Invariant Forms, Pressure and Rigidity for Anosov Flows
An Anosov flow on a manifold $M$ is a smooth flow $\phi^t$ with
- an invariant decomposition $TM = X \oplus E^u \oplus E^s$ (where $X = \dot{\phi} \neq 0$ is the generator of the flow and $E^u$ and $E^s$ are called the unstable and stable subbundles) and
- a Riemannian metric on $M$ such that $D\phi^t \big|_{E^s}$ and $D\phi^{-t} \big|_{E^u}$ are contractions whenever $t > 0$. 
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Forms, Pressure and Rigidity for Anosov Flows

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Anosov Flows

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- The canonical 1-form $A$ of an Anosov flow $\varphi^t$ is defined by $A(X) = 1$ and $E^u, E^s \subset \ker A$.

- A canonical time-change is defined using a closed 1-form $\alpha$ by replacing the generator $X$ of the flow by the vector field $X/(1 + \alpha(X))$, provided $\alpha$ is such that the denominator is positive.
Local Charts

Lemma
There exist local coordinates adapted to the invariant laminations, coordinate systems $\Psi : M \times (-\epsilon, \epsilon)^{2n+1} \to M$ such that $\Psi_p \Psi(p, \cdot)$ satisfies

- $\Psi_p$ is a $C^k$-diffeomorphism onto a neighborhood of $p$ for every $p \in M$. 

$\Psi_p$ depends continuously/Hölder-continuously/Zygmund-continuously on $p$ if the strong stable and unstable laminations do.

$\Psi_p$ preserves volume for each $p \in M$; if $\phi_t$ is transversely symplectic then $\Psi_p$ sends the standard symplectic structure to the one on transversals in $M$.

$\Psi_p(0) = p$.

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Geometric description
Longitudinal KAM cocycle

- Geometric description

**Theorem**

Let $M$ be a 3-manifold, $k \geq 2$, $\varphi: \mathbb{R} \times M \to M$ a $C^k$ volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:

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No stronger rigidity should be expected because $E^u \oplus E^s$ is smooth for all suspensions and contact flows.

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Let $\varphi$ be a $C^\infty$ flow on a closed manifold $M$. Denote by $X$ the generating vector field of $\varphi$. The flow $\varphi$ is said to be \textit{transversely symplectic} if there exists a $C^\infty$ closed 2-form $\omega$ on $M$ such that $\text{Ker}\omega = \mathbb{R}X$. The closed 2-form $\omega$ is said to be the \textit{transverse symplectic form} of $\varphi$. It is easy to see that $\omega$ is $\varphi$-invariant.
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- Geodesic flows
- Contact Anosov flows
Longitudinal KAM cocycle, higher dimensions

Magnetic flows are important examples of transversely symplectic flows and are constructed as follows:

Let \((N, g)\) be a closed \(C^\infty\) Riemannian manifold and \(\Omega\) a \(C^\infty\) closed 2-form on \(N\). Let \(\alpha\) denote the \(C^\infty\) 1-form on \(TN\) obtained by pulling back the Liouville 1-form of \(T^*N\) via the Riemannian metric. For \(\lambda \in \mathbb{R}\), the twisted symplectic structure \(\omega_\lambda\) is defined as \(\omega_\lambda = d\alpha - \lambda \pi^* \Omega\), where \(\pi: TN \to N\) denotes the canonical projection. Let \(H: TN \to \mathbb{R}\) be the Hamiltonian function defined as \(H(v) = \frac{1}{2} g(v,v)\) for any \(v \in TN\). The energy level \(H^{-1}(1/2)\) is the unit sphere bundle \(SN\).

