On "complexity" of probability preserving transformations.

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0. Entropy and complexity

For $(X, \mathcal{B}, m, T)$ a probability preserving transformation, $P \subset \mathcal{B}$ a finite partition, the entropy of the process (T, P) is

- $h(T, P) := \lim_{n\to\infty} \frac{1}{n} H(P_n)$

where $P_n := \bigvee_{j=0}^{n-1} T^{-j}P$ & $H(Q) := -\sum_{A\in Q} m(A) \log m(A)$.
The entropy of the probability preserving transformation T is

- $h(T) := \sup_{P \text{ finite}} h(T, P)$.

☺ (Kolmogorov-Sinai) If P generates $\mathcal{B}$ under T, then $h(T) = h(T, P)$.

- Would like a "complexity sequence" a_n so that $a_n = e^{nh(T)(1+o(1))}$ when $0 < h(T) < \infty$ and which also measures "complexity" when $h(T) = 0$.

NB When $0 < h(T) < \infty$, $a_n := H(P_n)$ for P any finite, generator works, but not when $h(T) = 0$.

1. Hamming balls and complexity

For $(X, \mathcal{B}, m, T)$ a probability preserving transformation, $P \subset \mathcal{B}$ a countable partition, $n \geq 1$ define the $P - n$ `Hamming pseudo metric` on X:

$$d_n^{(P)}(x, y) := \frac{1}{n} \#\{0 \leq k \leq n-1 : \ P(T^k x) \neq P(T^k y)\}$$

where $z \in P(z) \in P$; and the $P - n$ `Hamming balls`

$$B(n, P, z, \epsilon) := \{x \in X : \ d_n^{(P)}(x, z) \leq \epsilon\} \in \sigma(P_n).$$

For $P \in \mathfrak{P}$, the `complexity of` T `with respect to` P is the collection $\{K(P, n, \epsilon)\}_{n \geq 1, \ \epsilon > 0} \subset \mathbb{N}$ defined by
$K(P, n, \epsilon) := \min \{\#F : \ F \subset X, \ m(\bigcup_{z \in F} B(n, P, z, \epsilon)) > 1 - \epsilon\}$.
Call a_n a `complexity sequence for` T if

$$\frac{\log K(P, n, \epsilon)}{a_n} \underset{n \to \infty, \ \epsilon \to 0}{\longrightarrow} 1 \ \forall \text{ generators } P.$$

2. Complexity basics

¶1 (monotonicity) $K(P,n,\epsilon)\uparrow$ as $P,\ n\ \uparrow\ \&\ \epsilon\ \downarrow$.

¶2 (stability) For $k\geq 1,\ \epsilon>0$ and large $n\geq 1$:

$$K(P_k,n,2\epsilon)\leq K(P,n,\tfrac{\epsilon}{k})\leq K(P_k,n,\tfrac{\epsilon}{2}).$$

¶3 (continuity) Let
$P=\{P_n\}_{n\geq 1},\ Q=\{Q_n\}_{n\geq 1}\in\mathfrak{P}:=\{\texttt{ordered partitions}\}$
with $\sum_{n\geq 1} m(P_n\Delta Q_n)<\delta$, then $\exists\ N$ such that $\forall\ n\geq N,\ \epsilon>0$

$$K(Q,n,\epsilon)\leq K(P,n,2\epsilon+2\delta).$$

¶4 (compactness) For any $P\in\mathfrak{P}$, $d_k>0,\ n_k\to\infty,\ \exists\ k_\ell\to\infty$ and $Y\in[0,\infty]$ such that

$$\frac{K(P,n_{k_\ell},\epsilon)}{d_{k_\ell}}\ \xrightarrow[\ell\to\infty,\ \epsilon\to 0]{}\ Y.$$

3. Complexity sequences when $0 < h(T) < \infty$.

Let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation with $0 < h(T) < \infty$, then $h(T)n$ is a complexity sequence for T.

