Riesz products and Multiplicative Gibbs measures

Ai-Hua FAN

University of Picardie, France

Warwick, April 18-22, 2011

Outline

Motivation and Problem

I. Multirecurrence and Multiple ergodic averages

Theorem (Furstenberg-Weiss, 1978) If

- (X, d) a compact metric space.
- $T_i : X \to T$ continuous, $T_i T_j = T_j T_i$ $(1 \le i, j \le d)$.

Then there exists $x \in X$ and $(n_k) \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} T_i^{n_k} x = x, \quad \forall i = 1, 2, \cdots, \ell.$$

Applied to $X = \{0, 1\}^{\mathbb{N}}$, $T_i = T^i$, T being the shift.

Theorem (Szemeredi, 1975) If $\Lambda \subset \mathbb{N}$ satisfies

$$\limsup_{N\to\infty} \frac{|\Lambda \cap [1, N]|}{N} > 0,$$

Then Λ contains arithmetic progressions of arbitrary length.

II. Multiple ergodic theorem

Multiple ergodic averages

$$\frac{1}{n}\sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x)$$

- Lesigne
- Bourgain
- Furstenberg
- Host-Kra
- ...

III. Setting

- $T : X \to X$ topological dynamical system
- $f_1, \cdots, f_\ell$ continuous functions on X ($\ell \geq 2$)
- Denote, if the limit exists

$$A_{f_1,\cdots,f_\ell}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x).$$

- For α, denote

$$E(\alpha) = \{x \in X : A_{f_1,\cdots,f_\ell}(x) = \alpha\}.$$

Problem : What is the size of $E(\alpha)$?

N.B. The case $\ell = 1$ is classical. The case $\ell \geq 2$ is a challenging problem.

IV. Spectrum of Birkhoff averages

(X, T) : System satisfying specification property.
$\Phi : X \to \mathbb{B}$ (a Banach space) : continuous.

$$\mathcal{M}_\Phi(\alpha) := \left\{ \mu \in \mathcal{M}_{\rm inv} : \int \Phi d\mu = \alpha \right\}.$$

Theorem (Fan-Liao-Peyrière, DCDS 2008)

(a) If $\mathcal{M}_\Phi(\alpha) = \emptyset$, we have $X_\Phi(\alpha) = \emptyset$.
(b) If $\mathcal{M}_\Phi(\alpha) \neq \emptyset$, we have the conditional variational principle

$$h_{\rm top}(X_\Phi(\alpha)) = \sup_{\mu \in \mathcal{M}_\Phi(\alpha)} h_\mu.$$

Related works, Fan-Feng, Fan-Feng-Wu, Olivier, Barreira-Schmeling, Feng-Lau-Wu, Taken-Verbytzky, Olsen, et al.

On groups $\{-1, 1\}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{N}}$

I. A special case on $\{-1,1\}^{\mathbb{N}}$

- $X = \{-1,1\}^{\mathbb{N}}$
- T the shift : $(x_n)_{n\geq 0} \mapsto (x_{n+1})_{n\geq 0}$.
- $f_i(x) = x_1$ the projection on the first coordinates $(i = 1, 2, \cdots, \ell)$
- for $\theta \in \mathbb{R}$, denote

$$B_\theta := \left\{ x \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

Theorem (Fan-Liao-Ma, 2009)

For $\theta \notin [-1,1]$, $B_\theta = \emptyset$. For any $\theta \in [-1,1]$, we have

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\Big(\frac{1+\theta}{2}\Big),$$

where $H(t) = -t\log_2 t - (1-t)\log_2(1-t)$.

N.B. $\dim_H B_\theta \geq 1 - 1/\ell > 0$ if $\ell \geq 2$.

II. Proof using Riesz products

- Rademacher functions $r_n(x) = x_n$ are group characters
- Walsh functions

$$w_n = r_{n_1} \cdots r_{n_s}, \quad n = 2^{n_1-1}+2^{n_2-1}+\cdots+2^{n_s-1}, \quad 1 \le n_1 < n_2 < \cdots$$

  is a Hilbert space in $L^2(\{-1,1\}^{\mathbb{N}})$.

