Equidist on some (sigh) nilmanifolds (joint work with S. Forni)

General problem: Quantitative estimates on the convergence of Birkhoff averages for uniquely ergodic system.

- Surface flows (Forni 2003)
- Horocycle flows (F+ Forni 2003)
- Flows on nilmanifolds.

$N = \text{nilpotent connected, simply connected,}
\mathbb{H}^q$ lattice
$x \in \text{Lie}(N)$.

$\Phi^t_x (Fy) = Fy \exp t x$.

$N \triangleleft \mathbb{R}^d \quad N^{(i)} = \prod_i \mathbb{R}^d \quad N^{(i)} = \left\{ [N, N] \right\}
\mathbb{R}^d \rightarrow \phi_x \rightarrow \text{linear flow}$

Auslander, Green, Hahn

$\Phi^t_x$ uniquely ergodic $\phi^t_x$ uniquely ergodic

$X$ invariant vector.
\[ f = \text{space of metrics on } \mathbb{R}^n \]

\[ \text{Sufficient } \mathcal{M} \text{ with zero average.} \]

\[ L^2(N) = \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{i=1}^{n} \right)^2 \]

A) \[ \text{Hom}^3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & P \frac{r}{2} \\ 0 & 1 \end{pmatrix} \right\} \]

\[ \Gamma = \text{Hom}^3(\mathbb{Z}) \]

\[ X = \begin{pmatrix} 0 & 1 & x \\ 0 & x & 0 \end{pmatrix} \]

B) \[ M(\mathcal{M}) = \prod_{k=1}^{\infty} \frac{1}{B} \]

\[ B(y_1, y_{k+1}) = \left( y_1, y_2 + y_3, \ldots, y_{k+1} \right) \]

\[ \mathcal{M}(\mathcal{M}) = \text{Fix}(\mathcal{M}) \]

\[ \text{Lie}(\text{Fix}(\mathcal{M})) = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right\rangle \]

\[ X = \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \]

\[ \left[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = \frac{\partial}{\partial x_{i+j}} \]

(of local forms: \[ \sum_{n=0}^{\infty} e^{\text{can}^k} \])

\[ \text{Thm: Suppose } \alpha^k \text{ satisfies } \]

\[ \| \alpha^k \| \geq C \]

\[ \text{Then } \text{The flow } \Phi^t \text{ on } \mathcal{M}(\mathcal{M}) \text{ satisfies } \]

\[ T \in \mathcal{M}(\mathcal{M}) \text{ such that } \forall \gamma \in \mathcal{C}(\mathcal{M}) \text{ and sufficiently large } t \text{ we have.} \]
\[ \frac{1}{T} \int_0^T \Phi(t) \, dt \leq e^{\frac{1}{2}(\alpha(t) + 1)} + \varepsilon \]

Ingredients: \( \mathbb{C}^0 \) stability of \( \Phi_t \) and some procedure.

**Def.** A flow \( \Phi_t \) is \( \mathbb{C}^0 \) stable

if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( \| \Phi_t \|_{\mathbb{C}^0} < \delta \Rightarrow \| \Phi_s \|_{\mathbb{C}^0} < \varepsilon \) for all \( s \in [0, t] \).

\( \forall \delta \in \mathbb{D} = \text{space of } \mathbb{X} \)-invariant states \( \leq (\varepsilon, \varepsilon) \).

\( \mathbb{C}^0 \) stable with tame estimates,

\( \exists r \in \mathbb{R} \) such that

\( \| \Phi_t \|_{\mathbb{C}^0} \leq e^{r} \| g \|_{\mathbb{C}^0} \).

**Thm.** Not flows with chaotic time-reversal are \( \mathbb{C}^0 \) stable.

**Question:** How general is \( \mathbb{C}^0 \) stability in the homogeneous setting?

Let \( Y \) be the Birkhoff average along a segment of orbit of length \( L \) starting at \( x \), ending at \( y \).

\( Y = D + R \), \( D = \text{orthogonal projection in } \mathbb{X} \).
\[ f = f_D + f_R \quad f_D \in \ker D \quad A D e \in \mathbb{R}^2 \]

\[ R(f) = R(f_D + f_R) = R(f_D) = \gamma(f_D) \]

\[ \gamma = x g \quad \gamma(x g) = \frac{1}{n} (g(y) - g(x)) \]

\[ \| f \|_L^2 \leq \frac{2}{T} \left( |g(x)| + |g(y)| \right) \]

\[ \leq \frac{2}{T} \| g \|_{\infty} \leq \frac{2}{T} e_{g_{\text{ob}}} \| f \|_{\infty} \]

\[ \leq \frac{2}{T} e_{g_{\text{ob}}} \| f \|_{\infty}^{\sigma+r} \]

Thus: \[ \| R \|_{L^2} \leq \frac{e_{g_{\text{ob}}}}{T} \]

Rescaling \[ M = \text{Heis}^3(\mathbb{R}) \]

\[ \text{Aut}(\text{Heis}^3(\mathbb{R})) = \mathcal{S}_2(\mathbb{R}) \]

\[ X = \text{eigenvector} \quad A = (\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}) \in \mathcal{S}_2(\mathbb{Z}) \]

A preserved \[ 1 \]

\[ \Rightarrow A \text{ induce a diffeo } \phi_a \text{ of } M \]

which expands orbits by a factor \( d > 1 \)

\[ \phi_a \text{ leaves } X \text{ invariant} \]

\[ \phi_a \text{ as an homothety of ratio } \frac{1}{d} \text{.} \]

\[ \| x \|_{\text{length of orbit}} \]

Consider \( \exp \log A \)
Main ingredients

- Use Sobolev norms for subelliptic operators which control only horizontal directions.

- Control the structure of a box which injects in a manifold about the segment of length 1.