On helical flows: vanishing viscosity limit and global existence for ideal fluids

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Introduction

Helical flow: invariant under helical symmetry.

Helical symmetry $\longrightarrow$ rotation and simultaneous translation along axis of rotation.

Helical structures are quite common in fluid flow, especially turbulent flows.

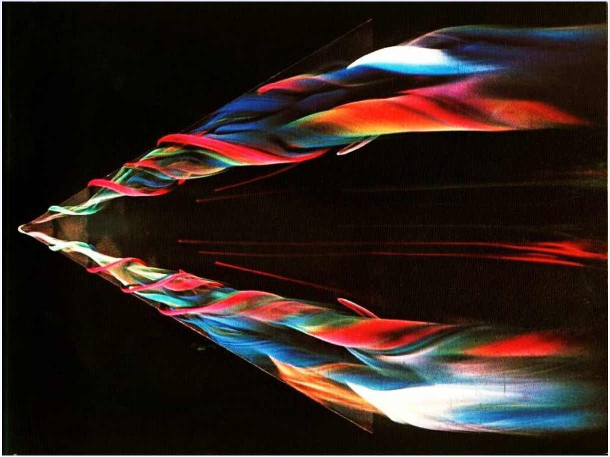

Figure: Vortices on the upper surface of a thin Delta wing. Henri Werl, ONERA. e-fluids Gallery of Images.

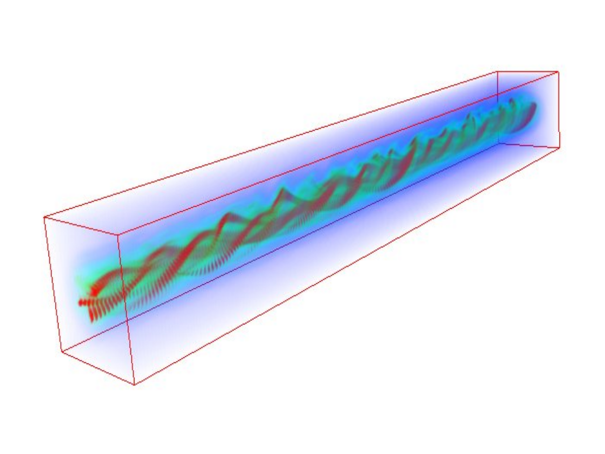

Figure: Volume visualization of vorticity magnitude in helical flow. Aeronautics and Astronautics, UT Austin.

Let $\mathcal{D} \subset \mathbb{R}^3$ be open set.

Navier-Stokes equations for incompressible fluid flow:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u, & \text{in } \mathcal{D} \times (0, \infty) \\ \text{div } u = 0, & \text{in } \mathcal{D} \times [0, \infty). \end{cases} \tag{1}$$

Above, $u = (u_1, u_2, u_3)$ is velocity, p is (scalar) pressure, and $(u \cdot \nabla) u$ means:

$$\sum_{i=1}^{3} u_i \partial_{x_i} u_j, \quad j = 1, 2, 3.$$

$\nu = 0 \longleftrightarrow$ Euler equations of incompressible, ideal fluid flow.

Flows with special symmetries

Axisymmetric flows: introduce rotation

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Say flow is axisymmetric if $u(R_\theta x) = R_\theta u(x)$ for every θ.

Alternatively, write u is cylindrical coordinates, cylindrical frame:

$$u = u^r \mathbf{e_r} + u^\theta \mathbf{e}_\theta + u^z \mathbf{e_3}.$$

Say flow is axisymmetric if components are independent of θ.

Relevant quantity: swirl,

$$u_\theta \equiv u \cdot \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \quad r = \sqrt{x^2 + y^2}.$$

Vanishing swirl is conserved by both Navier-Stokes and Euler.

Axisymmetric no-swirl [$u_\theta = 0$]:

$\nu \geq 0$ global $\exists$, smooth data.
Similar to $2D$, no vortex stretching.

Axisymmetric with swirl:

have vortex stretching; global $\exists$ ($\nu > 0$) if fluid domain far from symmetry axis.

Flows wth helical symmetry

Introduce helical translation

$$S_{\theta,\kappa}X = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \kappa\theta \end{bmatrix}.$$

Say u is helically symmetric if $u(S_{\theta,\kappa}x) = R_\theta u(x)$.