Let \(\phi_\lambda\) be the restriction to \(SN\) of the Hamiltonian flow of \(H\) with respect to \(\omega_\lambda\). \(\phi_\lambda\) is a transversely symplectic flow with respect to \(\omega_\lambda|_{SN}\), which is said to be the magnetic flow of the pair \((g, \lambda \Omega)\).
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\(\varphi^\lambda\) is a transversely symplectic flow with respect to \(\omega_\lambda \mid_{SN}\), which is said to be the \textit{magnetic flow} of the pair \((g, \lambda \Omega)\).
An Anosov flow is said to be *uniformly quasiconformal* if

\[
K_i(x, t) := \frac{\|d\varphi^t\big|_{E^i}\|}{\|d\varphi^t\big|_{E^i}\|^*}
\]

is bounded on \(\{u, s\} \times M \times \mathbb{R}\), where \(\|A\|^* := \min_{\|v\| = 1} \|Av\|\) is the *conorm* of a linear map \(A\).
Theorem (Fang)

Let $M$ be a compact Riemannian manifold and $\varphi : \mathbb{R} \times M \rightarrow M$ a transversely symplectic Anosov flow with $\dim E^u \geq 2$ and $\dim E^s \geq 2$. Then $\varphi$ is quasiconformal if and only if $\varphi$ is up to finite covers $C^\infty$ orbit equivalent either to the suspension of a symplectic hyperbolic automorphism of a torus, or to the geodesic flow of a closed hyperbolic manifold.
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Theorem (Fang)

Let $\varphi$ be a $C^\infty$ volume-preserving quasiconformal Anosov flow. If $E^s \oplus E^u \in C^1$ and $\dim E^u \geq 3$ and $\dim E^s \geq 2$ (or $\dim E^s \geq 3$ and $\dim E^u \geq 2$), then $\varphi$ is up to finite covers and a constant change of time scale $C^\infty$ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical time change of the geodesic flow of a closed hyperbolic manifold.
Theorem (Fang - F - Hasselblatt 2010)

Let $M$ be a compact Riemannian manifold of dimension at least 5, $k \geq 2$, $\varphi : \mathbb{R} \times M \to M$ a uniformly quasiconformal transversely symplectic $C^k$ Anosov flow.

Then $E^u \oplus E^s$ is Zygmund-regular and there is an obstruction to higher regularity that defines the cohomology class of a cocycle we call the longitudinal KAM-cocycle. This obstruction can be described geometrically as the curvature of the image of a transversal under a return map, and the following are equivalent:

1. $E^u \oplus E^s$ is “little Zygmund”

2. The longitudinal KAM-cocycle is a coboundary.

3. $E^u \oplus E^s$ is Lipschitz-continuous.

4. $\varphi$ is up to finite covers, constant rescaling and a canonical time-change $C^k$-conjugate to the suspension of a symplectic Anosov automorphism of a torus or the geodesic flow of a real hyperbolic manifold.
To show that 3 implies 4 we study the *canonical 1-form* of the time-change of a geodesic flow or of the suspension of an infranilmanifold automorphism, and because we only have Lipschitz-continuity at our disposal, we need to explore how smooth-rigidity results can be pushed to the lowest conceivable regularity. This requires two main results
Theorem (Hasselblatt 2010)

Let $M$ be a compact locally symmetric space with negative sectional curvature and suppose $A$ is a Lipschitz continuous 1-form such that $dA$ is invariant under the geodesic flow. Then $A$ is $C^\infty$, and indeed $dA$ is a constant multiple of the exterior derivative of the canonical 1-form for the geodesic flow.

Note that the Lipschitz assumption ensures that $dA$ is defined almost everywhere and essentially bounded (V. M. Goldshtein, V. I. Kuzminov, I. A. Shvedov: Differential forms on a Lipschitz manifold, (1982)). This is all we use. For comparison, we state an earlier result of Hamenstädt:

Theorem (Hamenstadt)

If the Anosov splitting of the geodesic flow of a compact negatively curved manifold is $C^1$ and $A$ is a $C^1$ form such that $dA$ is invariant, then $dA$ is proportional to the canonical 1-form of the geodesic flow.
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Corollary

Let $M$ be a compact locally symmetric space with negative sectional curvature and consider a time-change whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is $C^\infty$, and the time-change is a canonical time-change.

