Local homogeneity and dimensions of measures

Antti Käenmäki

Department of Mathematics and Statistics
University of Jyväskylä
Finland

Warwick, 21st April 2011

Local homogeneity and dimensions of measures

Antti Käenmäki

Department of Mathematics and Statistics
University of Jyväskylä
Finland

Warwick, 21st April 2011

This talk mainly exhibits a recent work with
Tapio Rajala (Pisa) and Ville Suomala (Jyväskylä).

Local homogeneity and dimensions of measures

1. **Local homogeneity**

2. **Upper conical density results**

3. **Dimension estimates for porous measures**
   - Large porosity
   - Small porosity

Dimension of a set

Let X be a metric space, $A \subset X$, $x \in A$, $0 < \delta < 1$, and $r > 0$.

Question

How many balls of radius δr are needed to cover $A \cap B(x, r)$?

Dimension of a set

Let X be a metric space, $A \subset X$, $x \in A$, $0 < \delta < 1$, and $r > 0$.

Question

How many balls of radius δr are needed to cover $A \cap B(x, r)$?

If $\dim_{\mathrm{A}}(A) < s$, where $\dim_{\mathrm{A}}$ is the Assouad dimension, then the number of such balls is at most a constant times δ^{-s}.

Dimension of a set

Let X be a metric space, $A \subset X$, $x \in A$, $0 < \delta < 1$, and $r > 0$.

Question

How many balls of radius δr are needed to cover $A \cap B(x, r)$?

If $\dim_{\mathrm{A}}(A) < s$, where $\dim_{\mathrm{A}}$ is the Assouad dimension, then the number of such balls is at most a constant times δ^{-s}.

Can a similar idea be used to define a dimension for measures?

Dimension of a set

Let X be a metric space, $A \subset X$, $x \in A$, $0 < \delta < 1$, and $r > 0$.

Question

How many balls of radius δr are needed to cover $A \cap B(x,r)$?

If $\dim_A(A) < s$, where $\dim_A$ is the Assouad dimension, then the number of such balls is at most a constant times δ^{-s}.

Can a similar idea be used to define a dimension for measures?

Let μ be a measure on a doubling metric space X, $x \in X$, and $\delta, \varepsilon, r > 0$. Define

$$\hom_{\delta,\varepsilon,r}(\mu,x) = \{\#\mathcal{B} : \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x,r) \text{ so that } \mu(B) > \varepsilon\mu\big(B(x,r)\big) \text{ for all } B \in \mathcal{B}\}$$

Local homogeneity of a measure

and let the *local homogeneity of* μ *at* x be

$$\hom_\delta(\mu, x) = \lim_{\varepsilon \downarrow 0} \limsup_{r \downarrow 0} \hom_{\delta,\varepsilon,r}(\mu, x).$$

Local homogeneity of a measure

and let the *local homogeneity of μ at x* be

$$\text{hom}_\delta(\mu, x) = \lim_{\varepsilon \downarrow 0} \limsup_{r \downarrow 0} \text{hom}_{\delta,\varepsilon,r}(\mu, x).$$

The *local homogeneity dimension of μ at x* is

$$\dim_{\text{hom}}(\mu, x) = \liminf_{\delta \downarrow 0} \frac{\log^+ \text{hom}_\delta(\mu, x)}{-\log \delta}.$$

Local homogeneity of a measure

and let the *local homogeneity of μ at x* be

$$\hom_\delta(\mu, x) = \lim_{\varepsilon \downarrow 0} \limsup_{r \downarrow 0} \hom_{\delta,\varepsilon,r}(\mu, x).$$

The *local homogeneity dimension of μ at x* is

$$\dim_{\hom}(\mu, x) = \liminf_{\delta \downarrow 0} \frac{\log^+ \hom_\delta(\mu, x)}{-\log \delta}.$$

Roughly speaking, the local homogeneity dimension $\dim_{\hom}(\mu, x)$ is the least possible exponent s so that large parts of $B(x, r)$ in terms of μ can always be covered by δ^{-s} balls of radius δr for all small $\delta, r > 0$.