This follows from

Theorem(`Blume, Ferenczi, Katok, Thouvenot`) For a finite $P \in \mathfrak{P}$,

$$\frac{1}{n}\log K(P, n, \epsilon) \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} h(T, P). \qquad (\bigstar)$$

4. Proof of (★)

$$\texttt{Set}\quad \Phi_{n,\epsilon} := \min\{|F| :\ F \subset P_n,\ m(\bigcup_{a\in F} a) > 1-\epsilon,$$

$$\mathcal{Q}(P,n,\epsilon) := \max\{\#\{c \in P_n :\ d_n(a,c) \le \epsilon\} :\quad a \in P_n\}$$

where d_n is Hamming distance on P_n then

$$\frac{\Phi_{n,\epsilon}}{\mathcal{Q}(P,n,\epsilon)} \le K^{(T)}(P,n,\epsilon) \le \Phi_{n,\epsilon} \qquad (🚲)$$

By SMB $\frac{1}{n}\log_2 \Phi_{n,\epsilon} \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} h(T,P)$.

$$\texttt{and}\quad \frac{1}{n}\log \mathcal{Q}(P,n,\epsilon) \le \frac{1}{n}\log\left(|P|^{\epsilon n}\binom{n}{\epsilon n}\right) \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} 0.\ ☑$$

5. Examples

¶1 **Finite complexity theorem** (Ferenczi '97) T has finite complexity in the sense that

$$\frac{K(P,n,\epsilon)}{a(n)} \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} 0\ \forall\ P \in \mathfrak{P},\ a(n) \to \infty$$

iff T has discrete spectrum. cf Kushnirenko '66, Ratner '71

$\therefore$ no complexity sequence for nontrivial discrete spectrum $\because$
$K(P,n,\epsilon) \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} \infty$ for generating P.

¶2 **Complexity of Chacon transformation**(Ferenczi '97) If $P \in \mathfrak{P}$ is a generator, then $\frac{K(P,n,\epsilon)}{2n} \underset{n\to\infty,\ \epsilon\to 0}{\longrightarrow} 1$.

¶3 **Positive entropy dimension** (Ferenczi-Park '07)
$\forall\ \alpha \in (0,1),\ \exists$ an ergodic, probability preserving transformation with $K(P,n,\epsilon) = \exp[n^{\alpha(1+o(1))}]$ as $n \to \infty,\ \epsilon \to 0$.

6. Relative Complexity

For $\mathcal{C} \subset \mathcal{B}$ a factor and $P \in \mathfrak{P}$, the relative complexity of T with respect to P given $\mathcal{C}$ is the collection $\{K_{\mathcal{C}}(P, n, \epsilon)\}_{n \geq 1,\ \epsilon > 0}$ of $\mathcal{C}$-measurable random variables defined by

$$K_{\mathcal{C}}(P, n, \epsilon)(x) := \min\{\#F : \ F \subset X,\ m(\bigcup_{z \in F} B(n, P, z, \epsilon) \| \mathcal{C})(x) > 1 - \epsilon\}$$

where $m(\cdot \| \mathcal{C})$ denotes conditional measure with respect to $\mathcal{C}$.
As before,

$$\frac{1}{n} \log K_{\mathcal{C}}(P, n, \epsilon) \xrightarrow[n \to \infty,\ \epsilon \to 0]{m} h(T, P \| \mathcal{C}) \qquad (\bigstar)$$

where $\xrightarrow{m}$ denotes convergence in measure and $h(T, P \| \mathcal{C})$ denotes the relative entropy of the process (P, T) with respect to $\mathcal{C}$.

7. Relative complexity basics

¶1. `Distributional compactness`

- $\forall\ P \in \mathfrak{P},\ d_k > 0,\ \ n_k \to \infty,\ \ \exists\ k_\ell \to \infty$ and a random variable Y on $[0, \infty]$ such that

$$\frac{\log K_{\mathcal{C}}(P, n_{k_\ell}, \epsilon)}{d_{k_\ell}} \quad \underset{\ell \to \infty,\ \epsilon \to 0}{\overset{\mathfrak{d}}{\longrightarrow}} \quad Y \tag{a}$$

where $\overset{\mathfrak{d}}{\longrightarrow}$ denotes convergence in distribution.