Theorem (F - Hasselblatt 2010)

Let $\psi$ be a hyperbolic automorphism of a torus or an infranilmanifold $\Gamma \backslash M$. Then any essentially bounded invariant 2-form is almost everywhere equal to an $M$-invariant (hence smooth) closed 2-form.

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Let $\psi$ be a hyperbolic automorphism of a torus or a infranilmanifold $\Gamma \setminus M$ and consider a time-change of the suspension whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is $C^\infty$, and the time-change is a canonical time-change.
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Let $\psi$ be a hyperbolic automorphism of a torus or an infranilmanifold $\Gamma \backslash M$ and consider a time-change of the suspension whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is $C^\infty$, and the time-change is a canonical time-change.

**Theorem**

Let $(N, g)$ be a $n$-dimensional closed negatively curved Riemannian manifold and $\Omega$ a $C^\infty$ closed 2-form of $N$. For small $\lambda \in \mathbb{R}$, let $\varphi^\lambda$ be the magnetic Anosov flow of the pair $(g, \lambda \Omega)$. Suppose that $n \geq 3$ and $\varphi^\lambda$ is uniformly quasiconformal. Then $g$ has constant negative curvature and $\lambda \Omega = 0$. In particular, the longitudinal KAM-cocycle of $\varphi^\lambda$ is a coboundary.
Smooth Finsler metrics
Finsler manifolds of negative curvature

Smooth Finsler metrics

- Let \((M, F)\) be a \(C^\infty\) closed Finsler manifold of negative flag curvature.
- Let \(\varphi\) be its geodesic flow defined on the homogeneous bundle \(HM\).
- The lift of this Finsler structure to the universal covering space defines a possibly non-symmetric distance \(\tilde{d}\) on \(\tilde{M}\).
- We study the large scale metric geometry of \(\tilde{d}\)
Finsler manifolds of negative curvature

Preliminaries

\[ \pi:HM = TM_0/R \to M \]

Recall that the generator \( X \) of the geodesic flow is a Reeb field of a contact form \( A \) on \( HM \):

\[ dA(X, .) = 0 \]

\[ A(X) = 1 \]
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Theorem

Let $(M, F)$ be a closed $C^\infty$ Finsler manifold of negative flag curvature. Then its geodesic flow $\varphi : HM \to HM$ is Anosov. In addition the stable and unstable distributions of $\varphi$ are both transverse to $V(HM)$. 

It is well-known that contact Anosov flows are topologically transitive.
Theorem

Let \((M, F)\) be a closed \(C^\infty\) Finsler manifold of negative flag curvature. Then its geodesic flow \(\varphi : HM \to HM\) is Anosov. In addition the stable and unstable distributions of \(\varphi\) are both transverse to \(V(HM)\).

- It is well-known that contact Anosov flows are topologically transitive.
- There exists on \(HM\) a unique continuous \(\varphi\)-invariant 1-form \(\lambda_\varphi\) such that

\[
\lambda_\varphi(X) = 1 \quad \text{and} \quad \lambda_\varphi(E^{ss}) = \lambda_\varphi(E^{su}) = 0,
\]

which is said to be the canonical 1-form of \(\varphi\).

- \(A = \lambda_\varphi\)
Entropy

For any $\varphi$-invariant probability measure $\mu$ we denote by $h_\mu(\varphi)$ the metric entropy of $\varphi$ with respect to $\mu$.

We define the topological entropy of $\varphi$, $h_{\text{top}}(\varphi)$ by

$$h_{\text{top}}(\varphi) = \sup \left\{ h_\mu(\varphi) : \mu \text{ is a } \varphi - \text{invariant probability measure} \right\}.$$  

There is a unique ergodic fully supported probability measure for which the supremum is attained. This measure is called the Bowen-Margulis measure for $\varphi$ and is denoted by $\mu_{\text{BM}}$.