Properties of homogeneity dimension

Remark

If μ satisfies the density point property, then, for every μ-measurable $A \subset X$, we have

$$\dim_{\mathrm{hom}}(\mu|_A, x) = \dim_{\mathrm{hom}}(\mu, x)$$

for μ-almost all $x \in A$.

Properties of homogeneity dimension

Remark

If μ satisfies the density point property, then, for every μ-measurable $A \subset X$, we have

$$\dim_{\text{hom}}(\mu|_A, x) = \dim_{\text{hom}}(\mu, x)$$

for μ-almost all $x \in A$.

Remark

If X is also complete, then the function $x \mapsto \dim_{\text{hom}}(\mu, x)$ is Borel.

Properties of homogeneity dimension

Theorem (Rajala & Suomala & K. preprint)

If μ is a measure on a doubling metric space X, then

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq \dim_{\text{hom}}(\mu, x)$$

for μ-almost all $x \in X$.

Properties of homogeneity dimension

Theorem (Rajala & Suomala & K. preprint)

If μ is a measure on a doubling metric space X, then

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le \dim_{\mathrm{hom}}(\mu, x)$$

for μ-almost all $x \in X$.

Remark

If μ is s-regular measure, then

$$\dim_{\mathrm{hom}}(\mu, x) = \dim_{\mathrm{loc}}(\mu, x) = s$$

for μ-almost all $x \in X$.

Quantitative version of the result

The previous dimension result is obtained as a corollary to the following more quantitative result.

Quantitative version of the result

The previous dimension result is obtained as a corollary to the following more quantitative result.

Theorem (Rajala & Suomala & K. preprint)

Suppose X is a doubling metric space with a doubling constant N. If $0 < m < s$, then there exists a constant $\delta_0 = \delta_0(m, s, N) > 0$ such that for every $0 < \delta < \delta_0$ there is $\varepsilon_0 = \varepsilon_0(m, s, N, \delta) > 0$ so that for every measure μ on X we have

$$\limsup_{r \downarrow 0} \hom_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m}$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for μ-almost all $x \in X$ that satisfy $\overline{\dim}_{\text{loc}}(\mu, x) > s$.

Quantitative version of the result

The previous dimension result is obtained as a corollary to the following more quantitative result.

Theorem (Rajala & Suomala & K. preprint)

Suppose X is a doubling metric space with a doubling constant N. If $0 < m < s$, then there exists a constant $\delta_0 = \delta_0(m, s, N) > 0$ such that for every $0 < \delta < \delta_0$ there is $\varepsilon_0 = \varepsilon_0(m, s, N, \delta) > 0$ so that for every measure μ on X we have

$$\limsup_{r \downarrow 0} \hom_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m}$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for μ-almost all $x \in X$ that satisfy $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$.

This version is crucial in our applications.

Questions concerning loal homogeneity

Question

Does there exist any kind of set dimension related to $\dim_{\mathrm{hom}}(\mu, x)$ in a similar manner than e.g. $\dim_{\mathrm{p}}(\mu)$ is related to $\dim_{\mathrm{p}}(A)$?

Questions concerning loal homogeneity

Question

Does there exist any kind of set dimension related to $\dim_{\mathrm{hom}}(\mu, x)$ in a similar manner than e.g. $\dim_{\mathrm{p}}(\mu)$ is related to $\dim_{\mathrm{p}}(A)$?

Question

Is

$$\dim_{\mathrm{hom}}(\mu, x) = \lim_{\delta \downarrow 0} \frac{\log^{+} \mathrm{hom}_{\delta}(\mu, x)}{-\log \delta}$$

for μ-almost all x?

Questions concerning loal homogeneity

Question

Does there exist any kind of set dimension related to $\dim_{\mathrm{hom}}(\mu, x)$ in a similar manner than e.g. $\dim_{\mathrm{p}}(\mu)$ is related to $\dim_{\mathrm{p}}(A)$?

Question

Is

$$\dim_{\mathrm{hom}}(\mu, x) = \lim_{\delta \downarrow 0} \frac{\log^{+} \mathrm{hom}_{\delta}(\mu, x)}{-\log \delta}$$

for μ-almost all x?

Question

Is $x \mapsto \dim_{\mathrm{hom}}(\mu, x)$ a Borel function also in non-complete spaces?