¶2. `Generator theorem`

- If $\exists$ a countable T-generator P satisfying

$$\frac{\log K_{\mathcal{C}}(P, n_k, \epsilon)}{d_k} \quad \underset{k \to \infty,\ \epsilon \to 0}{\overset{\mathfrak{d}}{\longrightarrow}} \quad Y,$$

where Y is a random variable on $[0, \infty]$, then

$$\frac{\log K_{\mathcal{C}}(Q, n_k, \epsilon)}{d_k} \quad \underset{k \to \infty,\ \epsilon \to 0}{\overset{\mathfrak{d}}{\longrightarrow}} \quad Y \ \ \forall\ T\text{-generators } Q \in \mathfrak{P}. \qquad (\sqcup)$$

`Abbreviation:` $\quad (\sqcup) \iff \frac{1}{d_k} \log K_{\mathcal{C}}^{(T)}(n_k) \approx Y.$

8. Relative entropy dimension

Let $(X, \mathcal{B}, m, T)$ be a ppt, let $\mathcal{C} \subset \mathcal{B}$ be a factor and let $\mathcal{K} = \{n_k\}_k,\ n_k \to \infty$.

- *Upper relative entropy dimension of T with respect to $\mathcal{C}$ along $\mathcal{K}$* is

$$\overline{\texttt{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) := \inf\{t \geq 0 : \tfrac{\log K(P, n_k, \epsilon)}{n_k^t} \underset{k\to\infty,\ \epsilon\to 0}{\xrightarrow{m}} 0 \ \forall \ \ P \in \mathfrak{P}\}.$$

- *Lower relative entropy dimension of T with respect to $\mathcal{C}$ along $\mathcal{K}$* is

$$\underline{\texttt{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) := \sup\{t \geq 0 : \exists\, P \in \mathfrak{P},\ \tfrac{\log K(P, n_k, \epsilon)}{n_k^t} \underset{k\to\infty,\ \epsilon\to 0}{\xrightarrow{m}} \infty\}.$$

¶ If $\exists$ a countable T-generator P satisfying $\frac{1}{d_k} \log K_{\mathcal{C}}(P, n_k, \epsilon) \underset{k\,\to\,\infty,\ \epsilon\,\to\,0}{\xrightarrow{\mathfrak{d}}} Y$, where Y is a rv on $(0, \infty)$, then

$$\overline{\texttt{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) = \varlimsup_{k\to\infty} \frac{\log d_k}{\log n_k} \ \ \& \ \ \underline{\texttt{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) = \varliminf_{k\to\infty} \frac{\log d_k}{\log n_k}.$$

9. Relative isomorphism

Let $\mathcal{X}_i = (X_i, \mathcal{B}_i, m_i, T_i)$ $(i=1,2)$ be ppts with factors $\mathcal{C}_i \subseteq \mathcal{B}_i$. The ppts $\mathcal{X}_1, \mathcal{X}_2$ are relatively isomorphic over $\mathcal{C}_1, \mathcal{C}_2$ if $\exists$ a isomorphism $\pi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ with $\pi \mathcal{C}_1 = \mathcal{C}_2$.