If $\varphi$ is in addition volume-preserving, we denote by $\nu$ the unique $\varphi$-invariant Lebesgue probability measure.

$$h_{\text{top}}(\varphi) \geq h_{\text{vol}}(\varphi).$$
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- There is a unique ergodic fully supported probability measure for which the supremum is attained. This measure is called the \textit{Bowen-Margulis measure} for \( \varphi \) and is denoted by \( \mu_{BM} \).
- If \( \varphi \) is in addition volume-preserving, we denote by \( \nu \) the unique \( \varphi \)-invariant Lebesgue probability measure.
- \( h_{\text{top}}(\varphi) \geq h_{\text{vol}}(\varphi) \).
More generally, let $G$ be a Hölder continuous function on $\mathcal{N}$. We define the topological pressure of $\phi$ with respect to $G$ by

$$P(\phi, G) = \sup \left\{ h_\mu(\phi) + \int_{\mathcal{N}} G \, d\mu : \mu \text{ is a } \phi \text{–invariant probability} \right\}.$$ 

By the well-known variational principle (see [HK]), there exists again a unique ergodic fully supported $\phi$–invariant probability measure for which the supremum in the definition of $P(\phi, G)$ is attained. This measure is called the Gibbs measure of $\phi$ with respect to $G$. Clearly, $P(\phi, 0) = h_{\text{top}}(\phi)$ and the Gibbs measure of $\phi$ with respect to the function zero is just the Bowen-Margulis measure.
More generally, let $G$ be a Hölder continuous function on $N$. We define the topological pressure of $\varphi$ with respect to $G$ by

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Two continuous functions $G_1$ and $G_2$ are said to be $\phi$-cohomologous if $G_1 - G_2 = U'$ for some $U$ which is continuously differentiable with respect to $\phi$. If $G_1$ and $G_2$ are both H"older continuous then they have the same Gibbs measure if and only if $G_1 - G_2$ is $\phi$-cohomologous to a constant, $c$ say. In this case we have $P(\phi, G_1) = P(\phi, G_2) + c$. 

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Pressure
Cohomological pressure and cohomological Gibbs number

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a topologically transitive $C^\infty$ Anosov flow generated by $X$. We denote by $H_1(\mathbb{N}, \mathbb{R})$ the first de Rham cohomology group of $\mathbb{N}$. Let us recall firstly the Schwartzman's definition of a winding cycle. If $\mu$ is a $\varphi$-invariant probability measure then the $\mu$-winding cycle is a map $\Phi_\mu : H_1(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ defined by

$$\Phi_\mu(\alpha) = \int_{\mathbb{N}} \alpha(X) \, d\mu,$$

where $\alpha$ is a closed $C^\infty$ 1-form. Since $\mu$ is a $\varphi$-invariant, it is easy to see that $\Phi_\mu$ is a well-defined map.

We define $\Lambda : H_1(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ by

$$\Lambda(\alpha) = P(\varphi, \alpha(X)),$$

i.e. the topological pressure of $\varphi$ with respect to the function $\alpha(X)$. Immediately from the definition we obtain the relationship

$$\Lambda(\alpha) = \sup \{ h_\mu(\varphi) + \Phi_\mu(\alpha) : \mu \text{ is } \varphi-\text{invariant} \}$$

and hence that if $df$ is an exact form then $\Lambda(\alpha) = \Lambda(\alpha + df)$. Thus $\Lambda$ is well-defined.
Cohomological pressure and cohomological Gibbs number

Let \( \varphi : N \to N \) be a topologically transitive \( C^\infty \) Anosov flow generated by \( X \). We denote by \( H^1(N, \mathbb{R}) \) the first de Rham cohomology group of \( N \). Let us recall firstly the Schwartzman's definition of a winding cycle. If \( \mu \) is a \( \varphi \)-invariant probability measure then the \( \mu \)-winding cycle is a map \( \Phi_\mu : H^1(N, \mathbb{R}) \to \mathbb{R} \) defined by