1. **Local homogeneity**

2. **Upper conical density results**

3. **Dimension estimates for porous measures**
   - Large porosity
   - Small porosity

Conical densities

- Conical density theorems are used to derive geometric information from given metric information.

Conical densities

- Conical density theorems are used to derive geometric information from given metric information.
- The main applications deal with rectifiability and porosity.

Conical densities

- Conical density theorems are used to derive geometric information from given metric information.

- The main applications deal with rectifiability and porosity.

- The study of conical densities go back to Besicovitch (1938), Morse and Randolph (1944), Marstrand (1954), Federer (1969), Salli (1985), and Mattila (1988).

Conical densities

- Conical density theorems are used to derive geometric information from given metric information.

- The main applications deal with rectifiability and porosity.

- The study of conical densities go back to Besicovitch (1938), Morse and Randolph (1944), Marstrand (1954), Federer (1969), Salli (1985), and Mattila (1988).

- Recent work include Suomala and K. (2008), Csörnyei, Rajala, Suomala, and K. (2010), Suomala and K. (2011), and Rajala, Suomala, and K. (preprint).

Upper density result for Hausdorff measures

Theorem (Besicovitch 1938 and Marstrand 1954)

Suppose $0 \le s \le n$ and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$. Then

$$2^{-s} \le \limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{(2r)^s} \le 1$$

for $\mathcal{H}^s$-almost all $x \in A$.

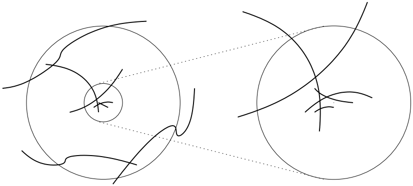
Upper density result for Hausdorff measures

Theorem (Besicovitch 1938 and Marstrand 1954)

Suppose $0 \le s \le n$ and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$. Then

$$2^{-s} \le \limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{(2r)^s} \le 1$$

for $\mathcal{H}^s$-almost all $x \in A$.



There are arbitrary small scales having a lot of A.

Upper density result for general measures

Theorem (Rajala & Suomala & K. preprint)

If μ is a Radon measure on a doubling metric space X and $A \subset X$ is μ-measurable, then

$$\limsup_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1$$

for μ-almost all $x \in A$.

Upper density result for general measures

Theorem (Rajala & Suomala & K. preprint)

If μ is a Radon measure on a doubling metric space X and $A \subset X$ is μ-measurable, then

$$\limsup_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1$$

for μ-almost all $x \in A$.

Again, there are arbitrary small scales having a lot of mass.

We remark that in $\mathbb{R}^n$, the limit above exists for all Radon measures.

It may happen that $\liminf_{r \downarrow 0} \mu\big(A \cap B(x,r)\big)/\mu\big(B(x,r)\big) = 0$.

Upper conical densities

- Note that most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls.

Upper conical densities

- Note that most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls.

- If we know that the measure is “scattered enough”, can we say how the measure is distributed on those scales where we have a lot of mass?

Upper conical densities

- Note that most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls.
- If we know that the measure is "scattered enough", can we say how the measure is distributed on those scales where we have a lot of mass?
- If the measure is purely unrectifiable and doubling, then the answer is yes. An example of Csörnyei, Rajala, Suomala, and K. (2010) shows that it is really needed that the measure is doubling.

Upper conical densities

- Note that most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls.

- If we know that the measure is "scattered enough", can we say how the measure is distributed on those scales where we have a lot of mass?

- If the measure is purely unrectifiable and doubling, then the answer is yes. An example of Csörnyei, Rajala, Suomala, and K. (2010) shows that it is really needed that the measure is doubling.

- Another possibility is to assume that the dimension of the measure is large enough.

