Non-L^1 functions with rotation sets of Hausdorff dimension one

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I worked for a long time with Henstock-Kurzweil integrals and, as a Ph. D. student, got interested in ergodic averages of non-L^1 functions.

P. Major : $\exists\ f : X \to \mathbb{R}$, and $S, T : X \to X$ two ergodic transformations on a probability space (X, μ) such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S^k x) = 0,\ \ \mu \text{ a.e. and } \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = a \neq 0,\ \ \mu \text{ a.e.}$$

By Birkhoff's Ergodic Theorem the above f cannot belong to $L^1(X, \mu)$.

My thesis advisor M. Laczkovich raised the question whether the two transformations S and T can be irrational rotations of the unit circle, $\mathbb{T}$.

In Major's construction the two transformations were conjugate a different approach was needed.

Z.B.: if $S, T : X \to X$ are two μ-ergodic transformations which generate a free $\mathbb{Z}^2$ action on the finite non-atomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ then for any $c_1, c_2 \in \mathbb{R}$ there exists a μ-measurable function $f : X \to \mathbb{R}$ such that

$$M_N^S f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) \to c_1, \text{ and } M_N^T f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) \to c_2,$$

μ almost every x as $N \to \infty$.

Two different irrational rotations generate a free $\mathbb{Z}^2$ action on $\mathbb{T}$ $\Rightarrow$ answer to Laczkovich's question.

$|A|$ denotes the Lebesgue measure of the measurable set $A \subset \mathbb{R}$, or on this page $A \subset \mathbb{R}^2$.

Recent results by Ya. Sinai and C. Ulcigrai.

Trigonometric sums

$$\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{1-e^{2\pi i(k\alpha+x)}}, \qquad (x,\alpha)\in(0,1)\times(0,1)$$

are considered.

$(0,1)\times(0,1)$ is endowed with the uniform probability distribution. It is proved that such trigonometric sums have a non-trivial joint limiting distribution in x and α as N tends to ∞, that is:

T.: *For any $\Omega \subset \mathbb{C}$ $\exists$*

$$\lim_{N\to\infty}\left|\left\{(\alpha,x):\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{1-e^{2\pi i(k\alpha+x)}}\in\Omega\right\}\right| = \mathcal{P}(\Omega)$$

with a suitable $\mathcal{P}$ probability measure on $\mathbb{C}$.

This result also applies to Birkhoff sums of a function with a singularity of type $1/x$ over a rotation, that is:

T.: *For any $a < b$ $\exists$*

$$\lim_{N\to\infty}\left|\left\{(\alpha,x): a \le \frac{1}{N}\sum_{k=0}^{N-1} f(x+n\alpha)\le b\right\}\right| = \mathcal{P}([a,b])$$

with a suitable $\mathcal{P}$ probability measure on $\mathbb{R}$.

Trying two answer Laczkovich's question first I proved the following theorem:

T.: *Let $f : \mathbb{R} \to \mathbb{R}$ be a given measurable function, periodic by 1.*

For an $\alpha \in \mathbb{R}$ put $M_n^\alpha f(x) = \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} f(x + k\alpha).$

Let Γ_f denote the set of those α's in $(0,1)$ for which $M_n^\alpha f(x)$ converges for almost every $x \in \mathbb{R}$.

Then from $|\Gamma_f| > 0$ it follows that f is integrable on $[0,1]$.

$|\Gamma_f| > 0 \Rightarrow f \in L^1$ and for all $\alpha \in [0,1] \setminus \mathbb{Q}$ the limit of $M_n^\alpha f(x)$ equals $\int_0^1 f$ by the Birkhoff Ergodic thm.

T.: *For any sequence of independent irrationals $\{\alpha_j\}_{j=1}^\infty$ there exists $f : \mathbb{R} \to \mathbb{R}$, periodic by 1 such that $f \notin L^1[0,1]$ and $M_n^{\alpha_j} f(x) \to 0$ for almost every $x \in [0,1]$.*

$\Rightarrow$ $\Gamma_f \setminus \mathbb{Q}$ can be dense for non-integrable functions.

R. Svetic: there exists a non-integrable $f : \mathbb{T} \to \mathbb{R}$ such that Γ_f is c-dense in $\mathbb{T}$.

(A set $S \subset \mathbb{T}$ is *c-dense* if the cardinality of $S \cap I$ equals continuum for every nonempty open interval $I \subset \mathbb{T}$.)

Question: Can Γ_f be of Hausdorff dimension one for non-L^1 functions?

T.: *There exist a measurable $f : \mathbb{R} \to \mathbb{R}$ periodic by one and a set $A \subset [0,1) \setminus \mathbb{Q}$ such that the Hausdorff dimension of A is one, for all $\alpha \in A$*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} f(x + k\alpha) = 0$$ *for almost every $x \in [0,1)$ and* $$\int_{[0,1)} |f| = +\infty.$$

Outline of the proof: First we define the sequences d_j and l_j converging to 0 and $K_j = 10j$ converging to ∞.

Then we define a subset A of the irrationals in $(0,1)$.

Suppose $\alpha \in A$ and its continued fraction development

is $$[a_{\alpha,1}, a_{\alpha,2}, ...] = \frac{1}{a_{\alpha,1} + \frac{1}{a_{\alpha,2} + \frac{1}{...}}},$$ and $p_{\alpha,n}/q_{\alpha,n}$ is its n'th convergent.

We define a sequence $n(j, \alpha) < n(j + 1, \alpha)$.

The $a_{\alpha,n(j,\alpha)}$ continued fraction partial denominators of α will be chosen in a very specific way so that $1/q_{\alpha,n(j,\alpha)}$ will be very close to l_j.

If n is within a block determined by $n(j-1, \alpha)$ and $n(j, \alpha)$, that is $n(j-1, \alpha) < n < n(j, \alpha)$ then we only assume that $a_{\alpha,n}$ is bounded by K_j.

