A dimension conservation principle

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Outline

- Introduction
  - Orthogonal projections ν^θ of the natural measure ν of the Sierpinski Carpet
  - Intersection of the Sierpinski carpet with a straight line
  - Rational slopes
- the rational case with detail
- The dimension of ν-typical slices

Outline

Introduction

Orthogonal projections ν^θ of the natural measure ν of the Sierpinski Carpet

Intersection of the Sierpinski carpet with a straight line

Rational slopes

the rational case with detail

The dimension of ν-typical slices

The Sierpinski carpet F is the attractor of the IFS

$$\mathcal{G} := \left\{ g_i(x, y) = \frac{1}{3}(x, y) + \frac{1}{3}\mathbf{t}_i \right\}_{i=1}^{8},$$

where we order the vectors $(u, v) \in \{0, 1, 2\} \times \{0, 1, 2\} \setminus \{(1, 1)\}$ in lexicographic order and write $\mathbf{t}_i$ for the i-th vector, $i = 1, \ldots, 8$.

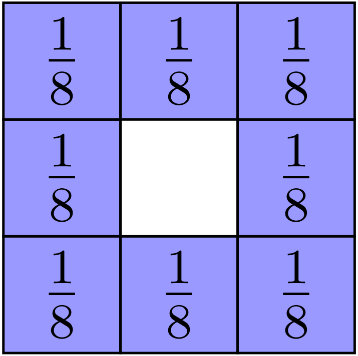


Figure: We call ν the equally distributed "natural" measure on the carpet F

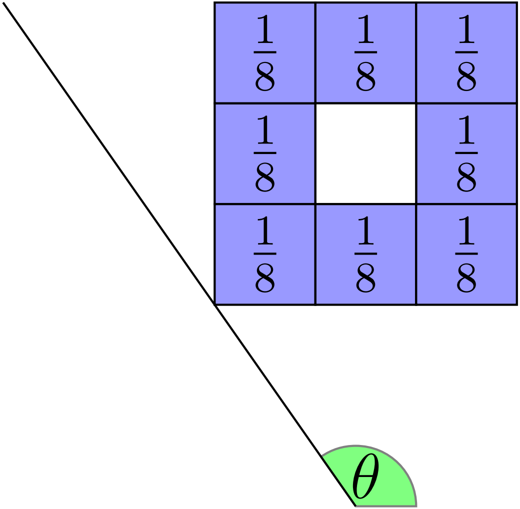
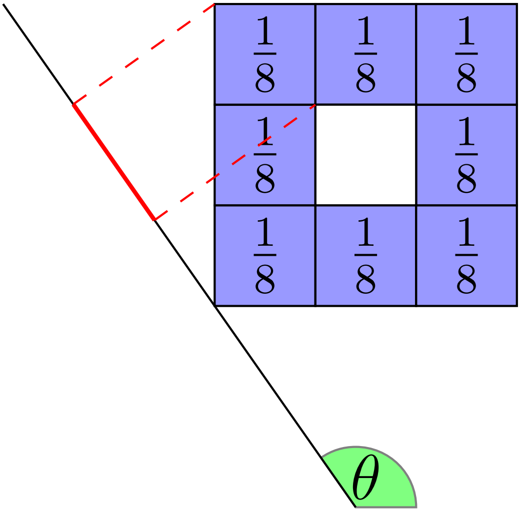


Figure: The θ projection to l_θ and the projected measure ν^θ supported by l_θ



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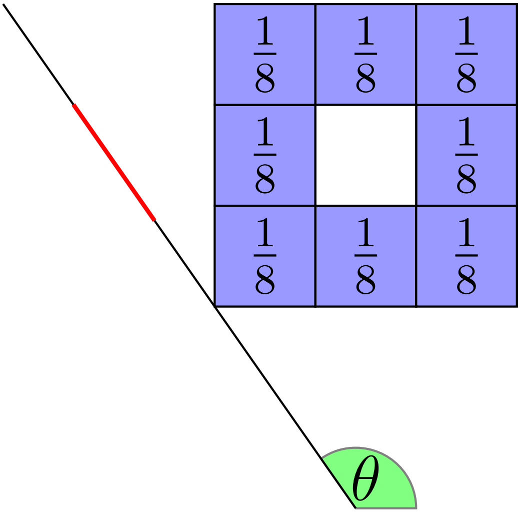

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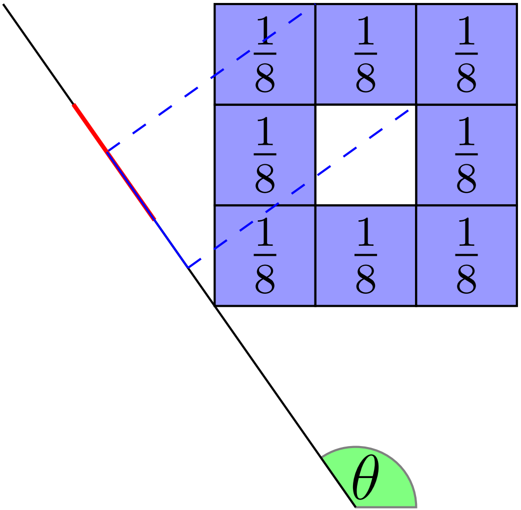