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- Let $\varphi : N \rightarrow N$ be a topologically transitive $C^\infty$ Anosov flow generated by $X$. We denote by $H^1(N, \mathbb{R})$ the first de Rham cohomology group of $N$. Let us recall firstly the Schwartzman’s definition of a winding cycle. If $\mu$ is a $\varphi$-invariant probability measure then the $\mu$-winding cycle is a map $\Phi_\mu : H^1(N, \mathbb{R}) \rightarrow \mathbb{R}$ defined by

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Cohomological pressure and cohomological Gibbs number

Definition

Following [Sharp], we define the cohomological pressure of $\varphi$, $P(\varphi)$ by $P(\varphi) = \inf \{ \Lambda(\alpha) : [\alpha] \in H^1(N, \mathbb{R}) \}$.

Theorem

([Sharp], Theorem 1) Let $\varphi : N \rightarrow N$ be a topologically transitive $C^\infty$ Anosov flow. Then the following two statements are equivalent:

(i) There exists a fully supported $\varphi$-invariant probability measure $\mu$ such that $\Phi_\mu \equiv 0$;

(ii) The function $\Lambda : H^1(N, \mathbb{R}) \rightarrow \mathbb{R}$ is bounded below (i.e. $P(\varphi) > -\infty$) and there exists a unique cohomological class $[\alpha] \in H^1(N, \mathbb{R})$ for which the infimum is attained.

If any (and hence both) of the above statements are true then we have

$$P(\varphi) = \sup \{ h_\mu(\varphi) : \mu \text{ is } \varphi - \text{invariant with } \Phi_\mu \equiv 0 \}$$

and $\Phi_{\mu_\alpha} \equiv 0$, where $\mu_\alpha$ denotes the Gibbs measure of $\alpha(X)$. 

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Cohomological pressure and cohomological Gibbs number

Definition

The cohomology class of $\alpha$ as in (ii) is said to be the Gibbs class of $\varphi$, and the Gibbs measure of $\varphi$ with respect to $\alpha(X)$ is said to be the cohomological Gibbs measure for $\varphi$. The cohomological Gibbs number of $\varphi$ is defined as

$$G(\varphi) = \int_N \alpha(X) d\mu_{BM} = \Phi_{\mu_{BM}}([\alpha]).$$

Remark If $(M, F)$ is reversible, for example in the Riemmanian case, then it is easy to verify (see [Pa3]) that $\Phi_{\mu_{BM}} \equiv 0$. So cohomological pressure and cohomological Gibbs number are interesting only for non-reversible Finsler manifolds of negative flag curvature.

Proposition

Let $\varphi$ be a contact $C^\infty$ Anosov flow. Then we have

$$h_{\text{top}}(\varphi) \geq P(\varphi) \geq h_{\text{vol}}(\varphi).$$
Canonical time changes

Definition

For any $C^\infty$ Anosov flow $\varphi : N \to N$ generated by $X$, a canonical time change of $\varphi$ is the flow generated by $\frac{X}{1 - \alpha(X)}$, where $\alpha$ is a closed $C^\infty$ 1-form on $N$ such that $1 > \alpha(X)$. We denote by $\varphi^\alpha$ the flow of $\frac{X}{1 - \alpha(X)}$. 
**Canonical time changes**

**Proposition**

Let $\varphi : N \to N$ be a contact $C^\infty$ Anosov flow generated by $X$. Let $\alpha$ be a closed $C^\infty$ 1-form on $N$ such that $1 > \alpha(X)$. Then we have $P(\varphi) = P(\varphi^\alpha)$.