- The subsystem

$$\xi_k = r_k r_{2k} \cdots r_{\ell k} \quad (k \ge 1)$$

  are dissociated in the sense of Hewitt-Zuckerman.

- The following Riesz product measure is well defined

$$d\mu_\theta = \prod_{k=1}^{\infty} (1 + \theta \xi_k(x)) dx.$$

II. Proof (continued)

Lemma 1 (Expectation)

If $f(x) = f(x_1, \cdots, x_n)$, we have

$$\mathbb{E}_\theta[f] = \int f(x) \prod_{k=1}^{\lfloor n/\ell \rfloor} \big(1 + \theta \xi_k(x)\big) dx.$$

Proof. Because r_n are Haar-independent. QED

II. Proof (continued)

Lemma 2 (Law of large numbers)

If $f(x) = \sum_{n=0}^{\infty} g_n x^n$ with $\sum_n |g_n| < \infty$, then for μ_θ-almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(\xi_k(x)) = \mathbb{E}_\theta[g(\xi_1)].$$

Proof. Apply Menchoff Theorem to $\sum_{k=0}^{\infty} \frac{1}{k}\Big(g(\xi_k) - \mathbb{E}_\theta[g(\xi_k)]\Big)$ and conclude by Kronecker theorem :

- $\xi_k^{2n}(x) = 1$, $\xi_k^{2n-1}(x) = \xi_k(x)\ \forall n \geq 1$.
- $g(\xi_k) = \sum_{n=0}^{\infty} g_{2n} + \xi_k \sum_{n=1}^{\infty} g_{2n-1}$.
- $\mathbb{E}_\theta(\xi_k) = \theta, \mathbb{E}_\theta(\xi_j\xi_k) = \theta^2, \quad (j \neq k)$.
- $\mathbb{E}_\theta[g(\xi_k)] = \sum_{n=0}^{\infty} g_{2n} + \theta \sum_{n=1}^{\infty} g_{2n-1}$.
- $g(\xi_j) - \mathbb{E}_\theta g(\xi_k)$ are orthogonal.

QED

II. Proof (continued) : Proof of Theorem

$\mu_\theta(B_\theta) = 1$ (Lemma 2 applied to $g(x) = x$) :

$$\mu_\theta\!-\!a.e.\ x \quad \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} \xi_k(x) = \mathbb{E}(\xi_1) = \theta.$$

By Lemma 1 (applied to 1_{I_n}) : $\forall x$, $\forall n \geq \ell$,

$$P_\theta(I_n(x)) = \frac{1}{2^n}\prod_{k=1}^{\lfloor n/\ell \rfloor}\big(1+\theta\xi_k(x)\big).$$

Notice that $\log(1+\theta\xi_k(x)) = -\sum_{n=1}^{\infty}\frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty}\frac{\theta^{2n-1}}{2n-1}\xi_k(x)$. Then for all points $x \in B_\theta$,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\log(1+\theta\xi_k(x)) = -\sum_{n=1}^{\infty}\frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty}\frac{\theta^{2n-1}}{2n-1}\theta.$$

The right hand side can be written as

$$\theta\log(1+\theta) - \frac{\theta-1}{2}\log(1-\theta^2) = \left[1 - H\left(\frac{1+\theta}{2}\right)\right]\log 2.$$

We conclude by Billingsley's theorem. QED

III. A special case on $\{0,1\}^{\mathbb{N}}$

- $X = \{0,1\}^{\mathbb{N}}$
- T the shift : $(x_n)_{n\geq 0} \mapsto (x_{n+1})_{n\geq 0}$.
- $f_i(x) = x_1$ the projection on the first coodinates $(i = 1, 2, \cdots, \ell)$
- for $\theta \in \mathbb{R}$, denote

$$A_\theta := \left\{ x \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

Remarks

- $f_i(T^i x) = x_i$ are not group characters.
- Riesz product method doesn't work and the study of A_θ is more difficult than B_θ.
- The study of A_θ was the motivation.