Definition of nonsymmetric cones

Let $G(n, n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$ and set $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Definition of nonsymmetric cones

Let $G(n, n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$ and set $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For $x \in \mathbb{R}^n$, $r > 0$, $V \in G(n, n-m)$, $\theta \in S^{n-1}$, and $0 < \alpha \leq 1$ define

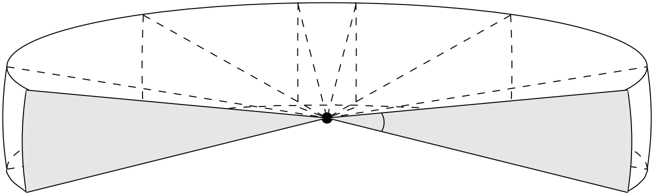
$$X(x, r, V, \alpha) = \{y \in B(x, r) : \operatorname{dist}(y - x, V) < \alpha|y - x|\},$$
$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \alpha|y - x|\}.$$

Definition of nonsymmetric cones

Let $G(n, n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$ and set $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For $x \in \mathbb{R}^n$, $r > 0$, $V \in G(n, n-m)$, $\theta \in S^{n-1}$, and $0 < \alpha \le 1$ define

$$X(x, r, V, \alpha) = \{y \in B(x, r) : \mathrm{dist}(y - x, V) < \alpha|y - x|\},$$
$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \alpha|y - x|\}.$$



The set $X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)$ when $n = 3$ and $m = 1$.

Upper conical density result for packing measures

Theorem (Suomala & K. 2008)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n, m, s, \alpha) > 0$ so that for every $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{P}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\alpha)\big)}{(2r)^s} \ge c$$

for $\mathcal{P}^s$-almost all $x \in A$.

Upper conical density result for packing measures

Theorem (Suomala & K. 2008)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n,m,s,\alpha) > 0$ so that for every $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{P}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\alpha)\big)}{(2r)^s} \ge c$$

for $\mathcal{P}^s$-almost all $x \in A$.

To our knowledge, this is the first upper conical density result for other measures than the Hausdorff measures.

Upper conical density result for general measures

Theorem (Rajala & Suomala & K. preprint)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n, m, s, \alpha) > 0$ so that for every Radon measure μ on $\mathbb{R}^n$ with $\underline{\dim}_{\mathrm{p}}(\mu) \ge s$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{\mu\big(B(x, r)\big)} > c$$

for μ-almost all $x \in \mathbb{R}^n$.

Upper conical density result for general measures

Theorem (Rajala & Suomala & K. preprint)

Suppose $0 \leq m < s \leq n$ and $0 < \alpha \leq 1$. Then there exists $c = c(n, m, s, \alpha) > 0$ so that for every Radon measure μ on $\mathbb{R}^n$ with $\underline{\dim}_{\mathrm{p}}(\mu) \geq s$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{\mu\big(B(x, r)\big)} > c$$

for μ-almost all $x \in \mathbb{R}^n$.

Assuming $\underline{\dim}_{\mathrm{H}}(\mu) \geq s$ instead of $\underline{\dim}_{\mathrm{p}}(\mu) \geq s$, the result was proved by Csörnyei, Rajala, Suomala, and K. (2010).

Upper conical density result for general measures

Theorem (Rajala & Suomala & K. preprint)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n, m, s, \alpha) > 0$ so that for every Radon measure μ on $\mathbb{R}^n$ with $\underline{\dim}_{\mathrm{p}}(\mu) \ge s$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{\mu\big(B(x, r)\big)} > c$$

for μ-almost all $x \in \mathbb{R}^n$.

Assuming $\underline{\dim}_{\mathrm{H}}(\mu) \ge s$ instead of $\underline{\dim}_{\mathrm{p}}(\mu) \ge s$, the result was proved by Csörnyei, Rajala, Suomala, and K. (2010).

The proof of this result uses a local homogeneity estimate.

A question concerning conical densities

Question

Does the upper conical density result hold in non-Euclidean spaces that have enough geometry (e.g. in the Heisenberg group)?

1. **Local homogeneity**

2. **Upper conical density results**

3. **Dimension estimates for porous measures**
   - Large porosity
   - Small porosity

Porosity

- If a set contains a lot of holes, then it should be small. Porosity is a quantity that measures the size and abundance of holes.

Porosity

- If a set contains a lot of holes, then it should be small. Porosity is a quantity that measures the size and abundance of holes.

- Porosity was introduced by Denjoy (1920). His definition is nowadays called *upper porosity*. Although upper porosity is useful in many connections, one cannot get nontrivial dimension estimates for upper porous sets.

