# BAYESIAN INVERSE PROBLEMS FOR BURGERS AND HAMILTON-JACOBI EQUATIONS WITH WHITE NOISE FORCING

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# BURGERS AND HAMILTON-JACOBI EQUATIONS

We consider Burgers equation with white noise forcing:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = f(x)\dot{W}(t), \quad u \in \mathbb{R}^d,\ x \in \mathbb{R}^d.$$

Assume: $f(x) = -\nabla F(x)$, and $u(t,x) = \nabla \phi(t,x)$;

$\phi$ satisfies the Hamilton-Jacobi equation:

$$\frac{\partial \phi(x,t)}{\partial t} + \frac{1}{2}|\nabla \phi(x,t)|^2 + F(x)\dot{W}(t) = 0.$$

• Burgers equation is a model for studying turbulence. It also has applications in non-equilibrium mechanics.

• We are interested in long time behavior: we consider the equations on $(-\infty, T]$.

• Suppose at $t_1, t_2, \ldots, t_m$, observations are made for velocity $u$ and velocity potential $\phi$ (subject to Gaussian noise);

we make inference on the white noise forcing on $(-\infty, T]$.

• Bayesian inverse problem for Navier-Stokes equations with model errors (stochastic forcing) is considered by Cotter, Dashti, Robinson and Stuart (2009)

# SOLUTION FORMULA

- Given an initial condition $\phi(x, t_0) = \phi_0(x)$, $\phi(\cdot, t)$ is determined by *Lax operator*:

$$\phi(\cdot, t) = \mathcal{K}^W_{t_0,t}\phi_0.$$

- Lax-Oleinik formula:

$$\phi(x, t) = \inf\left\{\phi_0(\gamma(t_0)) + \int_{t_0}^{t} \frac{1}{2}|\dot{\gamma}(\tau)|^2 - F(\gamma(\tau))\dot{W}(\tau)d\tau\right\},$$

where inf is taken with respect to all *absolutely continuous curves* $\gamma$ s.t. $\gamma(t) = x$.

- We are interested in solutions that exist for all time, i.e.

$$\phi(\cdot, t) = \mathcal{K}^W_{t_0,t}\phi(\cdot, t_0), \quad \forall t_0 < t.$$

# PERIODIC SETTING

• E, Khanin, Mazel and Sinai (2000) and Iturriaga and Khanin (2003):

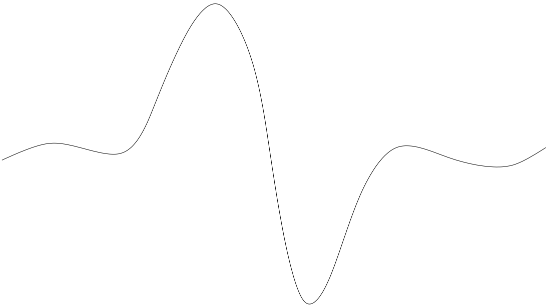
there is a unique solution $\phi$ (within an *additive constant*) that exists for all time, i.e.

$$\phi(\cdot, t) = \mathcal{K}^{W}_{t_0,t}\phi(\cdot, t_0), \quad \forall\, t_0 < t.$$

• For all $t$: $\phi(\cdot, t)$ is continuous, and Lipschitz.

• There is a unique spatially periodic solution $u(t, x)$ for the Burgers equation that exists for all time.

# NON-PERIODIC SETTING



- For potential $F(x)$ with a "big" maximum and a "big" minimum, H. and Khanin (2003) show that there is a solution $\phi$ and a solution $u$ that exist for all time.

- They are limit of finite time solutions with zero initial conditions.

# BAYESIAN INVERSE PROBLEM FOR H-J EQUATION

• Formulation: As $\phi$ is uniquely determined within a constant, and is continuous

$$\mathcal{G}(W) = \{\phi^W(x_i, t_i) - \phi^W(x_0, t_0),\ i = 1, \ldots, m\} \in \mathbb{R}^m,$$

is uniquely determined by $W$.