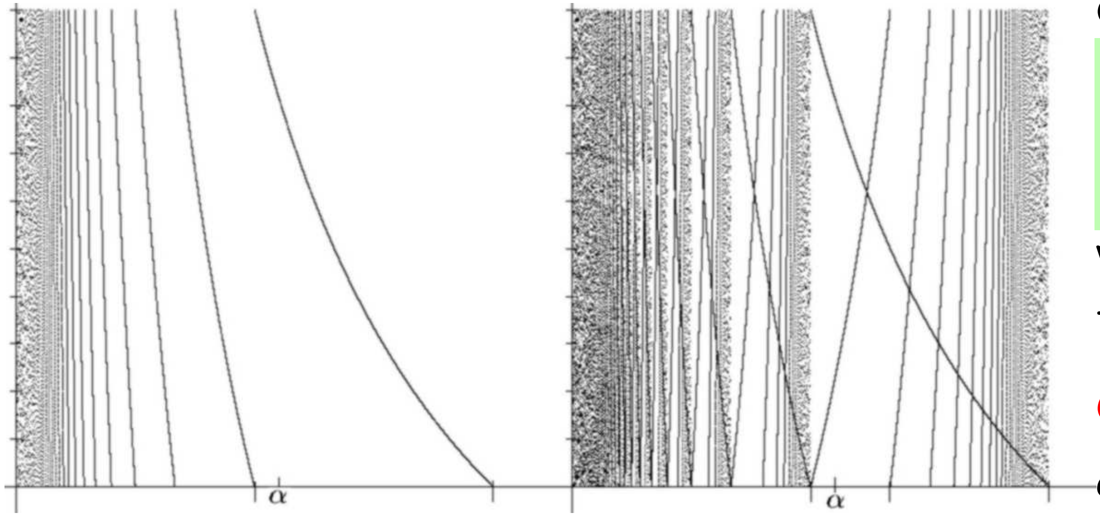
The Hausdorff dimension of A equals one.

The function f is defined as the sum of the functions f_j.
The functions f_j vanish outside a set B_j of length h_j.
The set B_j is subdivided into an even number of intervals of length l_j and f_j equals $\pm 1/h_j$ alternately on these subintervals.
This will provide us sufficient cancellation for the ergodic sums with respect to $\alpha \in A$ rotations.
On the other hand, we have $\int |f_j| = 1$.
We show that the measure of those x's for which

$$\sup_{K>0} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x + k\alpha) \right| \geq 1/j^2$$ is not greater than $1/j^2$.

This weak maximal type inequality will imply the main result.

Question: Suppose $0 < t < 1$ and $f(x) = \frac{1}{x|\log |x||^t}$, when $|x| \leq 1/2$, $f(0) = 0$, and f is periodic by one. What can be said about the Hausdorff dimension of the rotation set Γ_f?
In case it is still zero for all $t \in (0,1)$ one could continue by asking the same question for functions defined as above, but for which we have $f(x) = \frac{1}{x \log |x|| \log | \log |x|||^t}$, when $|x| \leq 1/2$.



Suppose $\alpha \in [0,1)$ irrational, then its continued fraction development:

$$\alpha = [a_{\alpha,1}, a_{\alpha,2}, \ldots] = \frac{1}{a_{\alpha,1} + \frac{1}{a_{\alpha,2} + \frac{1}{\ldots}}},$$

with $a_{\alpha,n} \in \mathbb{N}$.

The Gauss map is given by

$G(\alpha) = \{\frac{1}{\alpha}\}$, and

$a_{\alpha,n} = \lfloor (G^{n-1}(\alpha))^{-1} \rfloor$.

Set $\alpha_n = [a_{\alpha,n+1}, a_{\alpha,n+2}, \ldots] = G^n(\alpha)$.

The convergents of α are

$p_{\alpha,n}/q_{\alpha,n} = [a_{\alpha,1}, a_{\alpha,2}, \ldots, a_{\alpha,n}]$.

The numbers $p_{\alpha,n}$ and $q_{\alpha,n}$ can be defined by the following recursion:

$$p_{\alpha,-1} = q_{\alpha,0} = 1, \;\; q_{\alpha,-1} = p_{\alpha,0} = 0,$$

$$p_{\alpha,n} = a_{\alpha,n} p_{\alpha,n-1} + p_{\alpha,n-2}, \;\; q_{\alpha,n} = a_{\alpha,n} q_{\alpha,n-1} + q_{\alpha,n-2}, (n \in \mathbb{N}).$$

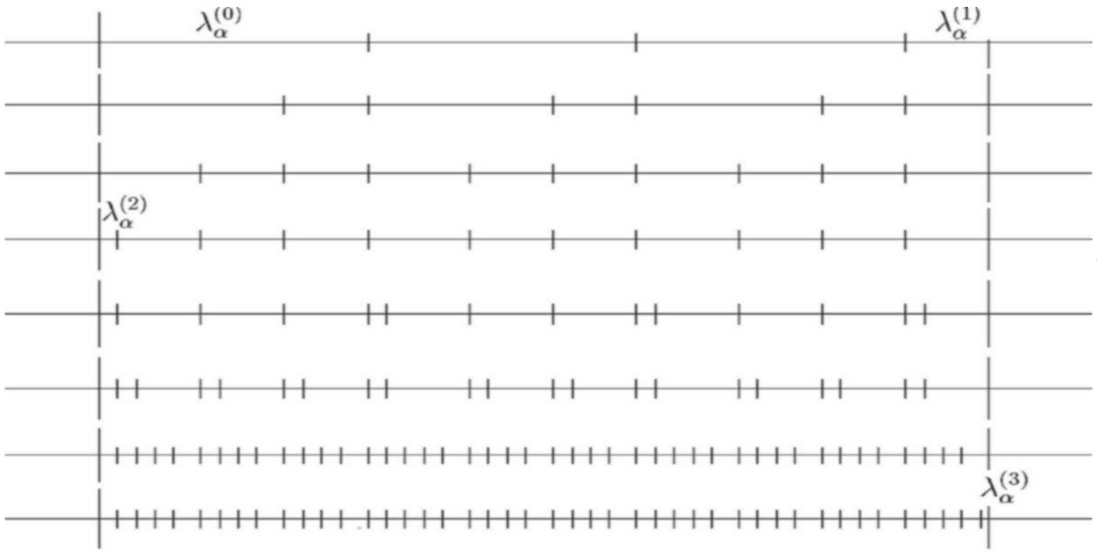
$\lambda_\alpha^{(n-1)} \stackrel{\text{def}}{=} |q_{\alpha,n-1}\alpha - p_{\alpha,n-1}| = \frac{1}{q_{\alpha,n}+q_{\alpha,n-1}\alpha_n}$. To be more precise,