Porosity

- If a set contains a lot of holes, then it should be small. Porosity is a quantity that measures the size and abundance of holes.
- Porosity was introduced by Denjoy (1920). His definition is nowadays called *upper porosity*. Although upper porosity is useful in many connections, one cannot get nontrivial dimension estimates for upper porous sets.
- Dimension estimates obtained from *lower porosity* were used by Sarvas (1975), Trocenko (1981), and Väisälä (1987) in connection with the boundary behavior of quasiconformal mappings.

Porosity

- If a set contains a lot of holes, then it should be small. Porosity is a quantity that measures the size and abundance of holes.

- Porosity was introduced by Denjoy (1920). His definition is nowadays called *upper porosity*. Although upper porosity is useful in many connections, one cannot get nontrivial dimension estimates for upper porous sets.

- Dimension estimates obtained from *lower porosity* were used by Sarvas (1975), Trocenko (1981), and Väisälä (1987) in connection with the boundary behavior of quasiconformal mappings.

- In lower porosity we have holes on all scales whereas in upper porosity we just know that there are arbitrary small scales having holes.

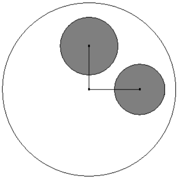
Porosity of sets

Let $A \subset \mathbb{R}^n$, $k \in \{1, \ldots, d\}$, $x \in A$, and $r > 0$. We define

$$\begin{aligned} \mathrm{por}_k(A, x, r) = \sup\{\varrho \geq 0 : &\text{ there are } y_1, \ldots, y_k \in \mathbb{R}^n \text{ such that} \\ &B(y_i, \varrho r) \subset B(x, r) \setminus A \text{ for every } i \\ &\text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } i \neq j\} \end{aligned}$$

and from this the (lower) *k-porosity of A at x* as

$$\mathrm{por}_k(A, x) = \liminf_{r \downarrow 0} \mathrm{por}_k(A, x, r).$$



Recent results

- For recent results on the dimension of porous sets, see Järvenpää, Järvenpää, Suomala, and K. (2005), Rajala (2009), Chousionis (2009), Järvenpää, Järvenpää, Rajala, Rogovin, Suomala, K. (2010), and Suomala and K. (2011).

Recent results

- For recent results on the dimension of porous sets, see Järvenpää, Järvenpää, Suomala, and K. (2005), Rajala (2009), Chousionis (2009), Järvenpää, Järvenpää, Rajala, Rogovin, Suomala, K. (2010), and Suomala and K. (2011).

- We also define porosity for measures. In applications, it is more convenient to consider measures instead of sets.

Recent results

- For recent results on the dimension of porous sets, see Järvenpää, Järvenpää, Suomala, and K. (2005), Rajala (2009), Chousionis (2009), Järvenpää, Järvenpää, Rajala, Rogovin, Suomala, K. (2010), and Suomala and K. (2011).

- We also define porosity for measures. In applications, it is more convenient to consider measures instead of sets.

- For recent results concerning porous measures, see Suomala and K. (2008), Beliaev, Järvenpää, Järvenpää, Rajala, Smirnov, Suomala, K. (2009), Rajala, Suomala, K. (preprint), and Shmerkin (preprint).

Porosity of measures

Let μ be a Radon measure on $\mathbb{R}^n$, $k \in \{1, \ldots, d\}$, $x \in \mathbb{R}^n$, $r > 0$, and $\varepsilon > 0$. We set

$$\begin{aligned}\operatorname{por}_k(\mu, x, r, \varepsilon) = \sup\{\varrho \geq 0 : {} & \text{there are } y_1, \ldots, y_k \in \mathbb{R}^n \setminus \{x\} \text{ such} \\ & \text{that } B(y_i, \varrho r) \subset B(x, r) \text{ and} \\ & \mu\big(B(y_i, \varrho r)\big) < \varepsilon \mu\big(B(x, r)\big) \text{ for every } i \\ & \text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } i \neq j\}\end{aligned}$$

and the *k-porosity of the measure μ at x* is defined to be

$$\operatorname{por}_k(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \operatorname{por}_k(\mu, x, r, \varepsilon).$$