Let $y$ be a noisy observation of $\mathcal{G}(W)$:

$$y = \mathcal{G}(W) + \sigma.$$

The prior probability $\mu_0$ is the Wiener measure on $C(-\infty, t_{\max}]$ $(t_{\max} = \max t_i)$.

Determine $\mu^y(W) = \mathbb{P}(W|y)$.

## BAYESIAN INVERSE PROBLEM FOR H-J EQUATION

Assuming a Gaussian noise $\sigma \sim \mathcal{N}(0, \Sigma)$, we aim to show:

- Bayes' formula holds:

$$\frac{d\mu^y}{d\mu_0} \propto \exp(-\Phi(W; y))$$

where

$$\Phi(W; y) = \frac{1}{2}|y - \mathcal{G}(W)|_\Sigma^2 = \frac{1}{2}\langle \Sigma^{-1/2}(y - \mathcal{G}(W)), \Sigma^{-1/2}(y - \mathcal{G}(W))\rangle.$$

- The posterior $\mu^y$ is well-posed; in particular

$$d_{\mathrm{Hell}}(\mu^y, \mu^{y'}) \leq c(r)|y - y'|_{\mathbb{R}^m},$$

when $|y|_{\mathbb{R}^m} \leq r$ and $|y'|_{\mathbb{R}^m} \leq r$.

# BANACH SPACE SETTING

For $y = \mathcal{G}(x) + \sigma$, $x \in X$ a **Banach space**:

Cotter, Dashti, Robinson and Stuart showed:

(I) If $\mathcal{G} : X \to \mathbb{R}^m$ is measurable, e.g. when it is continuous with respect to $x$, the Bayes' formula holds.

(II) When $\mu_0$ is Gaussian,
when $|y_1|_{\mathbb{R}^m} \leq r$, $|y_2|_{\mathbb{R}^m} \leq r$

$$|\Phi(x; y_1) - \Phi(x; y_2)| \leq K(r)(1 + \|x\|_X^q)|y_1 - y_2|_{\mathbb{R}^m}$$

then the posterior measure $\mu^y$ is well-posed, i.e.

$$d_{\mathrm{Hell}}(\mu^{y_1}, \mu^{y_2}) \leq c(r)|y_1 - y_2|_{\mathbb{R}^m}.$$

# METRIC SPACE SETTING

- Our space $C(-\infty, t_{\max}]$ is not Banach;

- It is a metric space with the metric:

$$D(W_1, W_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{-n \le t \le t_{\max}} |W_1(t) - W_2(t)|}{1 + \sup_{-n \le t \le t_{\max}} |W_1(t) - W_2(t)|}.$$

- We need to formulate Bayesian inverse problems for **metric** spaces.

- For a metric space $X$, condition (I) of Cotter et al. still holds: If $\mathcal{G}$ is continuous, then the Bayes' formula holds.

# METRIC SPACE SETTING

**For well-posedness:** Condition (II) needs to be generalized.

(III)
i) $\Phi$ is locally bounded: for $r > 0$, if $|y|_{\mathbb{R}^m} \leq r$

$$0 \leq \Phi(x; y) \leq M(r),$$

for $x \in X(r) \subset X$, $\mu_0(X(r)) > 0$.

ii) There is a $G : \mathbb{R} \times X \to \mathbb{R}$: $G(r, .) \in L^2(X, d\mu_0)$, and

$$|\Phi(x; y) - \Phi(x; y')| \leq G(r, x)|y - y'|_{\mathbb{R}^m},$$

when $|y|_{\mathbb{R}^m} \leq r$ and $|y'|_{\mathbb{R}^m} \leq r$.

Then

$$d_{\mathrm{Hell}}(\mu^y, \mu^{y'}) \leq c(r)|y - y'|_{\mathbb{R}^m}.$$

# PERIODIC H-J EQUATION

- We consider the **periodic case** first:

the forcing function $f(x)$ and forcing potential $F(x)$ are periodic; problems are on $\mathbb{T}^d$.

