The dimension of projections and convolutions, and a variant of Marstrand's Projection Theorem

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Summary

- I will quickly describe progress obtained in the last few years on the projections and convolutions of dynamically defined measures.
- The new results I want to emphasize are work (in progress) joint with/done by J.Erick López Velázquez and C."Gugu" Moreira.
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The projection theorem

Notation. $G(n,k)$ denotes the Grassmanian of k-planes in $\mathbb{R}^n$. We identify $V \in G(n,k)$ with the orthogonal projection onto V, and also with any linear map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ with kernel $V^\perp$.

Fantastic Theorem (Marstrand/Kaufman/Mattila)

Let E be a Borel set on $\mathbb{R}^n$, and let $1 \leq k < n$. Then:

- *If $\dim_H(E) > k$, then $\pi(E)$ has positive Lebesgue measure for almost every $\pi \in G(n,k)$.*
- *If $\dim_H(E) \leq k$, then $\pi(E)$ has Hausdorff dimension $\dim_H(E)$ for almost every $\pi \in G(n,k)$.*

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Remarks on the Projection Theorem

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- *One always has* $\dim_H(\pi(E)) \leq \min(\dim_H(E), k)$. *We call projections for which inequality occurs exceptional.*
- *The proofs are very non-constructive; they give no hint of how to find the exceptional set (which may be large in terms of topology and dimension).*
- *The dependence* $\pi \to \dim_H(\pi(E))$ *is in general ugly.*
- *Note the parameter space* $G(n, k)$ *has dimension* $k(n-k)$.
- *An analogous result holds for measures (for various notions of dimensions, such as correlation, Hausdorff dimension and exact dimension).*

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Projections of special sets and measures?

Question

For sets and measures with an arithmetic and/or dynamic origin, can one identify the precise set of exceptions in the Projection Theorem?

- *For example, if A, B are two dynamically defined sets, one is often interested in the dimension of the arithmetic sum $A + B$. This is one specific projection from the product, so a generic result is useless (well, not quite as we shall see).*
- *Furstenberg posed a number of conjectures of the following type: "For objects of dynamical origin, there are no exceptions to the projection theorem other than the evident ones".*

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- *In general, the answer is no, in the sense that not every incomplete family of projections will work.*
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The punchline

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Sometimes, answering the second question is key to answering the first.

For example, using a variant of Marstrand's Projection Theorem for a (small) class of linear projections, we are able to prove the following:

Theorem (J.E.López Velázquez, C. "Gugu" Moreira, P.S. 2012)

For $a \in (0,1)$ *let* C_a *be the middle-*$(1-2a)$ *Cantor set. If* $\log(a_1), \ldots, \log(a_n), 1$ *are rationally independent and* $\pi : \mathbb{R}^n \to \mathbb{R}^k$ *is a "transverse" linear map,*

$$\dim_H(\pi(C_{a_1} \times \cdots \times C_{a_n})) = \min\left(\sum_{i=1}^{n} \dim_H(C_{a_i}), k\right).$$

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Historical summary

Cases in which it was proved that all projections preserve dimension (other than trivial exceptions):

- Gugu Moreira (199?, unpublished): products of regular Cantor sets, one of them nonlinear ($\mathbb{R}^2 \to \mathbb{R}$)
- Y. Peres - P.S (2009, ETDS): products of self-similar sets ($\mathbb{R}^n \to \mathbb{R}$), planar self-similar sets with rotations ($\mathbb{R}^2 \to \mathbb{R}$).
- F. Nazarov, Y. Peres, P.S. (2011, Israel J.): products of measures on central Cantor sets ($\mathbb{R}^n \to \mathbb{R}$).
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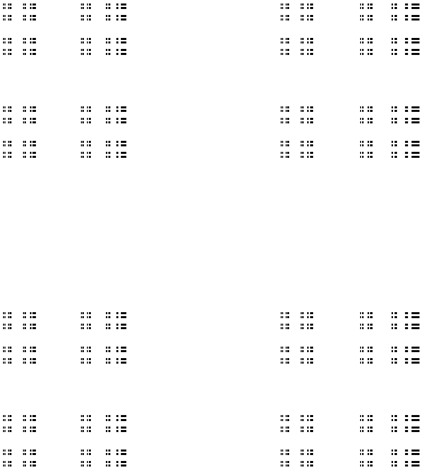
I had to include one picture in this talk



