Entropy and periodic points of principal algebraic actions

Klaus Schmidt

Vienna

Warwick, September 2010

Principal Actions

Let Γ be a countably infinite discrete group. An *algebraic* Γ*-action* is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Example 1: Let $X = \mathbb{T}^\Gamma$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let λ be the left shift-action on X, defined by $(\lambda^\gamma x)_\theta = x_{\gamma^{-1}\theta}$ for every $x = (x_\theta)_{\theta\in\Gamma} \in X$.

Example 2: Let again $X = \mathbb{T}^\Gamma$. The *right shift-action* $\gamma \mapsto \rho^\gamma$ of Γ on X is given by $(\rho^\gamma x)_\theta = x_{\theta\gamma}$. The actions λ and ρ commute.

Let $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma \in \mathbb{Z}[\Gamma]$, where the f_γ lie in $\mathbb{Z}$ and $\sum_{\gamma\in\Gamma} |f_\gamma| < \infty$. Define a group homomorphism $\rho^f\colon X \longrightarrow X$ by $\rho^f = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma$. Then ρ^f commutes with λ.

Let $X_f = \ker(\rho^f)$ and $\alpha_f = \lambda|_{X_f}$. This is the *principal* Γ*-action defined by* f. To avoid trivialities we always assume that f is not a unit in $\mathbb{Z}[\Gamma]$.

Problem: For fixed Γ, describe the dynamical properties of α_f in terms of the polynomial f.

Principal Actions Of $\mathbb{Z}$

For $\Gamma = \mathbb{Z}$, every $f = \sum_{n\in\mathbb{Z}} f_n n \in \mathbb{Z}[\mathbb{Z}]$ can be viewed as the Laurent polynomial $\sum_{n\in\mathbb{Z}} f_n u^n$. After multiplication by a power of u (which doesn't change X_f) we may assume that $f = \sum_{k=0}^{n} f_k u^k$ with nonzero f_0 and f_n.
If $f_n = |f_0| = 1$, α_f is (conjugate to) the toral automorphism given by the companion matrix

$$A_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{n-1} \end{pmatrix}$$

In general, α_f is (conjugate to) an automorphism of an n-dimensional solenoid (e.g., $f = 3 - 2u$ corresponds to 'multiplication by $3/2$' on the circle).
Dynamical properties like ergodicity or expansiveness are determined by the roots of f, and the entropy of α_f is given by
$h(\alpha_f) = \log\, |f_n| + \sum_{\{c : f(c)=0\}} \log^+ |c|$.

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables: $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$. Assume for simplicity that f is irreducible.

- If $d \geq 2$ then α_f is ergodic.
- α_f is mixing if and only if f is not of the form $u^{\mathbf{m}} c(\mathbf{n})$, where $c(\cdot)$ is cyclotomic.
- The entropy $h(\alpha_f)$ is given by the *logarithmic Mahler measure* of f: $\mathrm{m}(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})| \, dt_1 \cdots dt_d$ (Lind-S-Ward).
- $h(\alpha_f) > 0 \;\Leftrightarrow\; \alpha_f$ is mixing $\;\Leftrightarrow\; \alpha_f$ is Bernoulli (Ward, Rudolph-S).
- α_f is expansive if and only if $V_{\mathbb{C}}(f) = \{\mathbf{c} \in (\mathbb{C} \smallsetminus \{0\})^d : f(\mathbf{c}) = 0\}$ contains no points whose coordinates all have absolute value 1.
- $h(\alpha_f) = \limsup_{\Delta \searrow \{0\}} \frac{1}{|\mathbb{Z}^d/\Delta|} \log |\mathrm{Fix}_\Delta(X_f)/\mathrm{Fix}_\Delta^\circ(X_f)|$, where the limit is taken over all sequences of finite-index subgroups $(\Delta_n)_{n \geq 1}$ in $\mathbb{Z}^d$ with $\langle \Delta_n \rangle = \min\{\|\mathbf{n}\| : \mathbf{0} \neq \mathbf{n} \in \Delta_n\} = \infty$, and where $\mathrm{Fix}_\Delta(X_f) = \{x \in X_f : \alpha_f^{\mathbf{n}} x = x \text{ for every } \mathbf{n} \in \Delta_n\}$.
- If α_f is nonexpansive it is not known if $\limsup_{\Delta \searrow \{0\}}$ can be replaced by $\lim_{\Delta \searrow \{0\}}$.

Let Γ be countably infinite and discrete, $f \in \mathbb{Z}[\Gamma]$, and let α_f be the corresponding principal Γ-action on X_f.

Theorem (Hayes; S): If Γ is amenable and not virtually cyclic, and if f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, then α_f is ergodic.

Theorem (Deninger-S): α_f is expansive $\Leftrightarrow$ f is invertible in $\ell^1(\Gamma)$.

Problem: Find conditions on f which imply invertibility in $\ell^1(\Gamma)$.