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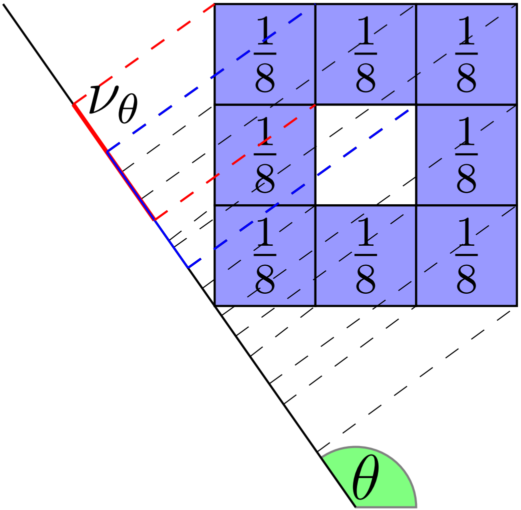

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Let $\Sigma_8 := \{1,\ldots,8\}^{\mathbb{N}}$ Let $\Pi : \Sigma_8 \to F$,

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} g_{i_1\ldots i_n}(0) \text{ and}$$

$$\nu := \Pi_* \mu_8$$

the natural measure on F, where
$\mu_8 := \left\{\frac{1}{8},\ldots,\frac{1}{8}\right\}^{\mathbb{N}}$ is the Bernoulli measure on Σ_8 given by.

$$\nu^\theta := \mathrm{proj}_*^\theta(\nu).$$

Clearly, ν^θ is the invariant measure for the IFS

$$\Phi^\theta := \left\{\varphi_i^\theta(t) = \frac{1}{3}\cdot t + \frac{1}{3}\cdot \mathrm{proj}^\theta(\mathbf{t}_i)\right\}_{i=1}^{8}$$

with equal weights. That is:

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$$\nu^\theta(B) = \sum_{k=1}^{8} \frac{1}{8} \nu^\theta \left(\left(\varphi_k^\theta\right)^{-1} (B) \right).$$

It follows from a theorem due to DJ Feng (2003) that for ν^θ-almost all $a \in I_\theta =$ we have:

$$d(\nu^\theta, a) := \lim_{r \to 0} \frac{\log \nu^\theta[a - r, a + r]}{\log r} = \dim_{\mathrm{H}} \nu^\theta. \tag{1}$$

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Let $E_{\theta,a} := \{(x, y) \in F : y - x \tan\theta = a\}$ be the intersection of the Sierpinski Carpet F with the line of slope θ through $(0, a)$. We shall study the dimension of $E_{\theta,a}$, $a \in [0, 1]$. We pay special attention to the case when $\tan\theta \in \mathbb{Q}$

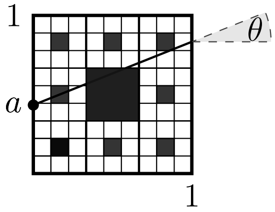

Figure: The intersection of the Sierpinski carpet with the line $y = \frac{2}{5}x + a$ for some $a \in [0, 1]$.

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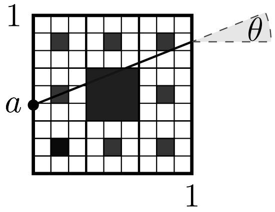

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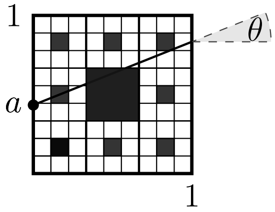

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Recall: F : Sierpinski carpet,

$E_{\theta,a} := \{(x,y) \in F : y - x\tan\theta = a\}$

Theorem (Marstrand)

For all θ, for $\mathcal{L}\mathrm{eb}_1$ almost all a we have

$$\dim_{\mathrm{H}}(E_{\theta,a}) \leq \dim_{\mathrm{H}} F - 1. \tag{2}$$

Theorem (Marstrand)

$\mathcal{L}\mathrm{eb}_2\{(\theta,a) : \dim_{\mathrm{H}}(E_{\theta,a}) = \dim_{\mathrm{H}}(F) - 1\} > 0.$

Actually, for $\mathcal{L}\mathrm{eb}_2$ a.a. (θ,a) if $E_{\theta,a} \neq \emptyset$ then $\dim_{\mathrm{H}}(E_{\theta,a}) = s - 1.$

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Theorem (Liu, Xi and Zhao (2007))

If $\tan(\theta) \in \mathbb{Q}$ *then,*

(a) *for Lebesgue almost a,* $\dim_{\mathrm{H}}(E_{\theta,a}) = \dim_{\mathrm{B}}(E_{\theta,a})$

(b) *The dimension of* $E_{\theta,a}$ *is the same constant for almost all* $a \in [0,1]$.

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Motivation

Conjecture (Liu, Xi and Zhao (2007))

For all θ such that $\tan\theta \in \mathbb{Q}$, *for almost all a we have* $\dim_{\mathrm{H}}(E_{\theta,a}) < \dim_{\mathrm{H}} F - 1$.

For $\tan\theta \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, this Conjecture was verified by Liu, Xi and Zhao.

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Theorem (Dimension conservation, Manning, S.)