A general framework, main application

Together with M. Hochman, we developed a unified framework that allows to recover, unify and substantially extend most of the previous results. Our main motivation was to resolve a conjecture of Furstenberg in full:

Theorem (M. Hochman and P.S., accepted in Ann. of math)

Let $A, B \subset [0,1]$ be closed sets, invariant under $x \to 2x \bmod (1)$ and $x \to 3x \bmod 1$ respectively. Then

$$\dim_H(A+B) = \min(\dim_H(A) + \dim_H(B), 1).$$

In fact the analogous result for measures also holds.

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A general framework, main idea

Although the main result of our paper is very technical, the main idea is the following:

Main Idea

If μ is a measure on $\mathbb{R}^n$ which displays a local form of statistical self-similarity then the map $\pi \to \dim(\pi\mu)$ is essentially lower semicontinuous.

Disclaimers

- We do not prove such a thing for any measure. What we really prove is that dim($\pi\mu$) is bounded below by a lower semicontinuous function that reflects the projection behavior of measures one sees when “zooming in” towards typical points of μ.
- Semicontinuity turned out to be less important than initially thought (one can obtain most of the results without going through it).
- Nevertheless, it is very convenient as a first approximation to assume semicontinuity.

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Dynamics on fractals

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An illustration: sums $C_a + C_b$

- Let $d = \min(\dim_H(C_a) + \dim_H(C_b), 1)$.
- The general semicontinuity framework "implies that" $t \to \dim_H(C_a + tC_b)$ is lower semicontinuous.
- Fix $\varepsilon > 0$ and let us look at the set

$$B_\varepsilon = \{t : \dim_H(C_a + tC_b) > d - \varepsilon\}.$$

- By semicontinuity and the Projection Theorem (black box), B_ε has nonempty interior.
- By self-similarity of C_a and C_b, B_ε is invariant under multiplication by b and by $1/a$.
- If $\log b / \log a \notin \mathbb{Q}$, we conclude that $B_\varepsilon = \mathbb{R} \setminus \{0\}$.

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Projections of $C_{a_1} \times \cdots \times C_{a_n}$

- Write $E = C_{a_1} \times \cdots \times C_{a_n}$, $d = \min(\dim_H(E), k)$. Let us try to understand why the previous argument does not work for projections onto $\mathbb{R}^k$, $k \geq 2$.
- We can define, as before,

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  Just as before, we know B_ε has nonempty interior.
- Self-similarity (and irrationality) tells us that B_ε is invariant under postcomposition with a dense set of diagonal matrices.
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- Unfortunately, the action of the diagonal group on $G(n,k)$ is not minimal!!! (e.g. for dimension reasons). So we cannot cover all of $G(n,k)$ in this way.

Projections of $C_{a_1} \times \cdots \times C_{a_n}$

- Write $E = C_{a_1} \times \cdots \times C_{a_n}$, $d = \min(\dim_H(E), k)$. Let us try to understand why the previous argument does not work for projections onto $\mathbb{R}^k$, $k \geq 2$.
- We can define, as before,

$$B_\varepsilon = \{\pi \in G(n,k) : \dim_H(\pi(E)) < d - \varepsilon\}.$$

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How to fix the argument

Main Idea

If we knew that the Projection Theorem holds for the family of linear maps $\{\pi \circ D : D \text{ is a diagonal matrix}\}$*, where* $\pi : \mathbb{R}^n \to \mathbb{R}^k$ *is a fixed projection, then the argument would be fixed, as the action of the diagonal group is, by definition, transitive in this family, and semicontinuity still holds.*

If it only was so simple

Remark

It is easy to see one cannot expect such a result for all maps π. For example, let A_1, A_2, B_1, B_2 be sets of equal Hausdorff and box dimension (so that the dimension of products is the sum of the dimensions).