In particular, helical symmetry $\Longrightarrow$ flow is $2\pi\kappa$-periodic in z.

$\kappa = 0$: axisymmetric flow.

Alternative: in cylindrical coordinates and cylindrical frame, flow is helical if each component depends on t, r and $\alpha\theta + (\kappa)^{-1}z$.

Investigated rigorously by a few authors: Dutrifoy 1999, Titi-Mahalov-Leibowich 1990, Ettinger-Titi, 2009 (incompressible); Sun-Jiang-Guo (compressible).

Introduce special tangent vector to helices:

$$\xi = (y, -x, \kappa).$$

Some properties due to helical symmetry: f scalar function; say

f is helical if $f(S_{\theta,\kappa}x) = f(x) \iff \frac{\partial f}{\partial \xi} = 0$;

u helically symmetric (vector field) $\iff$

$$\begin{cases} \frac{\partial u_1}{\partial \xi} = u_2 \\ \\ \frac{\partial u_2}{\partial \xi} = -u_1 \\ \\ \frac{\partial u_3}{\partial \xi} = 0. \end{cases}$$

Lemma

Let u be helically symmetric and set $\eta \equiv u \cdot \xi$. *Then*

$$\omega \equiv \text{ curl } u = \omega_3 \frac{\xi}{\kappa} + \left(\frac{\partial \eta}{\partial y}, -\frac{\partial \eta}{\partial x}, 0 \right) \frac{1}{\kappa},$$

where $\omega_3 = \partial_x u_2 - \partial_y u_1$.

Call $\eta = u \cdot \xi$ helical swirl.

If ω is vorticity then

$$\frac{D\omega}{Dt} + \frac{1}{\kappa}\omega_3(u_2, -u_1, 0) + \frac{1}{\kappa}(\partial_x \eta \partial_y u - \partial_y \eta \partial_x u) = \nu \Delta \omega.$$

In inviscid case vanishing helical swirl prevents vortex stretching. Indeed, for $\nu = 0$ have

$$\frac{D\omega_3}{Dt} = 0.$$

Not so if $\nu > 0$.

$\exists$ results for helical flows:

no helical swirl: well-posedness for smooth data (Dutrifoy, 1999);

$\exists\,!$ weak solution, Euler, if curl $u = \omega \in L^\infty$ (Ettinger-Titi, 2009). Proof is Yudovich-type argument.

any helical swirl: also, $\exists\,!$ strong solution, ν-Navier-Stokes, bounded (helical) domain, no-slip at boundary, $u_0 \in H^1$, (Mahalov-Titi-Leibowich, 1990). Proof relies on key *a priori* estimate for all helical vector fields vanishing at "infinity":

$$\|u\|_{L^4} \leq C\kappa^{1/4}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2};$$

Some problems

Questions:

(1) vanishing viscosity limit for rough(er) initial data (than velocity in H^s, $s > 5/2$ – classical result)?

(2) existence, inviscid case, for rough(er) initial data (than vorticity in L^∞), zero helical swirl?

(3) what happens to η^ν as $\nu \to 0$, if initially nonzero and small?

Set $\Omega = \mathbb{R}^2 \times \mathbb{R}/2\pi\kappa\mathbb{Z}$.

(Joint work with Quansen Jiu, M. C. Lopes Filho and Dongjuan Niu.)

Let $u_0^\nu \in H^1_{\text{per}}(\Omega)$ i.e.

$$u_0^\nu(x, y, z + 2\pi\kappa) = u_0^\nu(x, y, z).$$

Given: $\eta_0^\nu \equiv u_0^\nu \cdot \xi, \quad$ where $\xi = (y, -x, 1)$.

Suppose

$$u_0^\nu \to u_0^E \quad \text{strongly in } L^2 \text{ as } \nu \to 0.$$

Assume also $u_0^E \cdot \xi = 0$. [Hence $\eta_0^\nu \to 0$ sL^2 as $\nu \to 0$.]

Let $u^\nu = u^\nu(x, y, z, t)$ be ν-NS solution, initial data u_0^ν.

The η-equation

$$\partial_t \eta + u \cdot \nabla \eta = \nu \Delta \eta + 2\nu\omega_3, \tag{2}$$

$\omega_3 = \partial_x u_2 - \partial_y u_1$.