$$\lambda_\alpha^{(n-1)} = (-1)^{n-1}(q_{\alpha,n-1}\alpha - p_{\alpha,n-1}) = \frac{1}{q_{\alpha,n} + q_{\alpha,n-1}G^n(\alpha)}.$$

$$\frac{\lambda_\alpha^{(n)}}{\lambda_\alpha^{(n-1)}} = [a_{\alpha,n+1}, a_{\alpha,n+2}, \ldots],$$

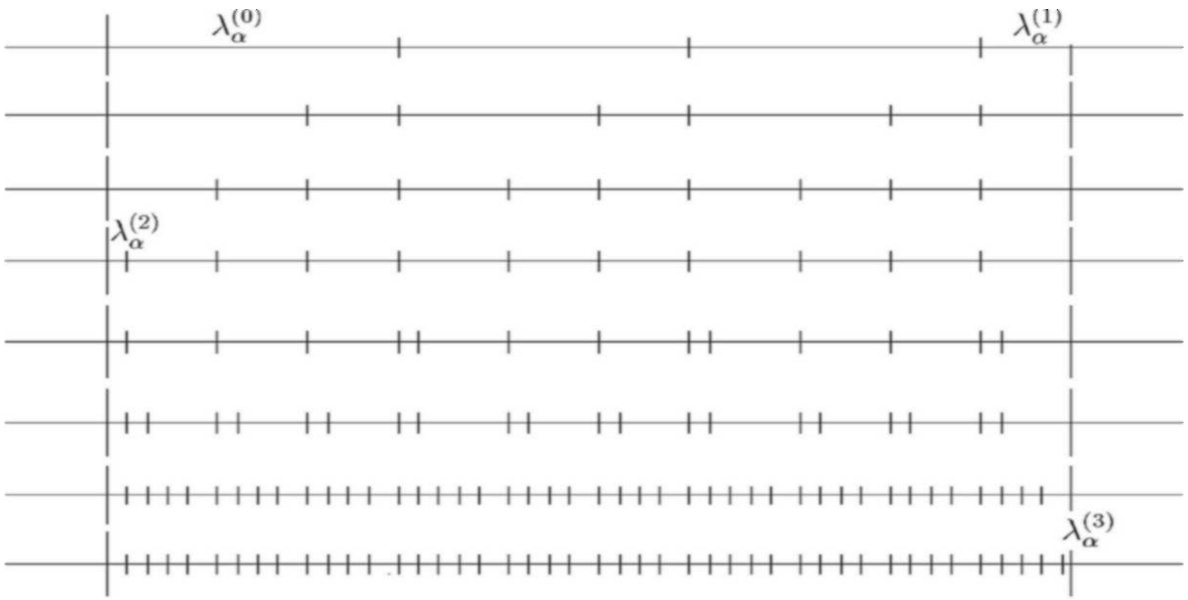
and

$$\frac{1}{a_{\alpha,n+1}+1} \leq \frac{\lambda_\alpha^{(n)}}{\lambda_\alpha^{(n-1)}} \leq \frac{1}{a_{\alpha,n+1}}.$$



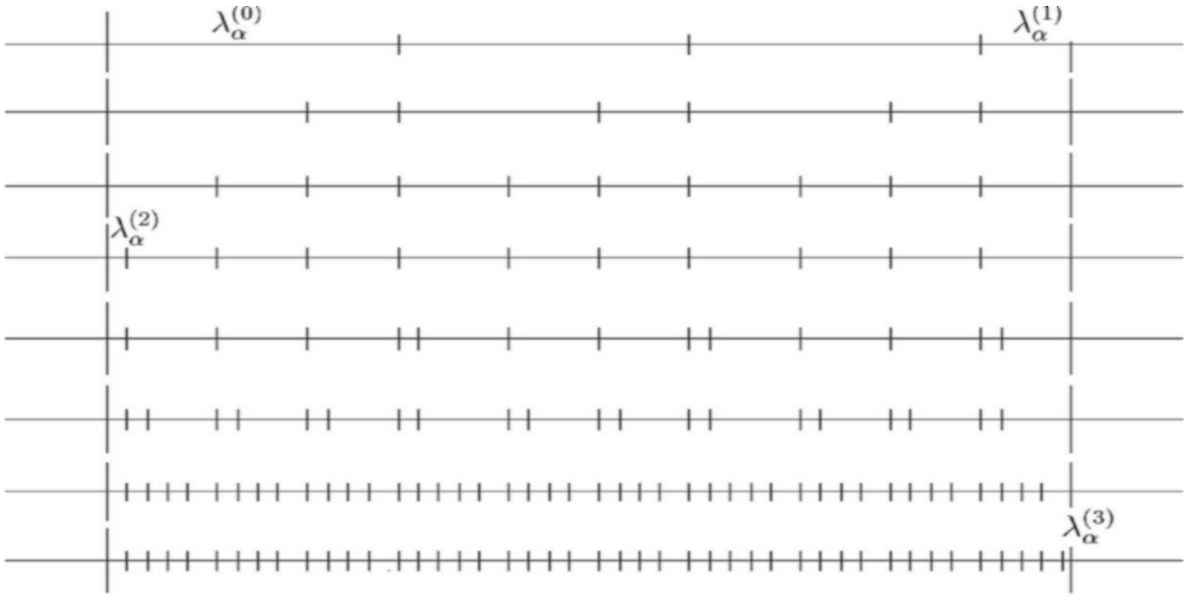
The intervals of length $\lambda_\alpha^{(n)}$ show up alternating on the sides of 0 modulo 1, to the right for even and to the left (close to 1 on the figure) for odd n's.