$\forall\, \theta \in [0, \pi/2)$ *and* $a \in I^\theta$ *if either of the two limits*

$$\dim_{\mathrm{B}}(E_{\theta,a}) = \lim_{n\to\infty} \frac{\log N_{\theta,a}(n)}{\log 3^n},$$

$$d(\nu^\theta, a) = \lim_{\delta\to 0} \frac{\log(\nu^\theta[a-\delta, a+\delta])}{\log \delta}$$

exists *then the other limit also exists, and, in this case,*

$$\dim_{\mathrm{B}}(E_{\theta,a}) + d(\nu^\theta, a) = s. \qquad (3)$$

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Theorem

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$$\dim_B(E_{\theta,a}) = s - \dim_H(\nu^\theta) \geq s - 1.$$

The assertion includes that the box dimension exists.

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$\tan\theta \in \mathbb{Q}$

Theorem

If $\tan\theta \in \mathbb{Q}$ *then, for Lebesgue almost all* $a \in I^\theta$*, we have*

$$d^\theta(\mathcal{L}\mathrm{eb}) := \dim_{\mathrm{B}}(E_{\theta,a}) = \dim_{\mathrm{H}}(E_{\theta,a}) < \frac{\log 8}{\log 3} - 1.$$

Corollary

If $\tan\theta \in \mathbb{Q}$ *then, for Lebesgue almost all* $a \in I^\theta$*, we have*

$$d(\nu^\theta, a) = \frac{\log 8}{\log 3} - d^\theta(\mathcal{L}\mathrm{eb}) > 1.$$

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Proposition

If $\tan\theta \in \mathbb{Q}$ *then there is a constant* $d^\theta(\nu^\theta)$ *such that for* ν^θ*-almost all* $a \in I^\theta$ *we have*

$$\dim_{\mathrm{H}}(E_{\theta,a}) = \underline{\dim}_{\mathrm{B}}(E_{\theta,a}) = \overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \geq s - 1. \tag{4}$$

The left hand side is ν^θ-almost everywhere constant.

Outline

Thm [MS]: $\tan\theta \in \mathbb{Q} \Longrightarrow \dim_{\mathrm{H}}(E_{\theta,a}) < \dim_{\mathrm{H}} F - 1$ for a.a. a.

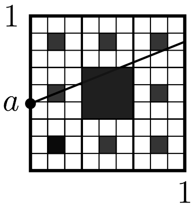
We define three matrices A_0, A_1, A_2 then we consider the Lyapunov exponent of the random matrix product

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} \cdots A_{i_n}\|_1,$$

where $A_{i_k} \in \{A_0, A_1, A_2\}$ chosen independently in every step with probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then we prove that

$$\gamma < \frac{log 8}{\log 3}.$$

$M/N = 2/5$



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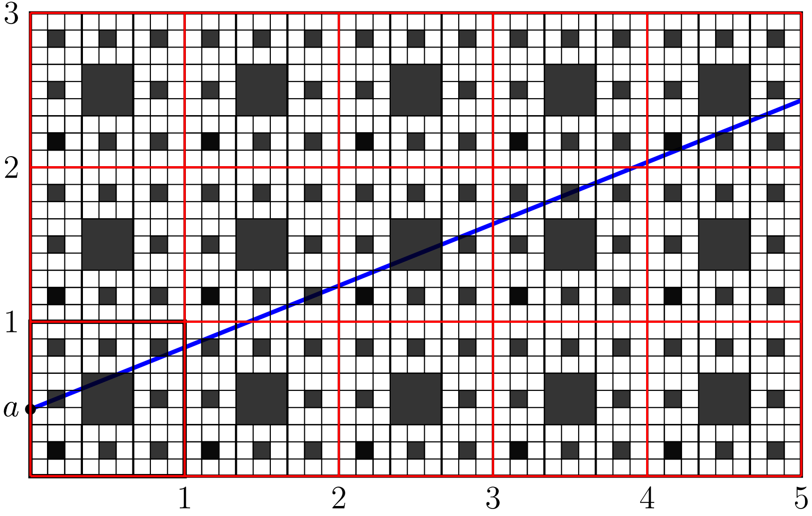

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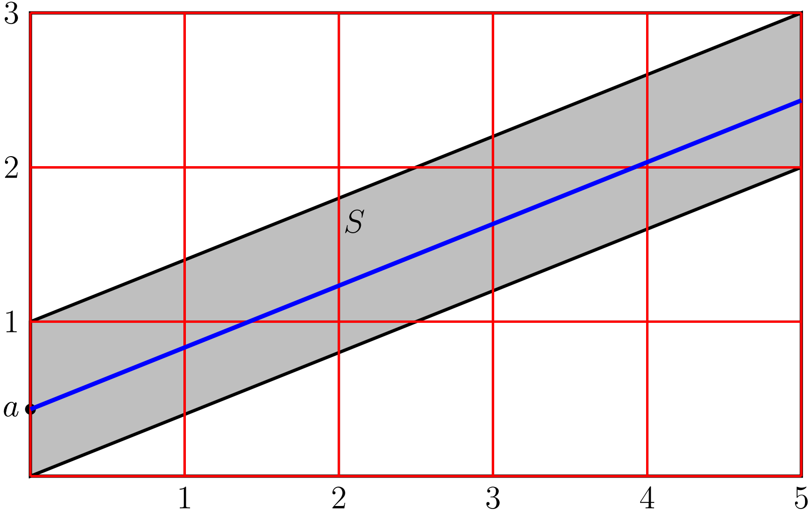

