(Super)diffusive asymtotics for perturbed Lorentz or Lorentz-like processes

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joint w. Péter Nándori and Tamás Varjú

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From laudatio for Dolgopyat

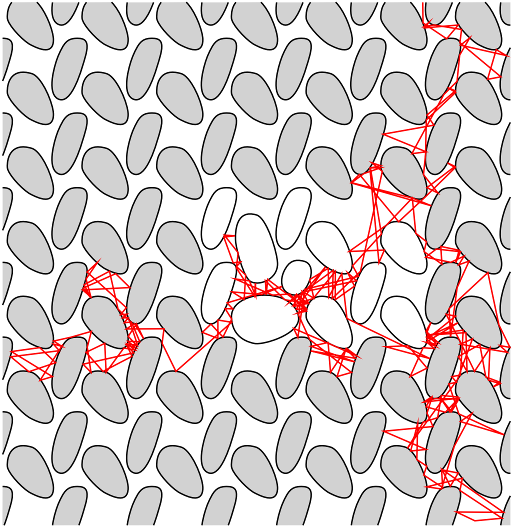
From Chernov's laudatio for Dolgopyat's 2009 Brin prize:

Physical systems are often inconvenient and unsuitable for direct application of conventional theories:

- dynamics may have ugly singularities, . . . ,
- **natural invariant measures may be infinite, etc.**
- etc.

A Lorentz orbit

Finite horizon, 'locally perturbed periodic'



Notions and notations: Lorentz Process

Lorentz process - billiard dynamics (uniform motion + specular reflection) (Ω, T, μ)

- $\hat{Q} = \mathbb{R}^d \setminus \cup_{i=1}^{\infty} O_i$ is the configuration space of the Lorentz flow (the billiard table), where the closed sets O_i are pairwise disjoint, strictly convex with $\mathcal{C}^3-$smooth boundaries
- $\Omega = Q \times S_+$ is its phase space for the billiard ball map (where $Q = \partial\hat{Q}$ and S_+ is the hemisphere of outgoing unit velocities)
- $T : \Omega \to \Omega$ its discrete time billiard map (the so-called Poincaré section map)
- μ the T-invariant (infinite) Liouville-measure on Ω

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Notions and notations:

Periodic Lorentz → Sinai Billiard

If the scatterer configuration $\{O_i\}_i$ is $\mathbb{Z}^d$-**periodic**, then the corresponding dynamical system will be denoted by $(\Omega_{per} = Q_{per} \times S_+, T_{per}, \mu_{per})$.

Then it makes sense to **factorize** it by $\mathbb{Z}^d$ to obtain a **Sinai billiard** $(\Omega_0 = Q_0 \times S_+, T_0, \mu_0)$. The natural projection $\Omega \to Q$ (and analogously for Ω_{per} and for Ω_0) will be denoted by π_q.

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Why are local perturbations interesting?

Local perturbations

- Lorentz, 1905: described the transport of conduction electrons in metals (still in the pre-quantum era). Natural to consider models with local impurities;
- Non-periodic models
  - M. Lenci, '96-
  - Sz., '08: Penrose-Lorentz process [finite but unbounded horizon!]
- It is not a skew-product any more.

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- Hard ball systems in the nonconfined regime have ∞H
- Crystals
- Non-trivial asymptotic behavior and new kinetic equ. (Bourgain, Caglioti, Golse, Wennberg, ...; '98-, Marklof-Strömbergsson, '08-)
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Stochastic properties: Correlation decay

Let $f, g \quad M(=\Omega_0, \textit{billiard phase space}) \to \mathbb{R}^d$ be piecewise Hölder.

Definition

- With a given $a_n : n \geq 1$ (M, T, μ) has $\{a_n\}_n$-correlation decay if $\exists C = C(f,g)$ such that $\forall f, g$ Hölder and $\forall n \geq 1$

$$\left| \int_M f(g \circ T^n) d\mu - \int_M f d\mu \int_M g d\mu \right| \leq C a_n$$

- The correlation decay is exponential (EDC) if $\exists C_2 > 0$ such that $\forall n \geq 1$

$$a_n \leq \exp(-C_2 n).$$

- The correlation decay is stretched exponential (SEDC) if $\exists \alpha \in (0,1), C_2 > 0$ such that $\forall n \geq 1$

$$a_n \leq C_1 \exp(-C_2 n^\alpha).$$

Diffusively scaled variant

Definition

Assume $\{q_n \in \mathbb{R}^d | n \geq 0\}$ is a random trajectory. Then its *diffusively scaled variant* $\in C[0,1]$ $(or \ \in C[0,\infty])$ is defined as follows: for $N \in \mathbb{Z}_+$ denote
$W_N(\frac{j}{N}) = \frac{q_j}{\sqrt{N}}$ $\quad (0 \leq j \leq N \ \ or \ \ j \in \mathbb{Z}_+)$ and define otherwise $W_N(t)(t \in [0,1] \ \ or \ \ \mathbb{R}_+)$ as its piecewise linear, continuous extension.