Property 1.: The points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$ on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are "almost equally spaced".



Denote by $\mathcal{P}(n)$ the partition obtained by considering the points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$. If $\mathcal{I} \subset \mathbb{T}$ is an arbitrary interval of length $\lambda_\alpha^{(n-2)}$ then there can be at most one $\mathcal{P}(n)$ partition subinterval $\mathcal{I}' \subset \mathcal{I}$ whose length is different from $\lambda_\alpha^{(n-1)}$. Moreover, the length of $\mathcal{I}'$ is larger than $\lambda_\alpha^{(n-1)}$ but less than $2 \cdot \lambda_\alpha^{(n-1)}$.

Property 2.:



If we add the point $q_{\alpha,n}\alpha$ to the partition points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$ then one short interval of length $\lambda_\alpha^{(n)}$ shows up adjacent to 0 modulo 1. Denote now by $\mathcal{I}$ an interval belonging to the partition $\mathcal{P}(n)$. If $0 \le k, k' < q_{\alpha,n+1}$, $k\alpha \in \mathcal{I}$ and $k'\alpha \in \mathcal{I}$ then $k - k'$ is an integer multiple of $q_{\alpha,n}$, that is, $k - k' = tq_{\alpha,n}$ for an integer t. If $k' = k + q_{\alpha,n} < q_{\alpha,n+1}$, $k\alpha, k'\alpha \in \mathcal{I}$, then the distance of $k\alpha$ and $k'\alpha$ equals $\lambda_\alpha^{(n)}$. Moreover, if $k'' = k + 2q_{\alpha,n} < q_{\alpha,n+1}$, $k''\alpha \in \mathcal{I}$ holds as well then $\{k''\alpha\} - \{k'\alpha\}$ and $\{k'\alpha\} - \{k\alpha\}$ are of the same sign.

Set $d_0 = 1$, $l_0 = 1/100$ and $K_j = 10j$ for $j \in \mathbb{N}$.

Suppose we have defined d_{j-1} and l_{j-1}.

Choose $0 < d_j < l_{j-1}/3 < d_{j-1}/100$ such that

$$\frac{1}{3 \cdot (32 \cdot 10^4 \cdot K_j^3 j^6)^2} = \frac{1}{3 \cdot (32 \cdot 10^7 j^9)^2} > \left(1 - \frac{4}{10j}\right)^{\log_2(8K_j^2/d_j^2)}$$

and for $j \geq 2$ we also have

$$\left(1 - \frac{4}{10(j-1)}\right)^{-3\log_2(8K_{j-1}^2/d_{j-1}^2)} < \left(1 - \frac{4}{10j}\right)^{-\log_2(8K_j^2/d_j^2)}.$$

Set $l_j = \dfrac{d_j}{16 \cdot 10^4 \cdot K_j^3 j^6}$.

Set $n(0, \alpha) = 0$ and suppose $j \geq 1$.

For any $\alpha \in [0,1) \setminus \mathbb{Q}$ choose $n(j, \alpha)$ so that

$$\frac{1}{q_{\alpha,n(j,\alpha)-2}} > d_j, \text{ but } \frac{1}{q_{\alpha,n(j,\alpha)-1}} \leq d_j.$$

By $\alpha_{n(j,\alpha)-1}, \alpha_{n(j,\alpha)-2} \in (0,1)$ we have

$$\lambda_\alpha^{(n(j,\alpha)-2)} = \frac{1}{q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}\alpha_{n(j,\alpha)-1}} < \frac{1}{q_{\alpha,n(j,\alpha)-1}} \le d_j \text{ and}$$

$$2\lambda_\alpha^{(n(j,\alpha)-3)} = \frac{2}{q_{\alpha,n(j,\alpha)-2} + q_{\alpha,n(j,\alpha)-3}\alpha_{n(j,\alpha)-2}} > \frac{2}{2q_{\alpha,n(j,\alpha)-2}} > d_j.$$

The choice of d_j implies that $n(j-1,\alpha) \le n(j,\alpha)$.

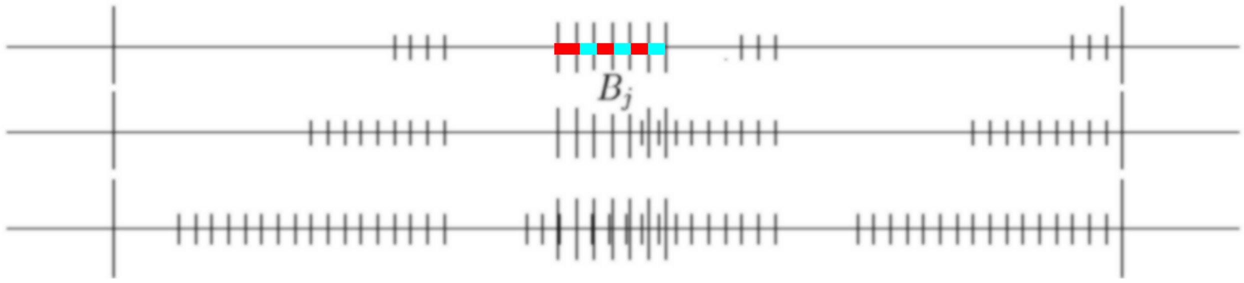
We denote by A the set of those $\alpha = [a_{\alpha,1}, a_{\alpha,2}, \ldots] \in [0,1) \setminus \mathbb{Q}$ for which $a_{\alpha,n} \le K_j$ holds for $n(j-1,\alpha) < n < n(j,\alpha)$, and

$$\frac{1}{q_{\alpha,n(j,\alpha)}} < l_j \le \frac{1}{q_{\alpha,n(j,\alpha)} - q_{\alpha,n(j,\alpha)-1}}.$$