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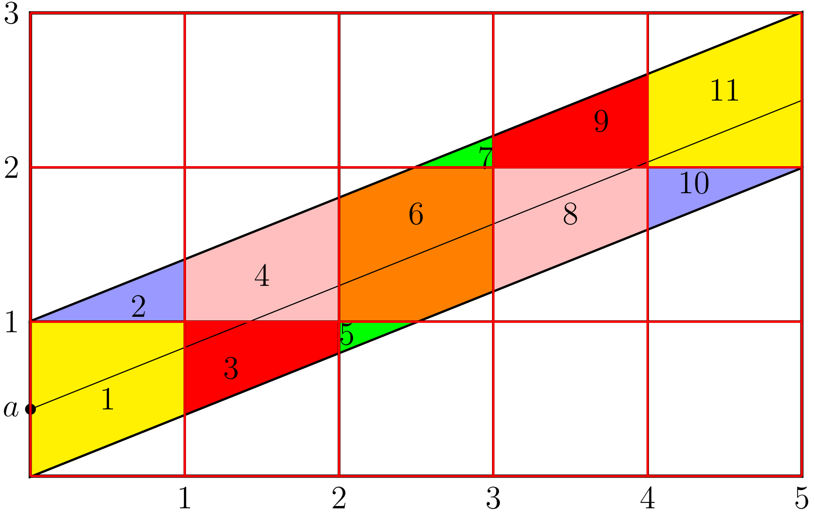

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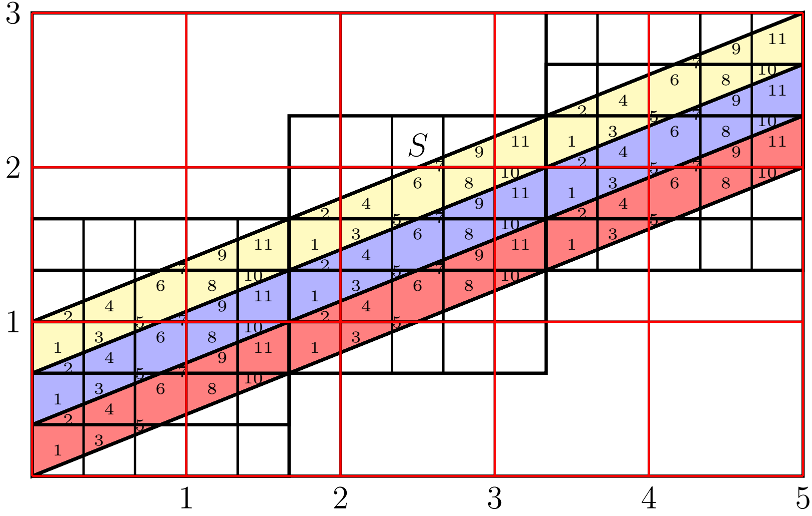

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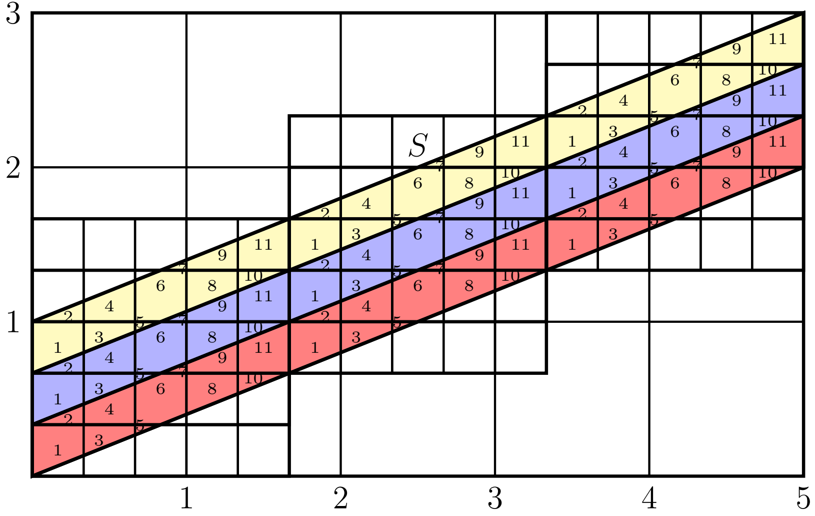

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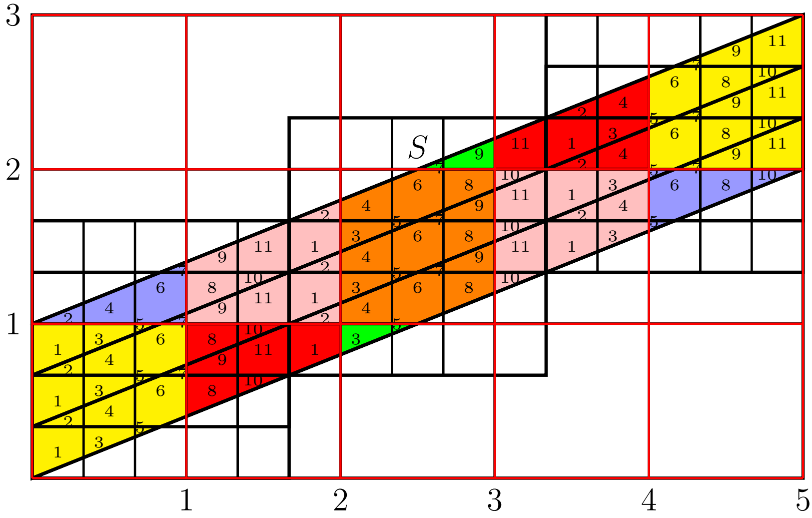