Suppose $\dim_H(A_1) = \dim_H(A_2) = 0.6$ *so* $\dim_H(t_1 A_1 \times t_2 A_2) = 1.2 > 1$, *and* $\dim_H(B_1) = \dim_H(B_2) = 0.2$. *Let* $\pi(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 + x_4)$.

We have $\dim_H(E) = 1.6$, *but for any diagonal map D on* $\mathbb{R}^n$, $\dim_H(\pi E) \leq 1 + 2 \times 0.2 = 1.4$.

So the "expected" dimension of $\pi D(E)$ *depends on the geometry of* π *and may be smaller than* $\min(\dim_H(E), k)$.

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The modified Projection Theorem

Theorem (Erick L.V. and Gugu M. 2012)

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be compact sets such that

$$\dim_H(A_1 \times \cdots \times A_n) = \sum_{i=1}^{n} \dim_H(A_i).$$

Denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$, and define

$$\mathbf{m} = \min_{I \subset \{1,\ldots,n\}} \left(\sum_{i \in I} \dim_H(A_i) + \dim(\pi(< e_i : i \in I^c >)) \right).$$

Then

$$\dim_H(\pi(t_1 A_1 \times \cdots \times t_n A_n)) = \min(k, \mathbf{m})$$

for a.e. $t_1, \ldots, t_n$.

Remarks

- There is an open dense set of "transversal" maps π for which $\mathbf{m} = \dim_H(\otimes_i A_i)$.
- The standard way to prove results of this kind is to use transversality. However transversality does not hold for this family of projections (exercise).
- The main difficulty arises in the case where the map is not transversal. This involves combinatorial/convexity ideas.
- There is a more general version for block-diagonal maps.

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Projections of products

Theorem (J.E.López Velázquez, C.G. Moreira, P.S. 2011)

Let $p_1, \ldots, p_n$ *be integers with* $\{\log p_1, \ldots, \log p_n, 1\}$ *rationally independent.*
Let $A_i \subset [0,1]$ *be invariant under* $x \to p_i x \bmod 1$. *Then for any* $\pi : \mathbb{R}^n \to \mathbb{R}^k$,

$$\dim(\pi(A_1 \times \cdots \times A_n)) = \min(k, \mathbf{m}).$$

Remarks

- *This is a stronger version than previously available of the fact that "expansions in different bases do not resonate geometrically". For example, if* $\lceil \dim(\otimes_i A_i) \rceil = k > 1$, *then the "right" dimension to project is* k.
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Sumsets in higher dimensions

Theorem (J.E.López Velázquez, C.G. Moreira, P.S. 2011)

Let A, B be self-similar sets on $\mathbb{R}^k$. Let $\mathcal{F}, \mathcal{G}$ be the semigroups generated by the maps in each of the corresponding IFS's. Suppose

$$\{FG^{-1} : F \in \mathcal{F}, G \in \mathcal{G}\} \quad \text{is dense in} \quad \mathbb{R} \times O_n.$$

Then $\dim_H(A + B) = \min(\dim_H(A) + \dim_H(B), k)$.

Remarks

- *Again the same result holds for convolutions of self-similar measures.*
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A nonlinear example

Theorem (J.E.López Velázquez, C.G. Moreira, P.S. 2011)

Let J_1, J_2 be hyperbolic Julia sets, at least one of them not linear and not contained in a finite union of real-analytic curves. Then

$$\dim_H(J_1 + J_2) = \min(\dim_H(J_1) + \dim_H(J_2), 2).$$

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Some questions

- is the modified projection theorem valid for all sets rather than just product sets? (for transversal projections, I have a more general condition, but I don't know if it is universal).
- Can one obtain projection theorems for other classes of projections? In particular, consider the case of a "sufficiently rich" subgroup $G \subset O(n)$, and consider the set of projections $\{\pi \circ g : g \in G\}$ for a fixed $\pi : \mathbb{R}^n \to \mathbb{R}^k$. Does the projection theorem hold, at least for a natural class of sets/measures invariant under the action of G?

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Thanks

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