- $$\mathcal{G}(W) = \{\phi(x_i, t_i) - \phi(x_0, t_0) : \ i = 1, \ldots, m\}.$$

- To show the validity of the Bayes' formula, we show that $\mathcal{G} : C(-\infty, t_{\max}] \to \mathbb{R}^m$ is continuous.

- To show well-posedness, we show conditions (III)(i) and (III)(ii).

# PERIODIC H-J EQUATION

- First we show that $\mathcal{G}$ is continuous.
- From the Lax-Oleinik formula: when $D(W_k, W) \to 0$, there are $c_k$ independent of $x_i$ and $t_i$ s.t.

$$\phi^{W_k}(x_i, t_i) - \phi^{W}(x_i, t_i) - c_k \to 0, \quad i = 0, 1, \ldots, m.$$

- $\mathcal{G}(W) = \{\phi^W(x_i, t_i) - \phi^W(x_0, t_0),\ i = 1, \ldots, m\}$,

$\mathcal{G} : C(-\infty, t_{\max}] \to \mathbb{R}^m$ is continuous.
- The Bayes' formula holds:

$$\frac{d\mu^y}{d\mu_0} \propto \exp(-\Phi(W; y)) = \exp(-\frac{1}{2}|y - \mathcal{G}(W)|_\Sigma^2).$$

# PERIODIC H-J EQUATION

- For well-posedness, we show (III)(i) and (III)(ii):

$$\text{(III)(i)} \qquad 0 \le \Phi(W; y) \le M(r) \ \text{ when } |y|_{\mathbb{R}^m} \le r;$$

$W \in X(r) \subset C(-\infty, t_{\max}]$, $X(r)$ of positive $\mu_0$ measure.

from the Lax operator

$$\begin{aligned} |\mathcal{G}(W)|_{\mathbb{R}^m} &= |\{\phi^W(x_i, t_i) - \phi^W(x_0, t_0)\}|_{\mathbb{R}^m} \\ &\le c\left(1 + \sum_{i=1}^{m} \max_{t_0 - 1 \le \tau \le t_i} |W(\tau) - W(t_i)|^2\right). \end{aligned}$$

$$\Phi(W; y) = \frac{1}{2}|y - \mathcal{G}(W)|_{\Sigma}^2 \le c(r + |\mathcal{G}(W)|_{\mathbb{R}^m})^2 \text{ when } |y|_{\mathbb{R}^m} \le r.$$

Fixing $M > 0$, the set $W$ s.t. $|\mathcal{G}(W)|_{\mathbb{R}^m} < M$ has a positive Wiener measure. (III)(i) is thus shown.

# PERIODIC H-J EQN

(III)(ii)

$$|\Phi(W;y)-\Phi(W;y')| \le G(r,W)|y-y'|_{\mathbb{R}^m}$$

when $|y|_{\mathbb{R}^m} \le r$, $|y'|_{\mathbb{R}^m} \le r$ and $G(r,\cdot)$ is in $L^2(C(-\infty,t_{\max}],d\mu_0)$.

- With $\Phi(W;y)=\frac{1}{2}|y-\mathcal{G}(W)|_{\Sigma}^2$:

$$\begin{aligned}|\Phi(W;y)-\Phi(W;y')|& \\ &\le \frac{1}{2}\|\Sigma^{-1/2}\|^2_{\mathbb{R}^m,\mathbb{R}^m}(|y|_{\mathbb{R}^m}+|y'|_{\mathbb{R}^m}+2|\mathcal{G}(W)|_{\mathbb{R}^m})|y-y'|_{\mathbb{R}^m} \\ &\le \|\Sigma^{-1/2}\|^2_{\mathbb{R}^m,\mathbb{R}^m}(r+|\mathcal{G}(W)|_{\mathbb{R}^m})|y-y'|_{\mathbb{R}^m}.\end{aligned}$$

With the bound in the previous slice, this is square integrable.

# NONPERIODIC H-J EQN

- We consider the **non-periodic** case where the forcing potential $F(x)$ has a "big" maximum and a "big" minimum.