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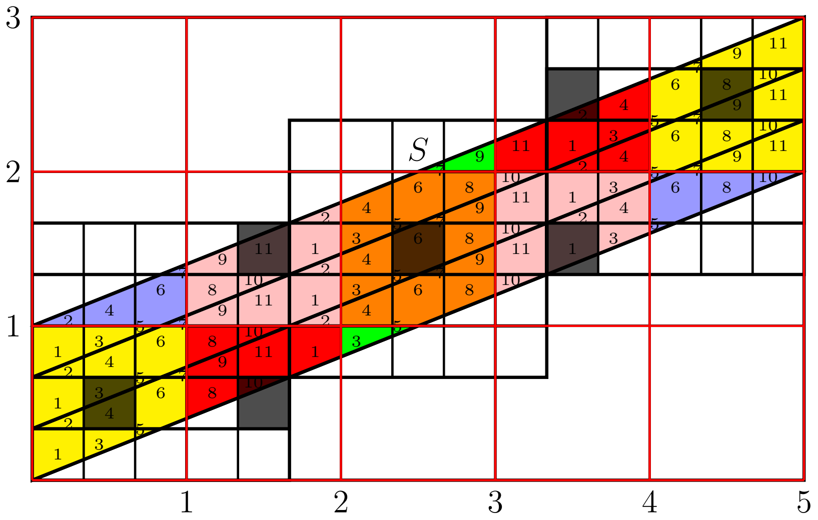

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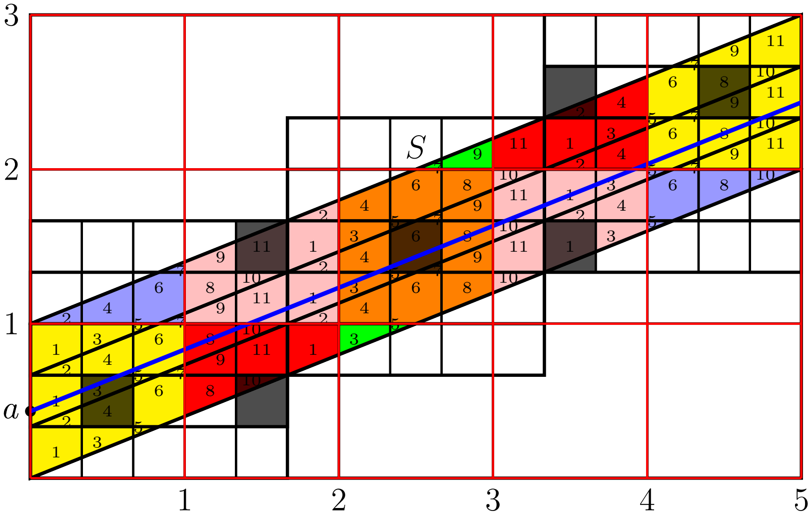
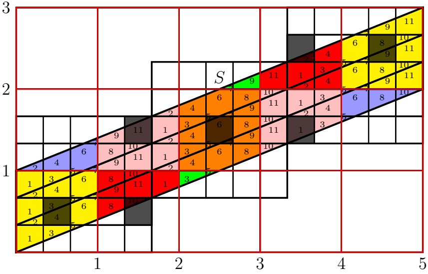
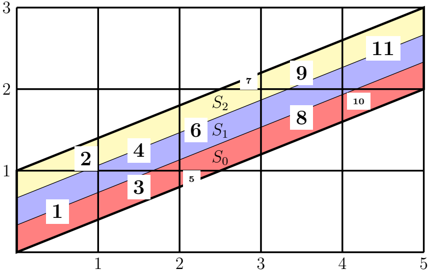


There are K:=2N+M-1 level zero shapes $Q_1, \ldots, Q_K$. For each "horizontal" (I mean non-vertical) stripes S_0, S_1, S_2 we define the $K \times K$ matrix A_0, A_1, A_2 respectively as follows:

$A_\ell(i,j) = 1$ iff the level zero shape i contains a level one shape j **in** stripe S_ℓ.