Corollary In this case,

$$\frac{1}{d_k} K_{\mathcal{C}_1}^{(T_1)}(n_k) \approx Y \iff \frac{1}{d_k} K_{\mathcal{C}_2}^{(T_2)}(n_k) \approx Y.$$

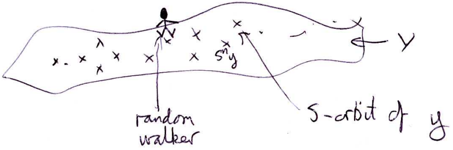
10. Definition of RWRS

Ingredients:

- $(\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots)$ iid rvs on $\mathbb{Z}$,
- $S_n = \sum_{k=0}^{n-1} \xi_k$, the random walk,
- $\Omega := \mathbb{Z}^{\mathbb{Z}}$, $\mu := \prod \operatorname{dist} \xi$, $R :=$ left shift on Ω
- $(Y, \mathcal{C}, \nu, S)$ ppt aka the scenery process

RWRS is the ppt $(X, \mathcal{B}, m, T)$ where

$X := \Omega \times Y$ $m := \mu \times \nu$, $T(x, y) := (Rx, S^{x_0} y)$



11. Relative entropy of RWRS over its base

¶Suppose the RWRS T has scenery S, and that the random walk is aperiodic and recurrent, then $h(T\|\mathtt{base}) = 0$.

Proof Fix a finite scenery partition β and let $P(x, y) := \alpha(x) \times \beta(y)$ where $\alpha(x) := [x_0]$, then

$$P_0^{n-1}(T)(x, y) = \alpha_0^{n-1}(R)(x) \times \beta_{V_n(x)}(y)$$

where $V_n(x) := \{s_k(x) := x_1 + x_2 + \cdots + x_k\}_{k=0}^{n-1}$.
By Spitzer's theorem, $E(\# V_n) = o(n)$, so

$$h((T, P)\|\mathtt{base}) \leftarrow \frac{1}{n} I(P_0^{n-1}(T)\|\mathtt{base})(x) \leq \frac{1}{n} H(\beta)\# V_n(x) \longrightarrow 0.$$

12. α-stability and random walks

Let $0 < \alpha < 2$. The random variable Y_α has standard symmetric α-stable (SαS) distribution if $E(e^{itY_\alpha}) = e^{-\frac{|t|^\alpha}{\alpha}}$.

For $1 < \alpha \leq 2$, $E(|Y_\alpha|) < \infty$ & $E(Y_\alpha) = 0$.

- The SαS process is a random variable B_α on $D([0,1]) := \{\texttt{CADLAG functions}\}$ with independent, SαS increments.
- B_2 is aka Brownian motion.
- The $\mathbb{Z}$-valued random walk is called:

– aperiodic if $E(e^{it\xi}) = 1 \iff t \in 2\pi\mathbb{Z}$;

– α-stable if $\exists\ a(n)$ such that $\frac{1}{a(n)} S_n \xrightarrow[n\to\infty]{\mathfrak{d}} Y_\alpha$.

The $a(n)$ are aka the normalizing constants of the random walk and are regularly varying with index $\frac{1}{\alpha}$.

13. Asymptotic complexity of aperiodic, α-stable RWRS

Theorem 2 Suppose that $(Z, \mathcal{B}(Z), m, T)$ RWRS with α-stable, aperiodic jumps ($\alpha > 1$), ergodic scenery $(Y, \mathcal{C}, \nu, S)$, $0 < h(S) < \infty$, then

$$\frac{1}{a(n)} \log K^{(T)}_{\mathcal{B}(\Omega)\times Y}(n) \quad \approx \quad \mathtt{Leb}\,(B_\alpha([0,1])) \cdot h \tag{1}$$

$$\mathtt{E\text{-}dim}_{\mathbb{N}}(T, \mathcal{B}(\Omega) \times Y) \;=\; \frac{1}{\alpha}. \tag{2}$$

Corollary Suppose RWRSs $(Z_i, \mathcal{B}_i, m_i, T_i)$ $(i = 1, 2)$ are relatively isomorphic over their bases.
If $(Z_1, \mathcal{B}_1, m_1, T_1)$ has α-stable jumps, pos. finite scenery entropy, then so does $(Z_2, \mathcal{B}_2, m_2, T_2)$ and

$$a_{\xi^{(2)}}(n) h(S^{(2)}) \underset{n\to\infty}{\sim} a_{\xi^{(1)}}(n) h(S^{(1)}).$$

14. Proof of theorem 2

Define $P \in \mathfrak{P}(Z, \mathcal{B}, m)$ by $P(x, y) := \alpha(x) \times \beta(y)$ where $\alpha(x) := [x_0]$, $\beta \in \mathfrak{P}(Y, \mathcal{C}, \mu)$ finite, S-generator.