The above property can be rephrased as

$$\frac{1}{a_{\alpha,n(j,\alpha)} \cdot q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}} < l_j \le \frac{1}{(a_{\alpha,n(j,\alpha)} - 1)q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}}.$$

Proposition.: $\dim_{\mathrm{H}} A = 1$.



Set $h_j = \dfrac{d_j}{100 \cdot K_j j^2}$ and $B_j = [\frac{1}{j} - 2h_j, \frac{1}{j} - h_j) \subset [0,1)$.

B_j are disjoint for $j = 1, 2, \ldots$ and $\dfrac{h_j}{l_j} = \dfrac{16 \cdot 10^4 \cdot K_j^3 j^6}{10^2 K_j j^2} = 16 \cdot 10^2 K_j^2 j^4$

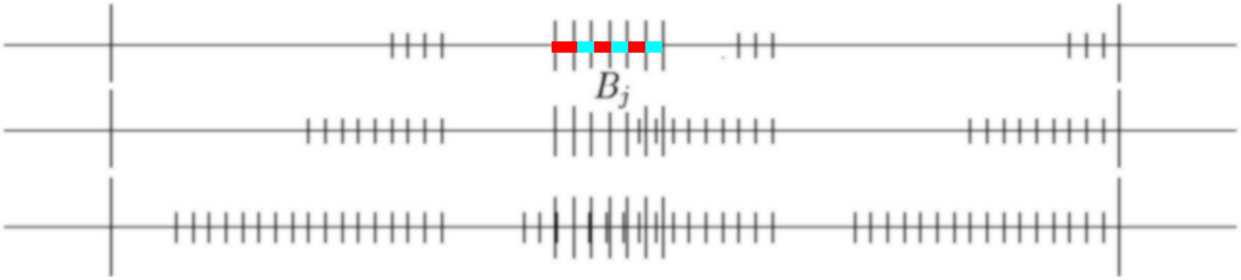
is an even integer.

Set $f_j(x) = 0$ if $x \in [0,1) \setminus B_j$.

For $t = 1, 2, \ldots, (h_j/l_j)$ set $f_j(x) = \dfrac{(-1)^t}{h_j}$ if $x \in [\frac{1}{j} - 2h_j + (t-1)l_j, \frac{1}{j} - 2h_j + tl_j)$.

extend the def. of f_j to $\mathbb{R}$ by making it periodic by one.

Clearly, $|f_j(x)| = 1/h_j$ for $x \in B_j$, $\int_{[0,1)} |f_j| = 1$ and $\int_{[0,1)} f_j = 0$.



Set $M^*(f_j, x, \alpha) = \sup_{K>0} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x + k\alpha) \right|.$

Denote by $X^*(f_j, \alpha)$ the set of those $x \in [0, 1)$ for which

$$M^*(f_j, x, \alpha) \geq \epsilon_j \stackrel{\text{def}}{=} \frac{1}{j^2}.$$

The next proposition establishes a weak maximal type inequality:

Proposition.: *If* $\alpha \in A$ *then* $|X^*(f_j, \alpha)| \leq \frac{1}{j^2}.$

This prop. $\Rightarrow$ that for $f = \sum_{j=1}^{\infty} f_j$ for any $\alpha \in A$ we have

$$\lim_{K\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} f(x + k\alpha) \right| = \limsup_{K\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} \sum_{j=1}^{\infty} f_j(x + k\alpha) \right| = 0.$$

The Hausdorff dimension estimate of A

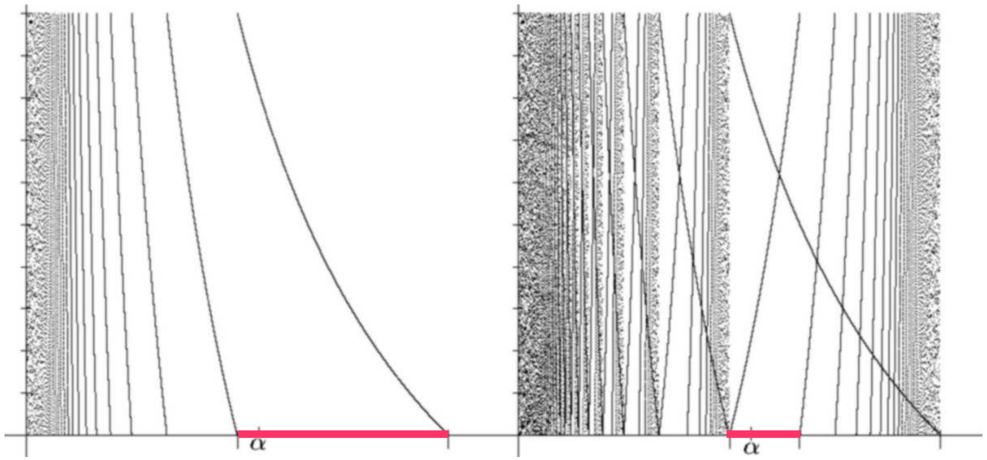
Suppose that μ is a finite Borel measure, a mass distribution on $\mathbb{R}$, the lower local dimension of μ at $\alpha \in \mathbb{R}$ equals

$$\underline{\dim}_{\mathrm{loc}}\,\mu(\alpha) = \liminf_{r\to 0+} \frac{\log_2 \mu(B(\alpha, r))}{\log_2 r}.$$

(It does not matter which base we use for the logarithm since changing the base multiplies the numerator and the denominator by the same constant.)