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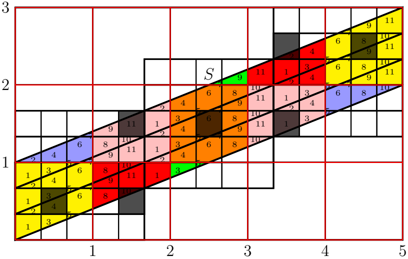

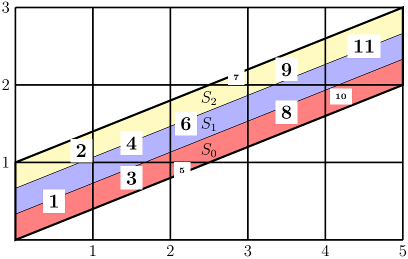

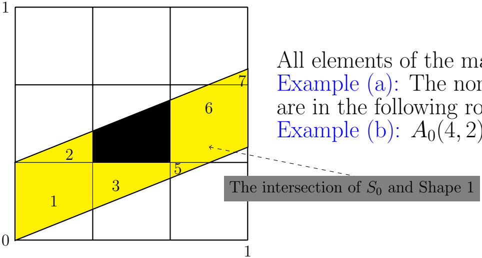

All elements of the matrices A_0, A_1, A_2 are either zero or one.
Example (a): The non zero elements of the first line of A_0 are in the following rows: $1, 2, 3, 5, 6, 7$.
Example (b): $A_0(4,2) = 1, \ \forall j \neq 2: \ A_0(4,j) = 0$.

$A_\ell(i, j) = 1$ iff the level zero shape i contains a level one shape j **in** stripe S_ℓ.

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \ldots \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \ldots \end{pmatrix}.$$

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Why do we need this?

For an $a = \sum_{k=1}^{\infty} a_k \cdot 3^{-k}$, with $a_k \in \{0,1,2\}$:

Observation: $A_{a_1 \dots a_n}(i,j)$ is the number of level n non-deleted squares that intersect $E_{\theta,a}$ within Q_i in a level n shape j. So, the number of level n-squares needed to cover $E_{\theta,a}$ is equal to $\|A_{a_1} \cdots A_{a_n}\|_1$, that is the sum of the elements of the non-negative $K \times K$ matrix $A_{a_1} \cdots A_{a_n}$. Since the size of the level n squares are $\sqrt{2} \cdot 3^{-n}$ this yields that

$$\overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \leq \frac{\overbrace{\lim_{n\to\infty} \frac{1}{n} \log \|A_{a_1} \cdots A_{a_n}\|_1}^{\gamma}}{\log 3}, \quad (5)$$

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To estimate the dimension of $E_{\theta,a}$ we need to understand the exponential growth rate of the norm of $A_{a_1\dots a_n} := A_{a_1}\cdots A_{a_n}$ which is the Lyapunov exponent of the random matrix product where each term in the matrix product is chosen from $\{A_0, A_1, A_3\}$ with probability $1/3$ independently:

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \log \|A_{a_1\dots a_n}\|_1, \text{ for a.a. } (a_1, a_2, \dots). \tag{6}$$

The limit exists (sub additive E.T.) and

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{a_1\dots a_n} \frac{1}{3^n} \log \|A_{i_1\dots i_n}\|_1. \tag{7}$$

Essentially what we need to prove it is that

$$\gamma < \log \frac{8}{3} \tag{8}$$

holds. Namely, by (5) $\overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \leq \frac{\gamma}{\log 3}$ and hence $\gamma < \log \frac{8}{3}$ is equivalent to

$$\overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \leq \frac{\gamma}{\log 3} < \frac{\log 8/3}{\log 3} = \frac{\log 8}{\log 3} - 1 = \dim_{\mathrm{H}}(F) - 1.$$

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Clearly, $\gamma \leq \log \frac{8}{3}$ holds. Namely, for

$$A_s := A_0 + A_1 + A_2 :$$

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{i_1 \ldots i_n} \frac{1}{3^n} \log \|A_{i_1 \ldots i_n}\|_1$$

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We needed to take higher iterates of the system (to get a system for which can verify that it is contracting on average in the projective distance) to prove the strict inequality.

- $\mathcal{CA}$: the set of $K \times K$ non-negative, column allowable (all columns contain non-zero elements) matrices.
- $\mathcal{CA}_p$: the set of those element of $\mathcal{CA}$ for which every row vector is either all positive or all zero.
- We prove (and this is an important part of our argument) that $\exists n_0$ and $(a'_1, \dots, a'_{n_0}) \in \{0, 1, 2\}^{n_0}$ s.t.

$$B_1 := A_{a'_1} \cdots A_{a'_{n_0}} \in \mathcal{CA}_p.$$

Clearly, $A_{i_1} \cdots A_{i_{n_0}} \in \mathcal{CA}$ holds for all $(i_1, \dots i_{n_0})$.

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Let $T := 3^{n_0}$, we have already defined the matrix B_1 now we define $B_2, \ldots, B_T$:

$$\{B_1, \ldots, B_T\} := \left\{A_{a_1 \ldots a_{n_0}}\right\}_{a_1 \ldots a_{n_0} \in \{0,1,2\}^{n_0}} .$$

For the vectors with all elements positive $\mathbf{x} = (x_1, \ldots, x_K) > \mathbf{0}$ and $\mathbf{y} = (y_1, \ldots, y_K) > \mathbf{0}$ we define the pseudo-metric

$$d(\mathbf{x}, \mathbf{y}) := \log \left[\frac{\max_i (x_i / y_i)}{\min_j (x_j / y_j)}\right] .$$

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d defines a metric on the simplex:

$$\Delta := \left\{ \mathbf{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K : x_i > 0 \text{ and } \sum_{i=1}^{K} x_i = 1 \right\}$$