Recall $\Pi_n(x) := \{a \in P_0^{n-1}(T) : \ m(a \| \mathcal{B}(\Omega) \times Y)(x) > 0\};$

$$\Phi_{n,\epsilon}(x) := \min\{\#F : \ F \subset \Pi_n(x) : \ m(\bigcup_{a \in F} a \| \mathcal{B}(\Omega) \times Y)(x) > 1 - \epsilon\};$$

$$\mathcal{Q}(P, n, \epsilon)(x) := \max\{\#\{c \in \Pi_n(x) : \ d_n^{(P)}(a, c) \le \epsilon\} : \ a \in \Pi_n(x)\}$$

where $d_n^{(P)}$ is (n, P)-Hamming distance on $P_0^{n-1}(T)$ then $\cdots$

15. Proof strategy

$$\frac{\Phi_{n,\epsilon}(x)}{\mathcal{Q}(P,n,\epsilon)(x)} \leq K^{(T)}_{\mathcal{B}(\Omega)\times Y}(P,n,\epsilon)(x) \leq \Phi_{n,\epsilon}(x). \tag{🚲}$$

We prove

$$\frac{1}{a(n)} \log_2 \Phi_{n,\epsilon} \xrightarrow[n\to\infty,\ \epsilon\to\ 0]{\mathfrak{d}} \mathtt{Leb}\,(B_\alpha([0,1]))h(S) \tag{⚽}$$

and

$$\frac{1}{a(n)} \log_2 \mathcal{Q}(P,n,\epsilon) \xrightarrow[n\to\infty,\ \epsilon\to 0]{m} 0. \tag{☢}$$

16. Proof sketch of ⚽

This uses `SMB along Følner sets` and the `functional CLT`, for α-stable, random walk $S_n = \sum_{k=1}^{n} \xi_k$:

$B_{\xi,n} \xrightarrow{\mathfrak{d}} B_\alpha$ in $D([0,1])$ where $B_{\xi,n}(t) := \frac{1}{a_\xi(n)} S_{[nt]}$ and $a_\xi(n)$ are the normalizing constants.

$$\therefore \ \frac{1}{a(n)} V_n \xrightarrow{\mathfrak{d}} \overline{B_\alpha([0,1])} \ \texttt{ in } \ \mathcal{H} := \{\texttt{compact sets in } \mathbb{R}\}.$$

Now a.s. $B_\alpha([0,1])$ is Riemann integrable so can approx. V_n by sets of form $(a(n)F) \cap \mathbb{Z}$ with F finite union of intervals

which latter are Følner sets. $\therefore$

$$\frac{1}{a(n)} I(P_0^{n-1}(T) \| \mathcal{B}(\Omega) \times Y)(x,y) = \frac{1}{a(n)} I(\beta_{V_n(x)})(y) \xrightarrow{\mathfrak{d}} \texttt{Leb}\,(B_\alpha([0,1]))$$

whence ⚽.

17. Proof of ☢ needs local time

For $1 < \alpha \leq 2$, `local time at` $x \in \mathbb{R}$ of SαS process B_α defined by

$$\mathfrak{t}_\alpha(t, x) := \lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_0^t 1_{[x-\epsilon, x+\epsilon]}(B_\alpha(s))ds.$$

- Boyland: a.s., $\mathfrak{t}_\alpha \in C_0([0,1] \times \mathbb{R})$.
- Eisenbaum-Kaspi: a.s. $\mathfrak{t}_\alpha(1, x) > 0$ for `Leb`-a.e. $x \in B_\alpha([0,1])$.