IV. A subset of B_0

For $\ell = 2$, define

$$X_0 := \left\{x \in \{0,1\}^{\mathbb{N}} : x_n x_{2n} = 0, \quad \text{for all } n\right\}.$$

Fibonacci sequence : $a_0 = 1,\ a_1 = 2, \quad a_n = a_{n-1} + a_{n-2}\ (n \geq 2)$.

Theorem (Fan-Liao-Ma, 2009)

$$\dim_B(X_0) = \frac{1}{2\log 2}\sum_{n=1}^{\infty}\frac{\log a_n}{2^n} = 0.8242936\cdots$$

Theorem (Kenyon-Peres-Solomyak, 2011)

$$\dim_H(X_0) = -\log_2 p = 0.81137\cdots \quad (p^3 = (1-p)^2).$$

Remarks

- $\dim_H(X_0) < \dim_B(X_0)$.
- A class of sets like X_0 is studied by Kenyon-Peres-Solomyak.

V. Combinatorial proof (of box dimension) Starting point

$$\dim_B X_0 = \lim_{n\to\infty} \frac{\log_2 N_n}{n}$$

where N_n is the cardinality of

$$\{(x_1 x_2 \cdots x_n) : x_k x_{2k} = 0 \ \text{ for } k \geq 1 \text{ such that } \ 2k \leq n\}.$$

Let $\{1, \cdots, n\} = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m$ with

$$C_0 := \{1,\ 3,\ 5,\ \ldots,\ 2n_0 - 1\},$$

$$C_1 := \{2\cdot 1,\ 2\cdot 3,\ 2\cdot 5,\ \ldots,\ 2\cdot(2n_1 - 1)\},$$

$$\cdots$$

$$C_k := \{2^k\cdot 1,\ 2^k\cdot 3,\ 2^k\cdot 5,\ \ldots,\ 2^k\cdot(2n_k - 1)\},$$

$$\cdots$$

$$C_m := \{2^m\cdot 1\},$$

The conditions $x_k x_{2k} = 0$ with k in different columns in the above table are independent. On each column, (x_k, x_{2k}) is conditioned to be different from $(1,1)$. Counting column by column, we get

$$N_n = a_{m+1}^{n_m} a_m^{n_{m-1}-n_m} a_{m-1}^{n_{m-2}-n_{m-1}} \cdots a_1^{n_0-n_1}.$$

Multiplicative Gibbs measures and Multiple ergodic averages

[part of Ph D. thesis of WU Meng]

I. Setting We are going to study some special cases concerning

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x, T^{qn} x)$$

where $f : \Sigma_m \times \Sigma_m \to \mathbb{R}$, $\Sigma_m = S^\infty$ with $S = \{0, 1, \cdots .m-1\}$.

- Assumption : $f(x, y) = \varphi(x_1, y_1)$ for $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$. φ may take values in $\mathbb{R}^d$.
- Object of study :

$$A_\varphi(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(x_n, x_{qn}).$$

$$E(\alpha) := \{x \in \Sigma_m : A_\varphi(x) = \alpha\}.$$

- Additive action of $\mathbb{N}$ on Σ_m : $(x_n) \to (x_{n+k})$.
- Multiplicative action of $q^{\mathbb{N}}$ on Σ_m : $(x_n) \to (x_{nq^k})$.

II. Partial result

Theorem

Assume that for each $i \in S$, the sequence $(\varphi(i,j))_{j \in S}$ is a permutation of $(\varphi(0,j))_{j \in S}$. Let

$$P(t) := \log_m \sum_{j=0}^{m-1} e^{\langle t, \varphi(0,j) \rangle}.$$

Then

$$\dim_H E(\alpha) = \frac{P(t_\alpha) - \langle \alpha, t_\alpha \rangle}{q} + \left(1 - \frac{1}{q}\right),$$

and t_α is the unique solution of $\nabla P(t_\alpha) = \alpha$.