We call it projective distance. For all $A \in \mathcal{CA}$ we define

$$\widetilde{A} : \Delta \to \Delta \qquad \widetilde{A}(\mathbf{x}) := \frac{\mathbf{x}^T \cdot A}{\|\mathbf{x}^T \cdot A\|_1}.$$

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$$\tau_B(A) := \sup_{\mathbf{x},\mathbf{y} \in \Delta,\ \mathbf{x} \neq \mathbf{y}} \frac{d(\mathbf{x}^T \cdot A, \mathbf{y}^T \cdot A)}{d(\mathbf{x}, \mathbf{y})}.$$

Lemma (Well known)

(a) *For* $\forall i = 1, \ldots, 3^n : \tau(B_i) < 1.$

(b) *The map* B_i *is a strict contraction in the projective distance:*

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Corollary of the Lemma:

So, the following IFS acting on the non-compact metric space (Δ, d) is contracting on average:

$$\left\{ \widetilde{B_1}, \ldots, \widetilde{B}_T \right\}$$

in the strong sense that the average of the Lipschitz constants is less than one.

recall : Δ : is the simplex:

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Definition

Suggested by a paper of Kravchenko (2006), on the complete metric space (Δ, d) we write $M(\Delta)$ for the set of all probability measures on Δ for which $\mu(\phi) < \infty$ holds for all real valued Lipschitz functions ϕ defined on (Δ, d). After Kantorovich, Rubinstein we define the distance of $\mu, \nu \in M(\Delta)$ by

$$L(\mu, \nu) := \sup \{\mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R}, \; \mathrm{Lip}(\phi) \leq 1\} .$$

Kravchenko (2006):

Proposition

The metric space $(M(\Delta), L)$ is complete.

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Suggested by a paper of Kravchenko (2006), on the complete metric space (Δ, d) we write $M(\Delta)$ for the set of all probability measures on Δ for which $\mu(\phi) < \infty$ holds for all real valued Lipschitz functions ϕ defined on (Δ, d). After Kantorovich, Rubinstein we define the distance of $\mu, \nu \in M(\Delta)$ by

$$L(\mu, \nu) := \sup \left\{ \mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1 \right\}.$$

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We introduce the operator $\mathcal{F} : M(\Delta) \to M(\Delta)$

$$\mathcal{F}\nu(H) := \frac{1}{T} \cdot \sum_{i=1}^{T} \nu\left(\widetilde{B}_i^{-1}(H)\right).$$

for a Borel set $H \subset \Delta$. Using $\nu \in M(\Delta)$, for every Lipschitz function ϕ we have

$$\mathcal{F}\nu(\phi) = \frac{1}{T} \cdot \sum_{i=1}^{T} \nu(\phi \circ \widetilde{B}_i).$$

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Lemma

(a) $\mathcal{F}$ *is a contraction on the metric space* $(M(\Delta), L)$.

(b) *There is a unique fixed point* $\nu \in M(\Delta)$ *of* $\mathcal{F}$ *and for all* $\mu \in M(\Delta)$ *we have* $L(\nu, \mathcal{F}^n \mu) \to 0$.

recall : $L(\mu, \nu) := \sup \{\mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1\}$,

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From now on we always write $\nu \in M(\Delta)$ for the unique fixed point of the operator $\mathcal{F}$ on $M(\Delta)$. That is

$$\nu(\phi) = \frac{1}{T^n} \cdot \sum_{i_1 \ldots i_n} \nu(\phi \circ \widetilde{B}_{i_1 \ldots i_n}). \tag{9}$$

holds for all Lipschitz functions ϕ and $n \geq 1$.

Following an idea of Furstenberg, it is a key point of our argument that we would like to give an integral representation of the Lyapunov exponent γ_B as an integral of a function φ to be introduced below against the measure ν.

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Lemma

Let γ_B be the Lyapunov exponent of the random matrix product formed from the matrices $B_1, \ldots, B_T$ taking each of the matrices with equal weight independently in every step. Then

$$n_0\gamma = \gamma_B = \int_\Delta \varphi(\mathbf{x}) d\nu(\mathbf{x})$$

where $\varphi : \Delta \to \mathbb{R}$ is defined by

$$\varphi(\mathbf{x}) := \frac{1}{T} \cdot \sum_{k=1}^{T} \log \|\mathbf{x}^T \cdot B_k\|_1, \qquad \mathbf{x} \in \Delta. \quad (10)$$

recall: ν is the unique invariant measure for the IFS $\left\{\widetilde{B}_1, \ldots, \widetilde{B}_T\right\}$

A good piece of news:

Lemma

We have $\mathrm{Lip}(\varphi) \leq 1$ *on the metric space* (Δ, d).

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$n_0\gamma = \gamma_B = \int_\Delta \varphi(\mathbf{x}) d\nu(\mathbf{x})$

We need to prove that:

$$\gamma_B < n_0 \cdot \log \frac{8}{3} \tag{11}$$

where $\gamma_B = n_0 \cdot \gamma$ is the Lyapunov exponent for the random matrix product formed from the matrices $B_1, \ldots, B_T$ each chosen independently with equal probabilities.