Proposition.: *Let $A \subset \mathbb{R}^n$ be a Borel set and let μ be a finite Borel measure. If $\underline{\dim}_{\mathrm{loc}}\,\mu(\alpha) \geq s$ for all $\alpha \in A$ and $\mu(A) > 0$ then $\dim_{\mathrm{H}} A \geq s$.*

There are many papers related to computing Hausdorff dimension of sets obtained by restrictions on the continued fraction partial denominators $a_{\alpha,n}$ of the numbers $\alpha = [a_{\alpha,1}, a_{\alpha,2}, \ldots]$ belonging to these sets. In the estimate of the Hausdorff dimension of our set A our bounds K_j on the $a_{\alpha,n}$ vary, and sometimes, for the $a_{\alpha,n(j,\alpha)}$'s there are very serious restrictions on the partial denominators. This is why we had to use a direct computation of the dimension, based on the estimate of the lower local dimension of a mass distribution on A.



The fundamental interval $I(n,\alpha)$ denotes the closed interval with endpoints $\frac{p_{\alpha,n}}{q_{\alpha,n}} = [a_{\alpha,1},...,a_{\alpha,n}]$ and $\frac{p_{\alpha,n}+p_{\alpha,n-1}}{q_{\alpha,n}+q_{\alpha,n-1}} = [a_{\alpha,1},...,a_{\alpha,n}+1]$.

We also put $I(0,\alpha) = [0,1]$.

$$|I(n,\alpha)| = \frac{1}{q_{\alpha,n}(q_{\alpha,n}+q_{\alpha,n-1})}.$$

The n'th iterate of the Gauss map, $G^n(\alpha)$ maps $I(n,\alpha)$ onto $[0,1)$.

Suppose $\alpha_0 = [a_{\alpha_0,1}, a_{\alpha_0,2}, ...]$, then G^n maps in a strict monotone way $I(n,\alpha_0)$ onto $[0,1)$. Denote by F_{n,α_0} the inverse of G^n restricted to $I(n,\alpha_0)$.

Then $$(-1)^{n-1}(q_{\alpha_0,n-1}F_{n,\alpha_0}(\alpha) - p_{\alpha_0,n-1}) = \frac{1}{q_{\alpha_0,n}+q_{\alpha_0,n-1}\alpha}$$

$\Rightarrow$ $$F'_{n,\alpha_0}(\alpha) = \frac{(-1)^n}{(q_{\alpha_0,n}+q_{\alpha_0,n-1}\alpha)^2},$$

$\Rightarrow$ F_{n,α_0}, and $G^n|_{I(n,\alpha_0)}$ both satisfy a bounded distortion property:

$$\forall n\in\mathbb{N}\quad \frac{F'_{n,\alpha_0}(\alpha)}{F'_{n,\alpha_0}(\beta)} \le 4,\ \forall\,\alpha,\beta\in[0,1] \text{ and } \frac{(G^n)'(\alpha)}{(G^n)'(\beta)} \le 4,\ \forall\,\alpha,\beta\in \text{int}(I(n,\alpha_0)).$$

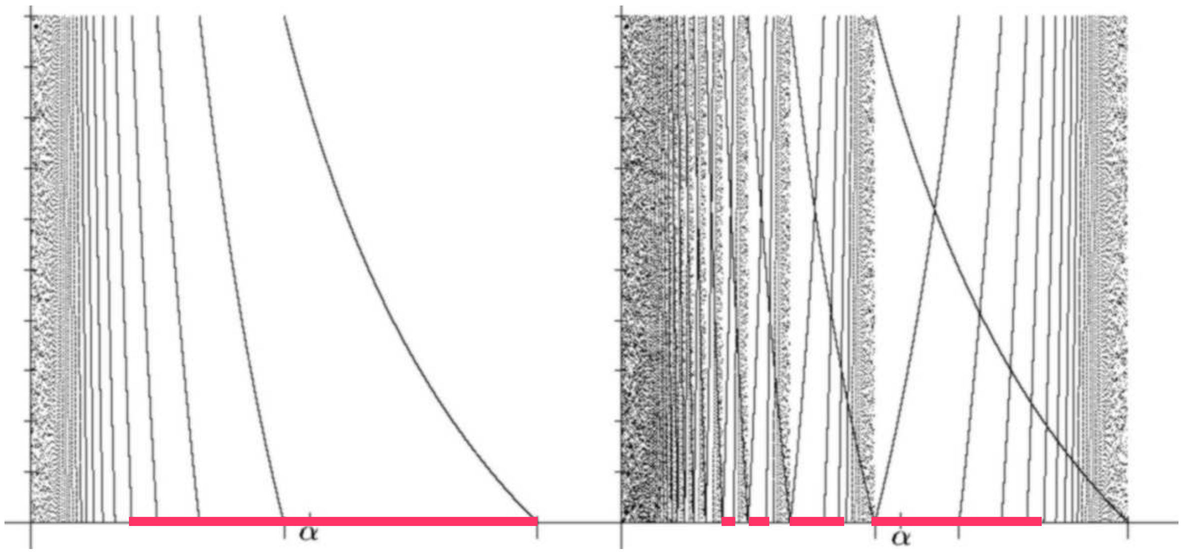
To define μ as a mass distribution it is sufficient to define it on the fundamental intervals of the form $I(n,\alpha)$, $n \in \mathbb{N}$, $\alpha \in [0,1) \setminus \mathbb{Q}$.
For any α we put $\mu(I(0,\alpha)) = \mu([0,1]) = 1.$
If $\alpha_0 \in [0,1) \setminus \mathbb{Q}$, $\text{int}(I(n,\alpha_0)) \cap A = \emptyset$ then we set $\mu(I(n,\alpha_0)) = 0$.
Suppose $\alpha_0 \in A$, $\alpha_0 = [a_{\alpha_0,1}, a_{\alpha_0,2}, ...]$. We need to define $\mu(I(n,\alpha_0))$ for all $n \in \mathbb{N}$.