Examples : $\varphi(x,y) = \phi(x + y \mod m)$

III. Notation

- (Associated matrices) $\varphi : S \times S \to \mathbb{R}^d$, $h : S \times S \to \mathbb{R}$, $t \in \mathbb{R}^d$

$$\Phi_h(t) := \left(h(i,j)e^{\langle t, \varphi(i,j)\rangle}\right)_{S\times S}.$$

$$\Phi(t) = \Phi_1(t).$$

- (Perron eigenvalue and eigenvectors of $\Phi(t)$)

$$\ell(t)\Phi(t) = \ell(t)\rho(t), \quad \Phi(t)w(t) = \rho(t)w(t)$$

$$\sum_i w_i(t) = 1, \quad \sum_i \ell_i(t)w_i(t) = 1.$$

IV. Pressure Definition (cas $d = 1$)

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log_m Z_n(t)$$

$$Z_n(t) = \sum_{x_1,\ldots,x_{qn}} \exp(t \sum_{j=1}^{n} \varphi(x_j, x_{jq}).$$

Theorem (Existence of Pressure)

$$P(t) = (q-1)^2 \sum_{k=1}^{\infty} \frac{\log_m ||\Phi(t)^k||_1}{q^{k+1}}$$

V. Gibbs measure For $n \geq 1$, μ_n is the probability measure uniformly distributed on each nq-cylinder and such that

$$\mu_n([x_1, ..., x_{qn}]) = \frac{1}{Z_n(t)} \exp(t \sum_{j=1}^{n} \varphi(x_j, x_{jq})).$$

Theorem (Existence of Gibbs measure)

For each t, the measures μ_n converge weakly to a probability measure μ_t, called Gibbs measure.

VI. Fundamental lemma

Theorem (Distribution of μ_t)

Let $N \geq 1$ and $F_1, ..., F_N$ be N arbitrary real functions defined on $S \times S$. We have

$$\lim_{n\to\infty} \int \prod_{j=1}^{N} F_j(x_j, x_{jq}) d\mu_n = \prod_{k=1}^{\lfloor \log_q N \rfloor} \prod_{\frac{N}{q^k} < i \leq \frac{N}{q^{k-1}}} \frac{1^t (\prod_{j=0}^{k-1} \Phi_{F_{iq^j}}(t)) w(t)}{\rho(t)^k}.$$

VII. Consequences of Fundamental lemma

- Existence of μ_t, Walsh-Fourier coefficients of μ_t
- Gibbs property

$$\mu_t[a_1, ..., a_N] = \frac{1}{\rho(t)^{\lfloor \frac{N}{q} \rfloor}} \exp\left(t \sum_{j=1}^{\lfloor \frac{N}{q} \rfloor} \varphi(a_j, a_{jq}) \right) \prod_{k=\lfloor \frac{N}{q} \rfloor + 1}^{N} w_{a_k}(t).$$

(product of an infinite number of Markov measures).

- Law of large numbers : Let $X_j := \varphi(x_j, x_{jq}) - \mathbb{E}_{\mu_t}\varphi(x_j, x_{jq})$, $Y_n = (X_1 + \cdots + X_n)/n$. Then

$$\mathbb{E}_{\mu_t} Y_n^2 = O((\log n)/n).$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(x_j, x_{jq}) \overset{\mu_t}{=} (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \frac{1^t \Phi(t)^j \Phi_\varphi(t) w(t)}{\rho(t)^{j+1}}.$$

VIII. What make it work

- Decompositions

$$\mathbb{N}^* = \bigsqcup_{q \nmid i} \Lambda_i, \quad \Lambda_i = \{iq^j\}_{j \geq 0}$$

$$[1,n] = \bigsqcup_{q \nmid I, i \leq n} \Lambda_i(n), \quad \Lambda_i(n) = \Lambda_i \cap [1,n].$$

- $\sharp\Lambda_i(n) = k$ iff $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$.
- The variables $x|_{\Lambda_i}$ $(q \nmid i)$ are independent.
- Perron-Frobenius Theorem

$$\frac{\Phi(t)^n}{\rho(t)^n} = w(t)\ell(t)(1 + O(\delta^n)) \quad (0 < \delta < 1).$$

- When Φ is "symmetric", eigenvectors $w(t)$ are constant vectors.

IX. Work to be done

If Φ is not "symmetric", the constructed Gibbs measure may be not optimal.

There is a long way to go.