Let $\mathbf{w} \in \mathbb{R}^K$ be the center of the simplex Δ:

$$\mathbf{w} := \frac{1}{K} \cdot \mathbf{e} \text{ where } \mathbf{e} := (1, \ldots, 1) \in \mathbb{R}^K.$$

We define the sequence of measures $\nu_n \in \mathcal{M}^1$ by $\nu_0 := \delta_{\mathbf{w}}$ and for $H \subset \Delta$:

$$\nu_n(H) := (\mathcal{F}^n \nu_0)(H) = \frac{1}{T^n} \cdot \sum_{i_1 \ldots i_n} \nu_0(\widetilde{B}^{-1}_{i_1 \ldots i_n}(H)),$$

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We prove that $\exists\varepsilon'$ s.t. for every n big enough:

$$\int_\Delta \varphi(\mathbf{x})d\nu_n(\mathbf{x}) = \frac{1}{T^m}\cdot\sum_{|\mathbf{i}|=m}\frac{1}{T}\sum_{j=1}^{T}\log\frac{\|B_j\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}$$

$$\leq n_0\cdot\log\frac{8}{3}-\varepsilon'$$

Then

$$\lim_{n\to\infty}\int_\Delta \varphi(\mathbf{x})d\nu_n(\mathbf{x}) = \int_\Delta \varphi(\mathbf{x})d\nu(\mathbf{x}) = \gamma_B$$

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$s-1=0.5849$	Leb − a.e.	ν − a.e.
$\frac{p}{q}=1$	0.5716	0.5961
$\frac{p}{q}=\frac{1}{2}$	0.5805	0.5893
$\frac{p}{q}=\frac{2}{3}$	0.5846	0.5853

Figure: $s=\frac{\log 3}{\log 2}$ the dimensions of Lebesgue typical and natural measure typical slices

Outline

The dimension of ν-typical slices

All new results from now are joint with **Balázs Bárány** (TU Budapest)

We have started to study the dimension of NOT only the Lebesgue but also the natural measure (ν_θ)-typical slices for a fixed angle θ of the **Sierpinski Gasket**. Our research started with the following observation

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Lemma

The dimension preservation Lemma holds for all self-similar IFS on the plane satisfying

- *the IFS is homogeneous (all contraction ratios are the same),*
- *the attractor is connected,*
- *te group of the rotations in the linear parts is finite.*

So, in particular these all holds for the Sierpinski Gasket.

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Using a change of coordinates it is enough to consider the slices of the carpet which is the attractor of the self-similar IFS $\{g_i(x)\}_{i=1}^{3}$

$$g_i(x) = \frac{1}{2}x + t_i,\ t_1 = (0,0)\,,\ t_2 = \left(0, \frac{1}{2}\right)\,,\ t_3 = \left(\frac{1}{2}, 0\right)\,.$$

Since we focus on natural measure typical slices, we use different approach. Namely, for this purpose, the matrices introduced by Liu, Xi and Zhao (2007) seems to be more suitable. We introduce them through a concrete example when $\tan\theta = \frac{3}{2}$.

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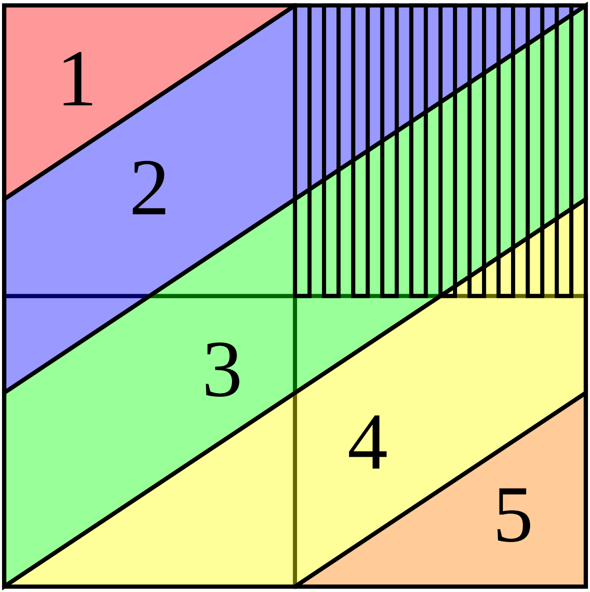

Figure: $\tan\theta = \frac{2}{3}$

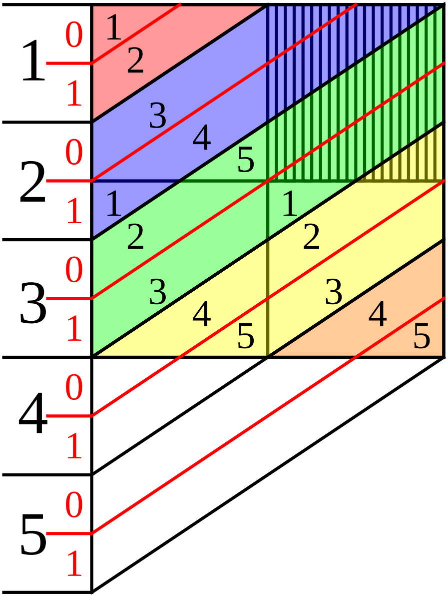

Figure: $\tan\theta = \frac{2}{3}$

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 A_0 A_1^3 A_0 = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \\ 3 & 4 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 & 2 \\ 3 & 3 & 1 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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