Very nice – try equation for ω_3:

$$\partial_t \omega_3 + u \cdot \nabla \omega_3 + \partial_x \eta \partial_y u_3 - \partial_y \eta \partial_x u_3 = \nu \Delta \omega_3.$$

Cannot control red term... – vortex stretching taking place.

Instead...

Helical decomposition

Let u be HELICAL and DIV-FREE.

Introduce

$$W \equiv \eta \frac{\xi}{|\xi|^2}, \quad \eta = u \cdot \xi; \qquad V \equiv u - W.$$

Lemma

V and W are both HELICAL and DIV-FREE. Of course, $V \cdot \xi = 0$ and $V \cdot W = 0$.

Corollary

We have $\Omega \equiv curl\ V = \Omega_3 \xi$.

Equation for V

$$\partial_t V + V \cdot \nabla V + \nabla p - \nu \Delta V$$

$$= -\frac{\eta}{|\xi|^2}\partial_\xi V - \left(V \cdot \nabla \left[\frac{\xi}{|\xi|^2}\right]\right)\eta - \frac{\eta^2}{|\xi|^2}\partial_\xi\left(\frac{\xi}{|\xi|^2}\right)$$

$$+2\nu\nabla\eta\cdot\nabla\left(\frac{\xi}{|\xi|^2}\right) + \nu\eta\Delta\left[\frac{\xi}{|\xi|^2}\right] - 2\nu\Omega_3\frac{\xi}{|\xi|^2} - 2\nu\left[\text{curl }\left(\frac{\eta\xi}{|\xi|^2}\right)\right]_3\frac{\xi}{|\xi|^2}.$$

AWFUL LOOKING BEAST!

Equation for Ω_3

(Summarized version.)

$$\partial_t \Omega_3 + V \cdot \nabla \Omega_3 - \nu \Delta \Omega_3$$

$$= D_{x,y}\left(\frac{(V_1,V_2)}{|\xi|}\,\eta\, g_1(\xi)\right) + D_{x,y}\left(|\eta|^2 g_2(\xi)\right) \qquad (3)$$

$$+\nu\left[D^2_{x,y}(\eta g_3(\xi)) + D_{x,y}(\eta g_4(\xi)) + D_{x,y}(\Omega_3\, g_5(\xi))\right],$$

with all $g_j' s$ bounded.

Now we estimate.

Start with energy estimate for u^ν

$\|u^\nu\|_{L^\infty((0,T);L^2(\Omega))} \leq C$, independent of ν, as long as $\boxed{\|u_0^\nu\|_{L^2} \leq C}$;

and, for η, deduce easily that
$\|\eta^\nu\|_{L^\infty((0,T);L^2(\Omega))} \leq C\sqrt{\nu}$, C independent of ν, as long as $\boxed{\|\eta_0^\nu\|_{L^2} \leq C\sqrt{\nu}}$,

$\|\nabla\eta^\nu\|_{L^2((0,T);L^2(\Omega))} \leq C$, independent of ν.

Next, multiply η-equation by $\eta|\eta|$ to get that

$$\frac{d}{dt}\|\eta\|_{L^3}^3 + 2\nu\int|\eta||\nabla\eta|^2 = 2\nu\int\eta|\eta|\omega_3$$

$$= 2\nu\int\eta|\eta|\left(\Omega_3 + \left[\operatorname{curl}\left(\frac{\eta\xi}{|\xi|^2}\right)\right]_3\right)$$

$$\leq C\nu\|\eta\|_{L^4}^2\|\Omega_3\|_{L^2} + C\nu\|\eta\|_{L^3}\left(\int|\eta||\nabla\eta|^2\right)^{1/2}$$

$$\leq C\nu^{3/2}\|\nabla\eta\|_{L^2}\|\Omega_3\|_{L^2} + C\nu\|\eta\|_{L^3}^3 + \nu\int|\eta||\nabla\eta|^2.$$

Use Gronwall to arrive at

$$\|\eta\|_{L^3} \leq C\sqrt{\nu}(1 + \|\Omega_3\|_{L^2(L^2)})^{1/3},$$

as long as $\boxed{\|\eta_0\|_{L^3} \leq C\sqrt{\nu}}$.