Oriented walks and Riesz products

I. Oriented walks on $\mathbb{Z}$

Let $t = (\epsilon_n(t))_{n\geq 1} \in \mathbb{D} := \{-1,+1\}^{\mathbb{N}}$. Consider

$$S_n(t) = \sum_{k=1}^{n} \epsilon_1(t)\epsilon_2(t)\cdots\epsilon_k(t).$$

- $"-1" =$ *left*, $"+1" =$ *right*
- At the time 0, an individual is at the origin and keeps the orientation to the right.
- If $\epsilon_1(t) = 1$, he forwards one step in the orientation he kept (to the right)
- If $\epsilon_1(t) = -1$, he returns back and then forwards one step (to the left).
- State at time $n+1$ is $(S_{n+1}, \xi_{n+1}) := (position, orientation)$

$$S_{n+1} = S_n + \epsilon_{n+1}\xi_n, \qquad \xi_{n+1} = \epsilon_{n+1}\xi_n.$$

- (S_n, ξ_n) is Markovian if (ϵ_n) is iid.

II. Oriented walks on $\mathbb{Z}^2$

Let $t = (\epsilon_n(t))_{n\geq 1} \in \mathbb{D} := \{-1, +1\}^{\mathbb{N}}$. Consider

$$S_n(t) = \sum_{k=1}^{n} e^{(\epsilon_1+\cdots+\epsilon_k)\alpha i} = \sum_{k=1}^{n} i^k \epsilon_1(t)\epsilon_2(t)\cdots\epsilon_k(t).$$

with $\alpha = \frac{\pi}{2}$.

- $"-1" =$ *turn to right* with $90°$
- $"+1" =$ *turn to left* with $90°$
- Orientations : 1 (rightward), i (upward), -1 (leftward), -i (downward)
- State at time $n+1$ is $(S_{n+1}, \xi_{n+1}) := (position, orientation)$

$$S_{n+1} = S_n + e^{\epsilon_{n+1}\pi/2i}\xi_n, \qquad \xi_{n+1} = e^{\epsilon_{n+1}\pi/2i}\xi_n.$$

III. Positions on $\mathbb{Z}^2$

For $z \in \mathbb{C}$ let

$$F_z = \left\{ t \in \mathbb{D} : \quad \lim_{n\to\infty} \frac{S_n(t)}{n} = z \right\}.$$

$$F_{\rm bd} = \{ t \in \mathbb{D} : \ S_n(t) = O(1) \ \text{ as } \ n \to \infty \}.$$

Let $\Delta = \{ Z = x + iy : \ |x| \le \frac{1}{2}, |y| \le \frac{1}{2} \}$, a unit square (not a disk !).

Theorem (Fan 2000)

(1) If $z \notin \Delta$, we have $F_z = \emptyset$.

(2) If $z = x + iy \in \Delta$, we have

$$\dim_H F_z = \dim_P F_z = \frac{1}{2} \left[H\left(\frac{1+2x}{2} \right) + H\left(\frac{1+2y}{2} \right) \right]$$

(3) $\dim_H F_{\rm bd} = 1$.

N. B. Fast Birkhoff average (like F_{bd}), see Fan-Schmeling, Pollicott, Jordan-Pollicott, ... Dynamicall diophantine approximation, see Fan-Schmeling, Persson-Schmeling,...

IV. Proof on $\mathbb{Z}^2$(sketch)

- $i^k = 1, i, -1, -i$ according to $k = 0, 1, 2, 3 \pmod 4$
- $c_k = a, b, c, d$ according to $k = 0, 1, 2, 3 \pmod 4$
- Riesz product

$$d\mu_c(t) = \prod_{k=1}^{\infty} (1 + c_k \epsilon_1(t)\epsilon_2(t) \cdots \epsilon_k(t))\, dt$$

- z= x+yi. Maximize on

$$\frac{a-c}{4} = x, \quad \frac{d-b}{4} = y.$$

V. Open questions

- Given an angle $0 < \alpha < 2\pi$. What is the behavior of

$$S_n(t) = e^{\epsilon_1 \alpha i} + e^{(\epsilon_1+\epsilon_2)\alpha i} + \cdots + e^{(\epsilon_1+\epsilon_2+\cdots+\epsilon_n)\alpha i} \ ?$$

  [S_n may not stay on a lattice.]

