{h,k} = a_h(u_k, u_k) - a_h(u_{h,k}, u_{h,k})$$

$$= a_h(u_k - u_{h,k}, u_k - u_{h,k}) + 2a_h(u_k, u_{h,k}) - 2a_h(u_{h,k}, u_{h,k})$$

$$= \underbrace{\|u_k - u_{h,k}\|_h^2}_{\mathcal{O}(h^{2r})} - \underbrace{\lambda_{h,k}\|i_h u_k - u_{h,k}\|_{0,\Omega}^2}_{\mathcal{O}(h^{4r})} + \underbrace{\lambda_{h,k}\left(\|i_h u_k\|_{0,\Omega}^2 - \|u_{h,k}\|_{0,\Omega}^2\right)}_{\mathcal{O}(h^2)?!}.$$

$r = \pi/\omega < 1$, $\omega > \pi$ is the maximal inner angle

THANK YOU!