**Proposition**

Let $\varphi$ be a contact $C^\infty$ Anosov flow with $\Phi_{\mu_{BM}} \equiv 0$. Let $\alpha$ be a closed $C^\infty$ 1-form on $N$ such that $1 > \alpha(X)$. Then the Gibbs class of $\varphi^\alpha$ is $[-h_{\text{top}}(\varphi) \cdot \alpha]$. 
Canonical time changes

Proposition

Let $\varphi$ be a topologically transitive $C^\infty$ Anosov flow and let $G$ be a Hölder continuous function on $N$. Let $f$ be any positive $C^\infty$ function on $N$. Then we have

$$P(\varphi, G) = P(\varphi^f, \frac{G}{f} - P(\varphi, G) \cdot \frac{1-f}{f}).$$

In addition the Gibbs measure of $\varphi$ with respect to $G$ is equivalent to that of $\varphi^f$ with respect to the function $\frac{G}{f} - P(\varphi, G) \cdot \frac{1-f}{f}$. 
Anosov splitting regularity of Finsler geodesic flows

**Theorem**
([Ha], Theorem B) Let φ be the geodesic flow of a closed negatively curved Riemannian manifold. If the Anosov splitting of φ is $C^2$, then the topological entropy of φ coincides with its metric entropy.
Anosov splitting regularity of Finsler geodesic flows

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G. Paternain (see [Pa2]) : Let \( g \) be a locally symmetric Riemannian metric on \( M \) and \( \theta \) be a small closed but non-exact \( C^\infty \) 1-form on \( M \). Let \( F = \sqrt{g} - \theta \) be the Randers metric and \( \varphi \) be its geodesic flow.
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- The Anosov splitting of \( \varphi \) is \( C^\infty \).
- \( \varphi \) is generated by \( \frac{X}{1 - \pi^*\theta(X)} \).
- The Gibbs class of \( \varphi \) is not trivial
- \( h_{\text{top}}(\varphi) > P(\varphi) = h_{\text{vol}}(\varphi) \).
Theorem
(Fang - Foulon 2009) Let $(M, F)$ be a closed $C^\infty$ Finsler manifold of negative flag curvature and $\varphi$ its geodesic flow. If the Anosov splitting of $\varphi$ is $C^2$, then the cohomological pressure of $\varphi$ coincides with its metric entropy.
Ingredients for the proof

**Definition**

We say that $\varphi$ is $d\lambda$-transitive if any continuous exact 2-form is a constant multiple of $dA$, where $A$ denotes the potential of the metric $F$.

**Proposition**

Let $\varphi$ be a contact $C^\infty$ Anosov flow such that $E^{ss}$ and $E^{su}$ are both orientable. If $\varphi$ is $d\lambda$-transitive and its Anosov splitting is $C^2$, then the cohomological pressure of $\varphi$ coincides with its metric entropy.

So the key point is to show

**Proposition**

Let $\varphi$ be the geodesic flow of a closed $C^\infty$ Finsler manifold $(M, F)$ of negative flag curvature. If the Anosov splitting of $\varphi$ is $C^1$, then $\varphi$ is $d\lambda$-transitive.
Action of the fundamental group

Let $\pi_1(M)$ be the fundamental group of $M$. For any $\gamma \in \pi_1(M)$, $\gamma$ acts naturally on $\tilde{M}$ and preserves the lifted Finsler metric $\tilde{F}$. Thus $\gamma$ acts naturally and Hölder continuously on the boundaries.

Definition

Let $X$ be a topological space and $\Phi : X \rightarrow X$ be a homeomorphism. Then $\Phi$ is said to have a north-south dynamic if $\Phi$ fixes exactly two points $\{a, b\} \subseteq X$ and for any $x \in X - \{a, b\}$, $\Phi^n(x) \rightarrow a$ and $\Phi^{-n}(x) \rightarrow b$ as $n \rightarrow +\infty$.

Proposition

Let $\gamma \in \pi_1(M)$. If $\gamma$ is not trivial, then the $\gamma$-action on $\partial^s\tilde{M}$ (respectively on $\partial^u\tilde{M}$) has a north-south dynamic.