Use this in the Ω_3-equation to find

$$\frac{d}{dt}\|\Omega_3\|_{L^2}^2 + \nu\|\nabla\Omega_3\|_{L^2}^2$$

$$\leq \frac{\nu}{2}\|\nabla\Omega_3\|_{L^2}^2 + \frac{C}{\nu}\left(\|\eta\|_{L^4}^4 + \|\eta\|_{L^3}^2\left\|\frac{V}{|\xi|}\right\|_{L^6}^2\right) + C\nu(\|\eta\|_{H^1}^2 + \|\Omega_3\|_{L^2}^2).$$

Estimate the V-term by

$$\left\| \frac{V}{|\xi|} \right\|_{L^6}^2 \leq \left\| \frac{V}{|\xi|} \right\|_{L^2}^{2/3} \left\| \nabla \left(\frac{V}{|\xi|} \right) \right\|_{L^2}^{4/3},$$

and

$$\left\| \nabla \left(\frac{V}{|\xi|} \right) \right\|_{L^2} \leq \left\| \operatorname{div} \left(\frac{V}{|\xi|} \right) \right\|_{L^2} + \left\| \operatorname{curl} \left(\frac{V}{|\xi|} \right) \right\|_{L^2}$$

$$\leq C(\|V\|_{L^2} + \|\Omega_3\|_{L^2}).$$

Hence,

$$\frac{d}{dt}\|\Omega_3\|_{L^2}^2 \leq \frac{C}{\nu}\left(\|\eta\|_{L^2}^2\|\nabla\eta\|_{L^2}^2 + \|\eta\|_{L^3}^2\|V\|_{L^2}^{2/3}(\|V\|_{L^2}^{4/3} + \|\Omega_3\|_{L^2}^{4/3})\right)$$
$$+C\nu(\|\eta\|_{H^1}^2 + \|\Omega_3\|_{L^2}^2).$$

Therefore,

$$\frac{d}{dt}\|\Omega_3\|_{L^2}^2 \leq C\left(\|\nabla\eta\|_{L^2}^2 + (1 + \|\Omega_3\|_{L^2(L^2)}^{2/3})(1 + \|\Omega_3\|_{L^2}^{4/3})\right)$$
$$+C\nu(\|\eta\|_{H^1}^2 + \|\Omega_3\|_{L^2}^2).$$

From here, use Gronwall to get

$$\|\Omega_3\|_{L^\infty(L^2)} \leq C,$$

as long as $\boxed{\|\Omega_3(t=0)\|_{L^2} \leq C}$.

Summary of available estimates for u^ν

Hypothesis': u_0^ν bdd in L^2; $\|\eta_0^\nu\|_{L^2} \leq C\sqrt{\nu}$; $\|\eta_0^\nu\|_{L^3} \leq C\sqrt{\nu}$; $\|\Omega_3(t=0)\|_{L^2} \leq C$. Then:

$\|u^\nu\|_{L^\infty(dt;L^2(dx))} \leq C$, independent of ν;

$$\text{curl } u^\nu = \text{ curl } V^\nu + \text{ curl } \eta^\nu \left(\frac{\xi}{|\xi|^2}\right) = \Omega_3^\nu \xi + \text{ curl } \eta^\nu \left(\frac{\xi}{|\xi|^2}\right);$$

$\|\Omega_3^\nu\|_{L^\infty(dt;L^2(dx))} \leq C$, independent of ν;

$\|\eta^\nu\|_{L^2(dt;H^1(dx))} \leq C$, independent of ν.

Navier-Stokes equation gives temporal estimates for equicontinuity.

Vanishing viscosity result

Theorem

Let $u_0^\nu \in H^1_{per}(\mathcal{D})$ such that

- $\|\eta_0^\nu\|_{L^2}, \ \|\eta_0^\nu\|_{L^3} \leq C\sqrt{\nu}$;
- $u_0^\nu \rightharpoonup u_0$ *weakly in* H^1_{per}.

Let u^ν be ν-NS solution initially u_0^ν. Then, passing to subsequences as needed,

$$u^\nu \to u$$

strongly in $C(dt; L^2)$; $u \in L^\infty(dt; L^2) \cap L^2(dt; H^1_{per})$ is a weak solution of 3D Euler with helical symmetry; $u \cdot \xi = 0$ a.e.