E. g. $\kappa(x) = \pi_q(Tx) - \pi_q(x) : M \to \mathbb{R}^d$, the free flight vector of a Lorentz process.
From now on $q_n = q_n(x) = \sum_{k=0}^{n-1} \kappa(T^k x), \quad n = 0, 1, 2, \dots$ is the Lorentz trajectory.

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Stochastic properties: CLT & LCLT

Definition

- *CLT and Weak Invariance Principle*

$$W_N(t) \Rightarrow W_{\mathcal{D}^2}(t),$$

the Wiener process with a non-degenerate covariance matrix $\mathcal{D}^2 = \mu_0(\kappa_0 \otimes \kappa_0) + 2\sum_{j=1}^{\infty} \mu_0(\kappa_0 \otimes \kappa_n)$.

- Local CLT Let x be distributed on Ω_0 according to μ_0. Let the distribution of $[q_n(x)]$ be denoted by Υ_n. There is a constant $\mathbf{c}$ such that

$$\lim_{n\to\infty} n\Upsilon_n \to \mathbf{c}^{-1} I$$

where I is the counting measure on the integer lattice $\mathbb{Z}^2$ and $\to$ stands for vague convergence.
In fact, $\mathbf{c}^{-1} = \frac{1}{2\pi\sqrt{\det \mathcal{D}^2}}$.

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2D, Periodic case: Some Results

		SEDC	EDC	CLT	LCLT
B-S, '81	M-partitions	X		X	
B-Ch-S, '91	M-sieves	X		X	
Y, '98	M-towers		X	X	
Sz-V, '04 (EThDS)					X
Ch-D, '09	standard pairs	X	X	X	?!

SEDC - Stretched Exponential Decay of Correlations
EDC - Exponential Decay of Correlations
CLT - Central Limit Theorem
LCLT - Local CLT

Locally perturbed FH Lorentz

- Sinai's problem, '81: locally perturbed FH Lorentz
- Sz-Telcs, '82: locally perturbed SSRW for $d = 2$ has the same diffusive limit as the unperturbed one

  Idea: local time $\rho(n)$ $(= \#\text{visits to origin until time } n)$ is $O(\log n)$ thus the $\sqrt{n}$ scaling eates perturbation up

  Method:
  - there are $\sim \rho(n) = O(\log n)$ time intervals spent at perturbation
  - couple the intervals spent outside perturbations to SSRW

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Theorem

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Preparatory work:

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Dolgopyat-Sz-Varjú, 08: recurrence properties of FH Lorentz (extensions of Thm's of Erdős-Taylor and Darling-Kac (on local times, first hitting times, etc.) from SSRW to FH Lorentz)

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Locally perturbed FH Lorentz 2.

Tools:

- Sz-Varjú, 04: local CLT for periodic FH Lorentz
- Chernov-Dolgopyat, 05-09:
  - standard pairs
  - growth lemma
  - Young-coupling

Methods:

- reduction to 1-D RW's
- Stroock-Varadhan's martingale method

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Standard pair

- A connected smooth curve $\gamma \subset \Omega_0$ is called an *unstable curve* if at every point $x \in \gamma$ the tangent space $\mathcal{T}_x\gamma$ belongs to the unstable cone $\mathcal{C}^u_x$.
- A *standard pair* is a pair $\ell = (\gamma, \rho)$ where γ is a homogeneous unstable curve and ρ is a homogeneous density on γ (homogeneous meaning good estimates!).

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Growth lemma: preliminary remarks

Sinai's philosophy: Expansion prevails partitioning

Viviane's formulation: Hyperbolicity dominates complexity

NB: P Bálint- IP Tóth, '08: for multidimensional FH S-billiards fulfilment of complexity condition implies exponential correlation decay

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Growth lemma, Ch-D, a form of Markov-property

Sinai billiard

Theorem

- *If* $\ell = (\gamma, \rho)$ *is a standard pair, then*

$$\mathbb{E}_\ell(A \circ T_0^n) = \sum_\alpha c_{\alpha n} \mathbb{E}_{\ell_{\alpha n}}(A)$$

where $c_{\alpha n} > 0$, $\sum_\alpha c_{\alpha n} = 1$ *and* $\ell_{\alpha n} = (\gamma_{\alpha n}, \rho_{\alpha n})$ *are standard pairs where* $\gamma_{\alpha n} = \gamma_n(x_\alpha)$ *for some* $x_\alpha \in \gamma$ *and* $\rho_{\alpha n}$ *is the pushforward of* ρ *up to a multiplicative factor.*

- *If* $n \geq \beta_3 |\log \mathrm{length}(\ell)|$, *then*

$$\sum_{\mathrm{length}(\ell_{\alpha n}) < \varepsilon} c_{\alpha n} \leq \beta_4 \varepsilon.$$

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Coupling lemma
Lorentz process

Assume that $|m_1|, |m_2| \to \infty$ and if ℓ_1, ℓ_2 are standard pairs satisfying

$$[\ell_j] = m_j, \quad \mathrm{length}(\ell_j) > |m_j|^{-100}, \quad j = 1, 2 \tag{1}$$

and

$$\frac{1}{2} < \frac{|m_1|}{|m_2|} < 2. \tag{2}$$

Lemma

Given $\zeta > 0$ and $\varepsilon > 0$ there exists R such that for any two standard pairs $\ell_1 = (\gamma_1, \rho_1), \ell_2 = (\gamma_2, \rho_2)$ satisfying the previous assumptions and $|m_j| > R$ the following holds.