Suppose that $\mu(I(n-1,\alpha_0))$ is defined and $\Gamma(n-1,\alpha_0) \stackrel{\text{def}}{=} \dfrac{\mu(I(n-1,\alpha_0))}{|I(n-1,\alpha_0)|}$.

We want to define $\mu(I(n,\alpha_0))$.

First suppose that we can find $j \in \mathbb{N}$ such that $n(j-1,\alpha_0) < n < n(j,\alpha_0)$.
Denote by $I_k(n,\alpha_0)$ the closed interval with endpoints $[a_{\alpha_0,1}, ..., a_{\alpha_0,n-1}, k]$ and $[a_{\alpha_0,1}, ..., a_{\alpha_0,n-1}, k+1]$.

Then $I(n,\alpha_0) = I_{a_{\alpha_0,n}}(n,\alpha_0)$ and $|I_{k+1}(n,\alpha_0)| < |I_k(n,\alpha_0)|$ for all $k \in \mathbb{N}$.



Then

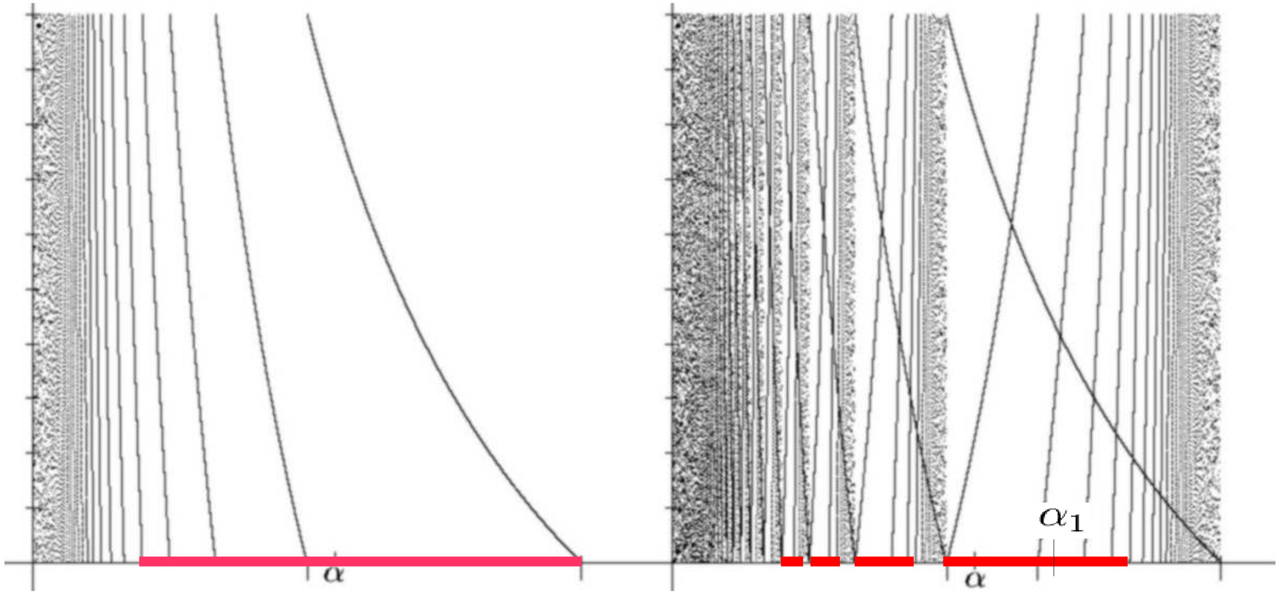
$$A \cap I(n-1,\alpha_0) \subset \bigcup_{k=1}^{K_j} I_k(n,\alpha_0) \stackrel{\text{def}}{=} I'(n,\alpha_0) = F_{n-1,\alpha_0}([\frac{1}{K_j},1]) = F_{n-1,\alpha_0}([\frac{1}{10j},1]).$$

By the bounded distortion property of F_{n-1,α_0} and by its strict monotonicity

$$\frac{|I(n-1,\alpha_0) \setminus I'(n,\alpha_0)|}{|I(n-1,\alpha_0)|} = \frac{|F_{n-1,\alpha_0}([0,\frac{1}{10j}])|}{|F_{n-1,\alpha_0}([0,1])|} \leq \frac{4}{10j}.$$

Therefore, $|I'(n,\alpha_0)| \geq (1-\frac{4}{10j})|I(n-1,\alpha_0)|.$

We put $\mu(I(n,\alpha_0)) \stackrel{\text{def}}{=} \frac{|I(n,\alpha_0)|}{|I'(n,\alpha_0)|}\mu(I(n-1,\alpha_0)).$