- A 3-dimensional generalization is the following

$$S_n(t) = \sum_{k=1}^{n} R^{\epsilon_1+\cdots+\epsilon_k} v$$

  where v is a vector and R is a rotation. The simplest is

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An open problem on Riesz products

I. Riesz product on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

- F. Riesz (1918) : singular BV function

$$F(x) = \lim_{N\to\infty} \int_0^x \prod_{n=1}^{N} (1 + \cos 2\pi 4^n t) dt$$

- Zygmund (1932) : $a_n = r_n e^{2\pi i \phi_n} \in \Delta$, $3\lambda_n \le \lambda_{n+1}$

$$F(x) = \lim_{N\to\infty} \int_0^x \prod_{n=1}^{N} (1 + r_n \cos 2\pi(\lambda_n t + \phi_n)) dt$$

- Notation

$$\mu_a := \prod_{n=1}^{\infty} (1 + r_n \cos 2\pi(\lambda_n t + \phi_n)) := \mu_F.$$

II On $\mathbb{T}$ (continued)

- Zygmund dichotomy (1932)

$$F \text{ singular } \Leftrightarrow (a_n) \notin \ell^2; \quad F \text{ a.c. } \Leftrightarrow (a_n) \in \ell^2.$$

- Peyrière criterion (1973)

$$\sum |a_n - b_n|^2 = \infty \Rightarrow \mu_a \perp \mu_b;$$

$$\sum |a_n - b_n|^2 < \infty \Rightarrow \mu_a \ll \mu_b.$$

N. B. The second implication is proved under $\sup |a_n| < 1.$

- Parreau (1990) : $\sup |a_n| < 1$ replaced by $|a_n| = |b_n|$.
- Kilmer-Saeki (1988) : "$\sum |a_n - b_n|^2$" not "sufficient".
- Equivalence problem

III Elegant Proof by Peyrière (singularity)

- (Banach-Steinhaus) $\exists \alpha \in \ell^2$:

$$\sum \alpha_n (\overline{a}_n - \overline{b}_n) = +\infty$$

- $\exists N_k$: almost everywhere convergence of

$$\sum_{n=1}^{N_k} \alpha_n \left(e^{i\lambda_n x} - \frac{1}{2}\overline{a}_n \right), \quad \sum_{n=1}^{N_k} \alpha_n \left(e^{i\lambda_n x} - \frac{1}{2}\overline{b}_n \right)$$

- Difference of these sums (at a convergent point) :

$$\frac{1}{2} \sum \alpha_n (\overline{a}_n - \overline{b}_n) < +\infty.$$

IV On a compact abelian G

- $\Gamma = \{\gamma_n\}(\subset \widehat{G})$ is dissociated if $\sharp W_n(\Gamma) = 3^n$

$$W_n := W_n(\Gamma) := \{\epsilon_1\gamma_1 + \cdots + \epsilon_n\gamma_n : \epsilon_j = -1, 0, 1\}$$

- Notation : $a = (a_n)_{n\geq 1} \subset \mathbb{C}, |a_n| \leq 1$

$$P_{a,n}(x) = \prod_{k=1}^{n}(1 + \mathrm{Re}\ a_k\gamma_k(x))$$

- Remarkable relation

$$W_{n+1} = W_n \sqcup (-\gamma_{n+1} + W_n) \sqcup (\gamma_{n+1} + W_n)$$

$$\widehat{P}_{a,n+1}(\gamma) = \widehat{P}_{a,n}(\gamma) \quad \forall\gamma \in W_n.$$

V On G (continued)

- Riesz product (Hewitt-Zuckerman,1966)

$$\mu_a = w^*-\lim P_{a,N}(x)dx =: \prod_{n=1}^{\infty}(1 + \mathrm{Re}\ a_n\gamma_n(x))$$

- Lacunary sequences (i.e. $\lambda_{n+1} \geq 3\lambda_n$) are dissociated (on $\mathbb{T}$)
  $\sum_1^n \epsilon_j\lambda_j = \sum_1^n \epsilon'_j\lambda'_j, \quad \epsilon_n \neq \epsilon'_n$
  $\lambda_n \leq |\epsilon_n - \epsilon'_n|\lambda_n \leq 2\sum_{j=1}^{n-1}\lambda_j$
  $\lambda_n \leq 2(3^{-(n-1)} + \cdots + 3^{-2} + 3^{-1})\lambda_n < \lambda_n.$