Obs. We got existence in H^1 “for free”.

Examine hypothesis. Most uncomfortable one is: $u_0^\nu \in L^2$.

Analogous to 2D flow. Indeed, if $\int \omega dx = 0$ then $|u| = \mathcal{O}(|x^2 + y^2|^{-1})$ at infinity. Otherwise, must deal with stationary flow with no decay – DiPerna-Majda decomposition.

If $\varphi \in C_c^\infty(0, +\infty)$ then the vector field

$$\Lambda \equiv \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, \frac{1}{\kappa} \right) \int_0^{\sqrt{x^2+y^2}} r\varphi(r)\, dr$$

is **div-free, helical**, has **vanishing helical swirl**, its **curl** is $\varphi(\sqrt{x^2+y^2})\xi$ and it is a **stationary** Euler solution. For large x, y third component is constant.

To deal with full space, subtract stationary flow and do energy estimates for what is left. (If $\int \omega dx = 0$ then OK.) This is quite messy, but works – evolve “stationary part” with heat flow.

Critical regularity

Assume $u_0 \cdot \xi = 0$, u_0 helical. Take smooth approximations $u_0^n \to u_0$ strongly in (some) L^r, some r.

(Substract stationary part if needed.) Have curl $u = \omega_3 \xi / \kappa$. Assume $\omega_{3,0} \in L^p$ some p. Which p?

"Easy" result: $u \in L^q$, compact if $q < 3p/(3-p)$. Can take $q = p' = p/(p-1)$ as long as $p > 3/2$. Then u compact in $L^{p'}$ so can pass to limit in nonlinear term.

Gives existence of weak solution if $\omega_3 \in L^p$, $p > 3/2$.

Alternative: use Delort-type symmetrization. Joint work with A. Bronzi and M. C. Lopes Filho.

Write: $u = K * (\omega_3 \xi)$.

Estimate $K = K(x)$ using Bessel-Fourier series.

Get $|K(x)| \leq C(|x|^{-2} + |\tilde{x}|^{-1})$, $\tilde{x} = (x_1, x_2, 0)$.

Symmetrize nonlinear term in weak form:

$$\int \nabla \psi(x) \int K(x-y) \times \xi(y) \omega_3(y)\, dy \omega_3(x) \xi(x)\, dx$$

$$= \int \int \mathcal{H}_\psi(x,y) \omega_3(x) \omega_3(y)\, dxdy,$$

where

$$\mathcal{H}_\psi = \frac{1}{2\kappa} K(x-y)\cdot(\xi(y)\times(\nabla\psi(x)-\nabla\psi(y))-(\xi(x)-\xi(y))\times\nabla\psi(y)).$$

With this get $|\mathcal{H}_\psi(x,y)| \leq C(|x-y|^{-1} + |\tilde{x}-\tilde{y}|^{-1})$.

Critical regularity becomes like 2D: $\omega \in L^p$, $p > 4/3$, "same as" for 2D.

Helical versus 2D versus axial symmetry

(Work in progress, joint with M. C. Lopes Filho, Dongjuan Niu, E. Titi.)
Where "is" helical flow wrt 2D flow or axisymmetric flow?

Have two theorems. First theorem: $u^{\nu,\kappa}$ converges to 2D Euler solution as $\kappa \to \infty$. Converge is strong in L^2, uniform in time; error is of the order $\sqrt{1/\kappa}$ in L^2.

Second theorem: $u^{\nu,\kappa}$ converges to *radially symmetric* 2D Euler solution as $\kappa \to 0$. Must take certain averages *along* helices due to oscillations.

Conclusion: helical flow "interpolates" 2D flow and radially symmetric 2D flow – stays far from axial symmetry and other 3D flows.

Some open problems

1. What is the analogue of Delort's result for axisymmetric flow, regarding concentration of kinetic energy, for helical flow with vortex sheet regularity initial data?

2. Existence for Euler with non-vanishing helical swirl?

3. What is the "rate of confinement" for helical, no-helical swirl, flow with non-negative ω_3?

4. What is 'Beale-Kato-Majda' criterion adapted to helical symmetry?

Thank you!