- To show Bayes' formula, we show that

$$\mathcal{G}(W) = \{\phi^W(x_i, t_i) - \phi^W(x_0, t_0),\ i = 1, \dots, m\}$$

is continuous from $C(-\infty, t_{\max}]$ to $\mathbb{R}^m$.

This is shown similarly as in the periodic case: there are constants $c_k$ so that:

$$\lim_{k\to\infty} \phi^{W_k}(x_i, t_i) - \phi^W(x_i, t_i) - c_k = 0.$$

- The Bayes' formula thus holds.

# NONPERIODIC H-J EQN

- The well-posedness of the posterior measure $\mu^y$, we show conditions (III)(i) and (III)(ii).

- We show the bound:

$$|\mathcal{G}(W)|_{\mathbb{R}^m} \le S(W),$$

where

$$S(W) = c + c\sum_{i=1}^{m} \sum_{l=T_i(W)}^{t_{\max}} (1 + \max_{l\le\tau\le l+1} |W(\tau) - W(l+1)|^2).$$

- The constant $T_i(W)$ depends on the Wiener path $W$.

# NONPERIODIC H-J EQN

- The Lax operator

$$\phi(\cdot, t) = \mathcal{K}^{W}_{s,t}\phi(\cdot, s);$$

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$$\phi(x_i, t_i) = \inf_{\gamma,\ \gamma(t_i)=x_i} \left\{ \phi(\gamma(s), s)) + \int_{t_0}^{t} \frac{1}{2}|\dot{\gamma}(\tau)|^2 - F(\gamma(\tau))\dot{W}(\tau) d\tau \right\}.$$

- We show that all the minimizers $\gamma$ are inside a compact set at a time;

$\gamma$ must be inside the compact set at a time larger than $T_i(W)$, which is independent of $s$.

# NONPERIODIC H-J EQN

• For the condition (III)(i): we show

$$\Phi(W;y) = \frac{1}{2}|y - \mathcal{G}(W)|_\Sigma^2 \le c(r + |\mathcal{G}(W)|_{\mathbb{R}^m})^2 \le c(r + S(W))^2$$

is less than $M(r)$ for $W \in X(r)$ of positive Wiener measure.

• There is a constant $T$ such that the set of paths $W$ with $T_i(W) > T$ has a positive Wiener measure.

• We can choose a constant $M$ s.t. out of these paths, the set of paths $W$ such that $S(W) < M$ has a positive Wiener measure.

# NONPERIODIC H-J EQN

- For the condition (III)(ii):

$$\begin{aligned} |\Phi(W;y)-\Phi(W;y')| &\leq \|\Sigma^{-1/2}\|^2_{\mathbb{R}^m,\mathbb{R}^m}(r+G(W))|y-y'|_{\mathbb{R}^m}. \\ &\leq \|\Sigma^{-1/2}\|^2_{\mathbb{R}^m,\mathbb{R}^m}(r+S(W))|y-y'|_{\mathbb{R}^m}. \end{aligned}$$

- To show that $G(r,W)=\|\Sigma^{-1/2}\|^2_{\mathbb{R}^m,\mathbb{R}^m}(r+S(W))$ is in $L^2(C(-\infty,t_{\max}],d\mu_0)$

we show $S(W)\in L^2(C(-\infty,t_{\max}],d\mu_0)$.

- This is achieved by using estimates for the convergence rates for the law of large numbers.

# BAYESIAN INVERSE PROBLEMS FOR BURGERS EQN

- There are shocks where the solution $u$ is discontinuous; $u$ is not defined everywhere, but $u(\cdot, t) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for all $t$.

- For $i = 1, \ldots, m$, let $l_i : L^1_{\mathrm{loc}}(\mathbb{R}^d) \to \mathbb{R}$ be continuous and bounded.

- Define

$$\mathcal{G}(W) = (l_1(u(\cdot, t_1)), \ldots, l_m(u(\cdot, t_m))) \in \mathbb{R}^m.$$

- Noisy observation

$$y = \mathcal{G}(W) + \sigma.$$

- Determine $\mu^y(W) = \mathbb{P}(W|y)$.