VI Group $\mathbb{D}_2 = \{-1, 1\}^{\mathbb{N}}$

- Rademacher-Bernoulli characters are dissociated :

$$\gamma_n(x) = x_n \qquad \forall x = (x_n) \in \mathbb{D}_2, \forall n \geq 1.$$

- Riesz products are Bernoulli product measures

$$\mu_a([x_1, \cdots, x_n]) = p_1(x_1) \cdots p_n(x_n)$$

$$p_n(\pm) = \frac{1}{2}(1 + \mathrm{Re}\ a_k \gamma_k(\pm)) = \frac{1}{2}(1 \pm a_k)$$

- $\to \mathbb{D}_m \ (m \geq 2)$

VII On $\mathbb{D}_2$ (continued)

- Kakutani dichotomy (1948) : $\mu_a \perp \mu_b$ iff

$$\prod_{n=1}^{\infty} \mathbb{E}\sqrt{(1+\mathrm{Re}\ a_k\gamma_k)(1+\mathrm{Re}\ b_k\gamma_k)} = 0$$

equivalently,

$$\sum (1 - \sqrt{p_n q_n} - \sqrt{(1-p_n)(1-q_n)}) = \infty$$

with

$$p_n = (1+a_n)/2, \quad q_n = (1+b_n)/2.$$

- Method → martingale : $\prod_{n=1}^{N} \sqrt{\frac{d\mu_{a,n}}{d\mu_{b,n}}}$ in $L^1(\mu_{b,n})$.

VIII Random Riesz products

- Random Riesz products of Rademacher type :

$$\prod_{n=1}^{\infty}(1 + \mathrm{Re} \pm \; a_n \gamma_n(x))$$

- Random Riesz products of Steinhaus type : $\forall \omega \in G^{\mathbb{N}}$

$$\mu_{a,\omega} := \prod_{n=1}^{\infty}(1 + \mathrm{Re}\; a_n \gamma_n(x + \omega_n))$$

- Homogeneous martingale (Kahane random multiplication) :

$$Q_n(x) := \prod_{k=1}^{n}(1 + \mathrm{Re}\; a_k \gamma_k(x + \omega_k)), \quad \forall x \in G.$$

IX Two conjectures

- Conjecture 1 : $\forall \omega \in G^{\mathbb{N}}$

$$\mu_{a,\omega} \perp \mu_{b,\omega} \Leftrightarrow \mu_a \perp \mu_b; \quad \mu_{a,\omega} \ll \mu_{b,\omega} \Leftrightarrow \mu_a \ll \mu_b.$$

- Conjecture 2 :

$$\mu_a \perp \mu_b \Leftrightarrow \prod_{n=1}^{\infty} I(a_n, b_n) = 0.$$

$$\mu_a \ll \mu_b \Leftrightarrow \prod_{n=1}^{\infty} I(a_n, b_n) > 0.$$

$$I(a_n, b_n) := \mathbb{E}\sqrt{(1 + \mathrm{Re}\ a_k\gamma_k)(1 + \mathrm{Re}\ b_k\gamma_k)}.$$

X Return to $\mathbb{T}$

- A distance $d(\cdot,\cdot)$ on the unit disk :

$$ds^2 = d\theta^2 + \frac{dr^2}{\sqrt{1-r}}, \quad z = re^{2\pi i\theta}.$$

$$d(z_1,z_2)^2 \asymp |z_1 - z_2|^2 \left(1 + \frac{\cos^2(\phi-\psi)}{\sqrt{2-|z_1+z_2|}}\right)$$

$\phi = \arg(z_1+z_2), \quad \psi = \arg(z_1 - z_2).$

- Conjecture 2 becomes

$$\sum d(a_n,b_n)^2 = \infty \Rightarrow \mu_a \perp \mu_b,$$

$$\sum d(a_n,b_n)^2 < \infty \Rightarrow \mu_a \ll \mu_b.$$