Coupling lemma

Lorentz process, continued

Lemma

Let $\bar{n} = |m_1|^{2(1+\zeta)}$*. There exist positive constants* $\bar{c}$ *and* $\bar{c}_{\beta j}$*, probability measures* $\bar{\nu}_1$ *and* $\bar{\nu}_2$ *supported on* $f^{\bar{n}}\gamma_1$ *and* $f^{\bar{n}}\gamma_2$ *respectively, and families of standard pairs* $\{\bar{\ell}_{\beta j}\}_\beta; j = 1, 2$ *satisfying*

$$\mathbb{E}_{\ell_j}(A \circ f^{\bar{n}}) = \bar{c}\bar{\nu}_j(A) + \sum_\beta \bar{c}_{\beta j}\mathbb{E}_{\bar{\ell}_{\beta j}}(A) \qquad j = 1, 2 \qquad (3)$$

with $\bar{c} \geq 1 - \varepsilon$*.*

Coupling lemma

Lorentz process, continued

Theorem

Moreover there exists a measure preserving map

$$\bar{\pi} : (\gamma_1 \times [0,1], f^{-\bar{n}}\bar{\nu}_1 \times \lambda) \to (\gamma_2 \times [0,1], f^{-\bar{n}}\bar{\nu}_2 \times \lambda)$$

where λ is the Lebesgue measure on $[0,1]$ such that if $\bar{\pi}(x_1, s_1) = (x_2, s_2)$ then for any $n \geq \bar{n}$

$$d(f^n x_1, f^n x_2) \leq C\theta^{n-\bar{n}},$$

where C, θ are the constants from our preliminary lemma.

Martingale approach
à la Stroock-Varadhan

Brownian motion is characterized by the fact that

$$\phi(W(t)) - \frac{1}{2}\int_0^t \sum_{ab=1,2} \sigma_{ab} D_{ab}\phi(W(s))ds \tag{4}$$

is a martingale for C^2-functions of compact support.

By Stroock-Varadhan it suffices to show that — the limiting process $\tilde{W}(t)$ of any convergent subsequence of the processes $W_N(.)$ — the process

$$\phi(\tilde{W}(t)) - \frac{1}{2}\int_0^t \sum_{ab=1,2} \sigma_{ab} D_{ab}\phi(\tilde{W}(s))ds \tag{5}$$

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Superdiffusive scaling

Reminder: $\kappa(x) = \pi_q(Tx) - \pi_q(x) : M \to \mathbb{R}^2$, the free flight vector of a Lorentz process.

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Now: for $N \in \mathbb{Z}_+$ denote

$$W_N\left(\frac{j}{N}\right) = \frac{q_j}{\sqrt{N \log N}} \qquad (0 \le j \le N \quad or \quad j \in \mathbb{Z}_+)$$

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∞H periodic Lorentz

- Bleher, '92:
  - $\mathbb{E}|\kappa(x)|^2 = \infty$
  - $\mathbb{E}|\kappa(x)\kappa(T^n x)| < \infty$ *if* $|n| \geq 1$.
  - Heuristic arguments for superdiffusive: $\sqrt{N \log N}$ scaling.
- Sz-Varjú, 07:
  - Rigorous proof for Bleher's conjecture (method: Young's towers & Fourier transform of P-F operator (NB: Aaronson-Denker)
  - Moreover: local limit law & Recurrence
  - Exact form of the limiting covariance
- Melbourne, '08, $O(1/t)$ corr. decay rate for the flow
- Chernov-Dolgopyat, '10: EDC & global LT for κ (method: Ch-D's standard pairs & Bernstein's method of freezing)

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Locally perturbed RW's

Paulin-Sz, '10: Local perturbations - under slight conditions - do not change the appropriate limit if jumps of the RW belong to the domain of attraction of a stable law of exponent $1 < \alpha \leq 2$.

Here transitions over 0 of type $(1,1) \to (-1,-1)$ do not get perturbed.

Nándori, '10: In a periodic RW with tail corresponding to $\sqrt{n \log n}$ scaling

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Dynamical tools for ∞H Lorentz

Nándori-Sz-Varjú, '10:

- Growth lemma
- Coupling lemma

NB: For Penrose-Lorentz process

- Growth lemma also holds
- Coupling lemma would require local limit law (for RW on Penrose lattice CLT is proved by Telcs, '10)

Moreover, by using the martingale method of D-Sz-V, '09

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- Growth lemma also holds
- Coupling lemma would require local limit law (for RW on Penrose lattice CLT is proved by Telcs, ’10)

Moreover, by using the martingale method of D-Sz-V, ’09

Nándori-Sz-Varjú, ’10:: third proof for global LT for ∞H periodic Lorentz (1st: Sz-V, ’07, 2nd, Ch-D, ’10).