18. Local time of α-stable rw:

$$N_{n,k} := \#\{1 \le j \le n : \; S_j = k\}.$$

Interpolated local time of rw:

$N_n \in C_0(\mathbb{R})$ `defined by` $N_n(t) := (\lceil t \rceil - t) N_{n,\lfloor t \rfloor} + (t - \lfloor t \rfloor) N_{n,\lceil t \rceil}$.

☺ **Functional CLT for local time** ([Borodin]):

$$(B_n, \mathfrak{t}_n) \xrightarrow[n\to\infty]{\mathfrak{d}} (B_\alpha(\cdot), \mathfrak{t}_\alpha(1,\cdot)) \text{ in } D([0,1]) \times C_0(\mathbb{R})$$

where $B_n(t) := \frac{1}{a(n)} S_n$ & $\mathfrak{t}_n(x) := \frac{a(n)}{n} N_n(a(n)x)$.

☺ **Corollary**: For $E \subset \mathbb{R}$ a finite union of closed, bounded intervals,

$$Y_{E,n} := \frac{a_\xi(n)}{n} \min_{k \in a_\xi(n) E} N_{n,k} = \min_{x \in E} N_n(x) \xrightarrow[n\to\infty]{\mathfrak{d}} \min_{x \in E} \mathfrak{t}_\alpha(1,x) =: \mathfrak{m}_E.$$

19. Vague idea of proof of ☢

Fix $\epsilon > 0$.

¶1 For a large set of $x \in \Omega$, $\exists\ E,\ F \subset \mathbb{R}$, finite unions of intervals with $\mathtt{Leb}\,(F \setminus E) < \epsilon$ such that $a(n)E \cap \mathbb{Z} \subset V_n(x) \subset a(n)F \cap \mathbb{Z}$ and $Y_{E,n}(x) > 0$.

¶2 For such x and large n, $a \in \Pi_n(x)$ is of form

$$a = (x_0^{n-1}, w) := [x_0^{n-1}] \times \bigvee_{j \in V_n(x)} S^{-j} w_j \quad (w_j \in \beta).$$

¶3 For such $n,\ x,\ \ a = (x_0^{n-1}, w),\ a' = (x_0^{n-1}, w') \in \Pi_n(x)$,
$\#\{j \in V_n(x) :\ w_j \neq w'_j\} \leq a(n)(\epsilon + \frac{d_n^{(P)}(a,a')}{Y_{E,n}(x)})$.

- $\mathtt{i.e.}$, for $a = (x_0^{n-1}, w),\ \mathcal{E} > 0$,

$\rho_{n,x} - \mathtt{diam}(B(n, P, a, \mathcal{E}) \cap \Pi_n(x)) \leq \Delta$ where $\rho_{n,x}((x_0^{n-1}, w), (x_0^{n-1}, w')) := \frac{1}{a(n)} \#\{j \in V_n(x) :\ w_j \neq w'_j\}$ & $\Delta = \epsilon + \frac{2\mathcal{E}}{Y_{E,n}(x)}$. As in the proof of ★,

¶4 $\dfrac{1}{a(n)} \log \mathcal{Q}(P, n, \delta)(x) \leq \dfrac{1}{a(n)} \log\left(|P|^{\Delta a(n)} \binom{|a(n)F|}{\Delta a(n)}\right) \rightsquigarrow 0.$ ☑

20. Problems

¶1 Theorem 2 does not apply to 1-dimensional RWRS whose jump random variables are 1-stable or to 2-dimensional random walks whose jump random variables are centered and in the domain of attraction of standard normal distribution on $\mathbb{R}^2$.

- Is it true that in both cases $\mathtt{E\text{-}dim}_{\mathcal{K}}(T, \mathcal{C}) = 1$?

¶2 What about RWRS's with the random walk non-lattice?

¶3 What about "smooth RWRS's"?

`Thank you for listening.`