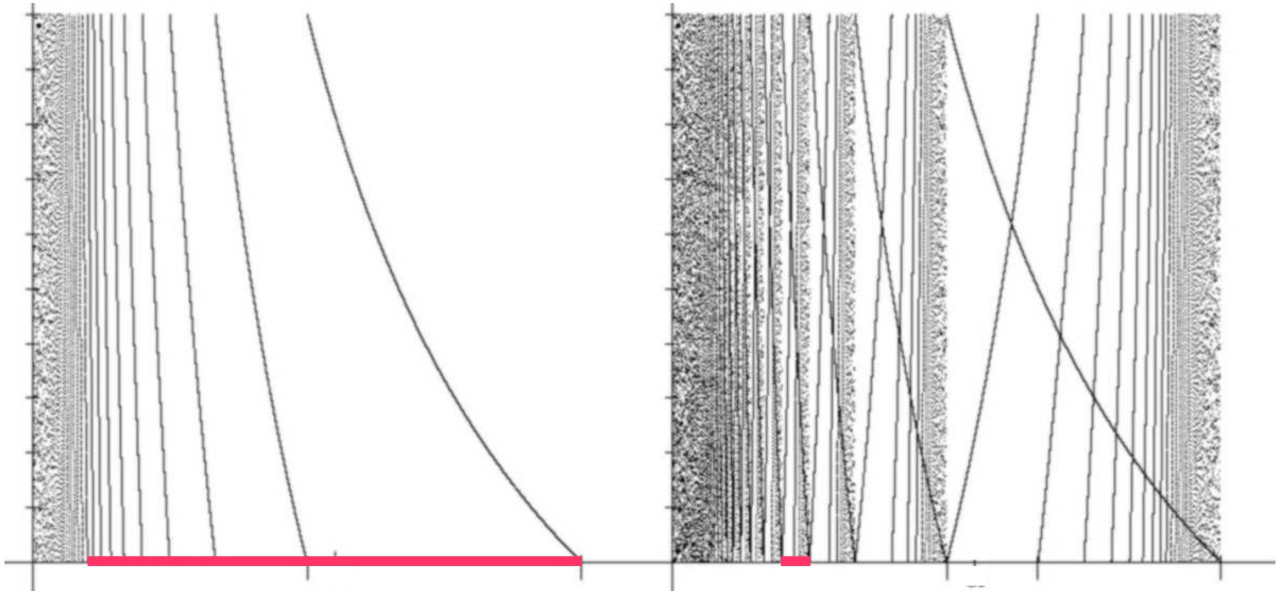
For all $k = 1, ..., K_j$ there exists $\alpha_1 \in \mathrm{int}(I_k(n, \alpha_0)) \cap A$.

$$\mu(I'(n,\alpha_0)) = \mu(\bigcup_{\alpha \in A \cap I(n-1,\alpha_0)} I(n,\alpha)) = \sum_{k=1}^{K_j} \mu(I_k(n,\alpha_0)) =$$

$$\frac{\sum_{k=1}^{K_j} |I_k(n,\alpha_0)|}{|I'(n,\alpha_0)|}\mu(I(n-1,\alpha_0)) = \mu(I(n-1,\alpha_0)).$$

$$\Rightarrow \quad \Gamma(n,\alpha_0) \overset{\mathrm{def}}{=} \frac{\mu(I(n,\alpha_0))}{|I(n,\alpha_0)|} = \frac{\mu(I(n-1,\alpha_0))}{|I'(n,\alpha_0)|} =$$

$$\frac{\mu(I(n-1,\alpha_0))}{|I(n-1,\alpha_0)|} \cdot \frac{|I(n-1,\alpha_0)|}{|I'(n,\alpha_0)|} \le \Gamma(n-1,\alpha_0)\left(1 - \frac{4}{10j}\right)^{-1}.$$



Missing cases: $\exists\ j \in \mathbb{N}$ for which $n = n(j, \alpha_0)$.

Put $\mu(I(n(j,\alpha_0),\alpha_0)) \stackrel{\text{def}}{=} \mu(I(n(j,\alpha_0)-1,\alpha_0))$.

Need estimates:

$$\frac{|I(n(j,\alpha_0),\alpha_0)|}{|I(n(j,\alpha_0)-1,\alpha_0)|} > \frac{1}{3a^2_{\alpha_0,n(j,\alpha_0)}} > \frac{1}{3\cdot(32\cdot 10^4 K_j^3 j^6)^2} > \left(1-\frac{4}{10j}\right)^{\log_2(8K_j^2/d_j^2)}.$$

$$\Gamma(n(j,\alpha_0),\alpha_0) \le \Gamma(n(j-1,\alpha_0),\alpha_0)\frac{\Gamma(n(j-1,\alpha_0)+1,\alpha_0)}{\Gamma(n(j-1,\alpha_0),\alpha_0)}\cdots$$

$$\frac{\Gamma(n(j,\alpha_0)-1,\alpha_0)}{\Gamma(n(j,\alpha_0)-2,\alpha_0)}\cdot\frac{\Gamma(n(j,\alpha_0),\alpha_0)}{\Gamma(n(j,\alpha_0)-1,\alpha_0)} < \left(1-\frac{4}{10j}\right)^{-3\log_2(8K_j^2/d_j^2)}$$

Suppose $\alpha_0 \in A$ and $0 < r < |I(n(2,\alpha_0),\alpha_0)|$.
Choose n such that $|I(n+1,\alpha_0)| \leq r < |I(n,\alpha_0)|$

Then one can obtain estimates like: $\Gamma(n,\alpha_0) < \left(1 - \dfrac{4}{10(j-1)}\right)^{7\log_2 r}.$

$$\Rightarrow \frac{\log_2 \mu(B(\alpha_0,r))}{\log_2 r} \geq \frac{\log_2(\Gamma(n,\alpha_0)) + 3\log_2(10^{19}j^{18}) + \log_2 r}{\log_2 r} > 1 - \frac{6}{j-1} - \frac{3\log_2(10^{19}j^{18})}{-\log_2 r}.$$

This implies $\liminf_{r\to 0+} \dfrac{\log_2 \mu(B(\alpha_0,r))}{\log_2 r} \geq 1. \Rightarrow \dim_{\mathrm{H}} A = 1.$