Porosity of measures

Let μ be a Radon measure on $\mathbb{R}^n$, $k \in \{1, \ldots, d\}$, $x \in \mathbb{R}^n$, $r > 0$, and $\varepsilon > 0$. We set

$$\begin{aligned}\mathrm{por}_k(\mu, x, r, \varepsilon) = \sup\{\varrho \geq 0 :\ & \text{there are } y_1, \ldots, y_k \in \mathbb{R}^n \setminus \{x\} \text{ such} \\ & \text{that } B(y_i, \varrho r) \subset B(x, r) \text{ and} \\ & \mu\big(B(y_i, \varrho r)\big) < \varepsilon\mu\big(B(x, r)\big) \text{ for every } i \\ & \text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } i \neq j\}\end{aligned}$$

and the *k-porosity of the measure μ at x* is defined to be

$$\mathrm{por}_k(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \mathrm{por}_k(\mu, x, r, \varepsilon).$$

An example of Smirnov et al. (2009) shows that even if $\mathrm{por}_1(\mu, x) > 0$ in a set of positive μ-measure, it is possible that $\mu(A) = 0$ for all $A \subset \mathbb{R}^n$ with $\inf_{x \in A} \mathrm{por}_1(A, x) > 0$.

Dimension estimate (when the porosity is large)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on $\mathbb{R}^n$ we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le n - k + \frac{c}{-\log\big(1 - 2\,\mathrm{por}_k(\mu, x)\big)}$$

for μ-almost all $x \in \mathbb{R}^n$.

Dimension estimate (when the porosity is large)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on $\mathbb{R}^n$ we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le n - k + \frac{c}{-\log\big(1 - 2\,\mathrm{por}_k(\mu, x)\big)}$$

for μ-almost all $x \in \mathbb{R}^n$.

The result is asymptotically sharp as $\mathrm{por}_k(\mu, x) \uparrow \frac{1}{2}$.

Dimension estimate (when the porosity is large)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on $\mathbb{R}^n$ we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq n - k + \frac{c}{-\log\bigl(1 - 2\,\mathrm{por}_k(\mu, x)\bigr)}$$

for μ-almost all $x \in \mathbb{R}^n$.

The result is asymptotically sharp as $\mathrm{por}_k(\mu, x) \uparrow \frac{1}{2}$.

For $k = 1$, the result was proved in Smirnov et al. (2009). The method used there does not work in the general case.

Dimension estimate (when the porosity is large)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on $\mathbb{R}^n$ we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq n - k + \frac{c}{-\log\bigl(1 - 2\,\mathrm{por}_k(\mu, x)\bigr)}$$

for μ-almost all $x \in \mathbb{R}^n$.

The result is asymptotically sharp as $\mathrm{por}_k(\mu, x) \uparrow \frac{1}{2}$.

For $k = 1$, the result was proved in Smirnov et al. (2009). The method used there does not work in the general case.

The proof of this result uses a local homogeneity estimate.

Dimension estimate (when the porosity is small)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on an s-regular metric space X satisfying the density point property we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s - \mathrm{por}_1(\mu, x)^s$$

for μ-almost all $x \in X$.

Dimension estimate (when the porosity is small)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on an s-regular metric space X satisfying the density point property we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s - \mathrm{por}_1(\mu, x)^s$$

for μ-almost all $x \in X$.

The proof of this result uses a local homogeneity estimate.

Observe that this is an application of the local homogeneity in metric spaces.

Questions concerning porosity

Question

Is the density point property needed in the small porosity dimension estimate?

Questions concerning porosity

Question

Is the density point property needed in the small porosity dimension estimate?

Question

Does the large porosity dimension estimate hold in non-Euclidean spaces that have enough geometry (e.g. in the Heisenberg group)?

Questions concerning porosity

Question

Is the density point property needed in the small porosity dimension estimate?

Question

Does the large porosity dimension estimate hold in non-Euclidean spaces that have enough geometry (e.g. in the Heisenberg group)?

Question

Do the previous theorems have counterparts for mean porous measures?

Thank you!

http://users.jyu.fi/~antakae/