Easy answer: If f has a dominant term, i.e., $|f_\gamma| > \sum_{\gamma' \in \Gamma \smallsetminus \{\gamma\}} |f_{\gamma'}|$ for some $\gamma \in \Gamma$, then α_f is expansive. Can one do better?

Remark: If α_f is expansive then the ideal $\mathbb{Z}[\Gamma]f$ contains an element with a dominant term.

Expansiveness is a good thing to have. Here are some useful consequences:

Theorem: If α_f is expansive and f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, then α_f is mixing.

Theorem: If Γ is amenable and α_f is expansive, then $h(\alpha_f) > 0$.

Problem: Is α_f Bernoulli under these hypotheses?

Assume that Γ is residually finite (i.e., that there exists a decreasing sequence $(\Delta_n)_{n\geq 1}$ of finite-index subgroups with $\bigcap_n \Delta_n = \{1\}$).

Theorem (Deninger-S): If Γ is amenable and α_f is expansive, then

$$h(\alpha_f) = \lim_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\mathrm{Fix}_\Delta(X_f)| = \log \det_{\mathcal{N}\Gamma}(\rho_f),$$

where the last term is the *Fuglede-Kadison determinant* of f, acting by right convolution on $\ell^2(\Gamma)$, and viewed as an element of the (left-equivariant) group von Neumann algebra $\mathcal{N}\Gamma$.

Hanfeng Li recently observed that this result only depends on the invertibility of ρ_f in $\mathcal{N}\Gamma$ (and not on that of f in $\ell^1(\Gamma)$).

Even more recently, this result was extended to the non-amenable case.

Theorem (Bowen): If Γ is non-amenable and α_f is expansive, then

$$h((\Delta_n)_{n\geq 1}, \alpha_f) = \lim_{n\to\infty} \frac{1}{|\Gamma/\Delta_n|} \log |\mathrm{Fix}_{\Delta_n}(X_f)| = \log \det_{\mathcal{N}\Gamma}(\rho_f),$$

where $h((\Delta_n)_{n\geq 1}, \alpha_f)$ is the *sofic entropy* of α_f w.r.t. the sequence $\Delta_n \searrow \{1\}$.

The current state of things:

If Γ is amenable and residually finite and α_f is nonexpansive, then

$$h(\alpha_f) \geq \limsup_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\mathsf{Fix}_\Delta(X_f)/\mathsf{Fix}^\circ_\Delta(X_f)|$$

$$= \limsup_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\det{}^*(\rho_f|_{\ell^2(\Gamma/\Delta)})| \leq \log \det{}^*_{\mathcal{N}\Gamma}(\rho_f)$$

Here $\det^*_{\mathcal{N}\Gamma}$ is the modified Fuglede-Kadison determinant for not necessarily invertible elements of $\mathcal{N}\Gamma$ introduced by Lück: consider ρ^{f^*f} as an element of $\mathcal{N}\Gamma$, and write $\rho^{f^*f} = \int \lambda\, dE(\lambda)$ for its spectral representation. Then $\log \det^*_{\mathcal{N}\Gamma}(\rho_f) := \int_{0^+}^{\infty} \log \lambda\, dF(\lambda)$, where $F(\lambda) = \mathrm{trace}(P(\lambda))$.

Conjecture: If f^*f is not a zero divisor in $\mathbb{Z}[\Gamma]$, then the first inequality is an equality.

Problem: If f^*f is not a zero divisor in $\mathbb{Z}[\Gamma]$, can the second inequality be replaced by an equality, and can 'lim sup' be replaced by 'lim'?

An Explicit Formula

Let $\Gamma \subset \mathrm{SL}(3,\mathbb{Z})$ be the discrete Heisenberg group, generated by the matrices

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the commutation relations

$$xz = zx, \quad yz = zy, \quad y^l x^k = x^k y^l z^{kl} = z^{kl}, \quad k, l \in \mathbb{Z}.$$

Every f in $\mathbb{Z}[\Gamma]$ can be written in the form

$$f = \sum_{(m_1,m_2,m_3)\in\mathbb{Z}^3} f_{(m_1,m_2,m_3)} x^{m_1} y^{m_2} z^{m_3}$$

with $f_{(m_1,m_2,m_3)} \in \mathbb{Z}$.

Theorem (Lind-S): Let $f = h_0(y,z) + x h_1(y,z)$ for some nonzero $h_0, h_1 \in \mathbb{Z}[y^{\pm 1}, z^{\pm 1}]$ such that α_f is expansive. Then

$$h(\alpha_f) = \int_0^1 \max\left\{\mathrm{m}(h_0(\cdot, e^{2\pi i t})), \mathrm{m}(h_1(\cdot, e^{2\pi i t}))\right\} dt,$$

where

$$\mathrm{m}(h) = \int_0^1 \log\, |h(e^{2\pi i s})|\, ds$$

is the logarithmic Mahler measure of a Laurent polynomial $h \in \mathbb{C}[u^{\pm 1}]$.