Characterization of α-limit sets for continuous maps of the interval

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Definition of α limit set

Definition 1 A complete negative trajectory *of a point* $x \in I$ *is an infinite sequence* $\{x_{-n}\}_{n=0}^{\infty}$ *such that* $x_0 = x$ *and* $f(x_{-(n+1)}) = x_{-n}$ *for any* $n \geq 0$.

Definition of α limit set

Definition 2 *Let* $\{x_{-n}\}_{n=0}^{\infty}$ *be a complete negative trajectory of a point* x *with respect to a map* $f \in C(I)$*. Then the set* $\alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ *of limit points of* $\{x_{-n}\}_{n=0}^{\infty}$ *is called the* α-limit set of $\{x_{-n}\}_{n=0}^{\infty}$.

Definition of α limit set

Lemma 1 *For any compact space* (X, d)*, any* $f \in C(X)$ *and any negative trajectory* $\{x_{-n}\}_{n=0}^{\infty}$*, the set* $\alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ *is nonempty, closed and invariant.*

Every α-limit set is ω-limit set

A set V is *right* (resp. *left*) *unilateral neighborhood* of $x \in I$ if there exists an $\varepsilon > 0$ such that $[x, x+\varepsilon) \subset V$ (resp. $(x-\varepsilon, x] \subset V$). If T is a side of x (i.e. T means ”right” or ”left”) then we can talk about *T-unilateral neighborhoods* of x.

Every α-limit set is ω-limit set

Let $U \subset I$ be the union of finitely many pairwise disjoint compact and non-degenerate intervals and let $K \subset U$. Then

- $f_U(K) := f(K) \cap U$,
- $f_U^n(K) := f_U(f_U^{n-1}(K))$, e.g. $f_U^2(K) := f(f(K) \cap U) \cap U$,
- $\tilde{K}_U := \bigcup_{i=1}^{\infty} f_U^i(K)$.

Every α-limit set is ω-limit set

- Let $A \subset I$ be a closed set and let $x \in A$.
- We say that a side T of x is *A-covering* if for any union of finitely many closed intervals U such that $A \subset \operatorname{Int} U$ and any closed T-unilateral neighborhood V of x there are finitely many components of $\tilde{V}_U$ such that the closure of their union covers A.
- If every $x \in A$ has A-covering side we call the set A *locally expanding* (with respect to f).

Every α-limit set is ω-limit set

Lemma 2 ([3, Theorem 2.12]) *Let $f \in C(I)$. A closed set A is an ω-limit set of f if and only if A is locally expanding.*

Lemma 3 ([3, Lemma 2.3]) *Let $K \subset U$ be an interval. Then $\tilde{K}$ is the union of two disjoint sets $\mathcal{A}, \mathcal{B}$ where:*

- *$\mathcal{A}$ is a finite union of disjoint intervals and*
- *$\mathcal{B}$ the union of orbits of finitely many pairwise pairwise disjoint wandering intervals.*

Moreover, if K is closed then so are all of the wandering intervals defining $\mathcal{B}$.

Every α-limit set is ω-limit set

Theorem 1 *For any $f \in C(I)$ and any negative trajectory $\{x_{-n}\}_{n=0}^{\infty}$, the set $\alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ is locally expanding.*

Every α-limit set is ω-limit set

Corollary 2 *Let $f \in C(I)$. Then any α-limit set $\alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ is an ω-limit set of f.*

Basic sets and α-limit sets

Lemma 4 ([8]) *Let M be a basic set of a map $f \in C(I)$ and let $\omega_f(x) \subset M$ for some $x \in I$. Then the set*

$$\{z \in M \ : \ \omega_f(z) = \omega_f(x)\}$$

is dense in M.

Basic sets and α-limit sets

A *portion* of a basic set M is the intersection of M with an interval J which is nonempty.

Lemma 5 ([1, Lemma 2.4]) *Let M be a basic set of $f \in C(I)$ and let J be an interval with endpoints in M such that $J \cap M$ is infinite. Then $\lim_{n\to\infty} f^n(J \cap M)$ exists (in the sense of Hausdorff metric) and contains the portion $(\min M, \max M) \cap M$.*

Basic sets and α-limit sets

Theorem 3 *Let M be a basic set and let $A = \omega_f(x)$ for some $x \in I$. If $A \subset M$ then $A = \alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ for some negative orbit $\{x_{-n}\}_{n=0}^{\infty} \subset I$.*

Zero entropy case

Lemma 6 ([9, Theorem 3.5]) *Let $f \in C(I)$ be a map with $h_{top}(f) = 0$ and let M be a maximal infinite ω-limit set. Then there is a sequence $\{I_n\}_{n=0}^{\infty}$ of compact periodic intervals such that for any n*

1. *I_n has period 2^n,*
2. *$I_{n+1} \cup f^{2^n}(I_{n+1}) \subset I_n$,*
3. *$\mathrm{Orb}(I_n) \supset M$,*
4. *$M \cap f^i(I_n) \neq \emptyset$ for every i,*

Zero entropy case

Lemma 7 ([4, Theorem 6.5]) *An infinite compact set $W \subset (0,1)$ is an ω-limit set of a map $f \in C(I)$ with zero topological entropy if and only if $W = Q \cup P$ where Q is a Cantor set and P is empty or countably infinite set disjoint with Q and satisfying the following two coditions:*

1. *every interval J contiguous to Q (i.e. $\mathrm{Int}\, J \cap Q = \emptyset$ and $\partial J \subset Q$) contains at most two points of P,*
2. *each of the intervals $[0, \min Q]$, $[\max Q, 1]$ contains at most one point of P.*

Zero entropy case

Theorem 4 *Let* $f \in C(I)$ *ans assume that* $h_{top}(f) = 0$*. If* M *is an infinite* ω*-limit set of* f *then any infinite* α*-limit set* $\alpha_f(\{x_{-n}\}_{n=0}^{\infty})$ *contained in* M *is perfect.*

Zero entropy case

Theorem 5 *For any* $f \in C(I)$ *with zero topological entropy the system of* α*-limit sets is the system of minimal sets of* f*.*

Zero entropy case

Theorem 6 *Any ω-limit set of a map $f \in C(I)$ which is contained in a basic set of f belongs to $\alpha(f)$.*

Zero entropy case

Theorem 7 *There is a map* $f \in C(I)$ *with zero topological entropy such that the set* $\alpha(f)$ *of* α*-limit sets of* f *is not closed in the Hausdorff metric.*

Zero entropy case

Theorem 8 *Let $f \in C(I)$ have zero topological entropy. Than the collection $\alpha(f)$ of α-limit sets is closed in the Hausdorff metric if and only if the set $\mathrm{Rec}(f)$ of recurrent points is closed.*

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