α-Farey and α-Lüroth maps - new types of phase transitions

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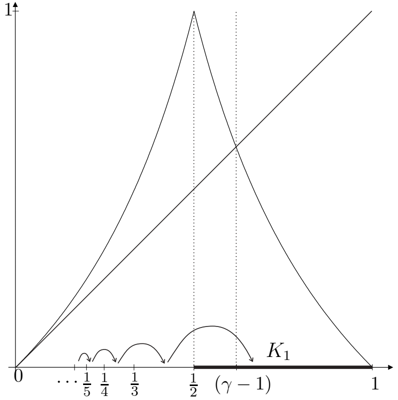
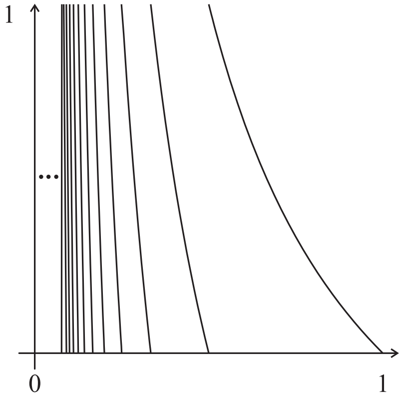
- The *Farey map* $F : [0,1] \to [0,1]$ is given by

$$F(x) := \begin{cases} \frac{x}{1-x}, & x \in \left[0, \frac{1}{2}\right], \\ \frac{1}{x} - 1, & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$



- The *Gauss map* $G : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}$ is given by

$$G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$



- G is invariant with respect to the (finite) Gauss measure $d\mu(x) := ((1+x)\log 2)^{-1} d\lambda(x)$.
- F is invariant with respect to the (infinite) measure $dm(x) := 1/x \cdot d\lambda(x)$.
- Fix $A_1 := (1/2, 1]$. For $x \in [0,1] \setminus \mathbb{Q}$ define the jump time

$$\varphi_{A_1}(x) := \inf\{n \in \mathbb{N}_0 : F^n(x) \in A_1\}$$

and let the jump transformation of the Farey map F with respect to A_1 for $x \in [0,1] \setminus \mathbb{Q}$ be given by

$$F_{A_1}(x) := F^{\varphi_{A_1}(x)+1}(x)$$

Fact

$$G = F_{A_1}.$$

Continued Fractions: Sum-level Result

- *n*-th Sum-Level-Set:

$$\mathscr{C}_n := \left\{ x \in [a_1, \ldots, a_k] : \sum_{i=1}^{k} a_i = n, \text{ for some } k \in \mathbb{N} \right\},$$

Theorem (K/Stratmann '10)

$$\lambda\left(\mathscr{C}_n\right) \sim \frac{\log 2}{\log n} \textit{ and } \sum_{k=1}^{n} \lambda\left(\mathscr{C}_k\right) \sim \frac{n \log 2}{\log n}.$$

Proof.

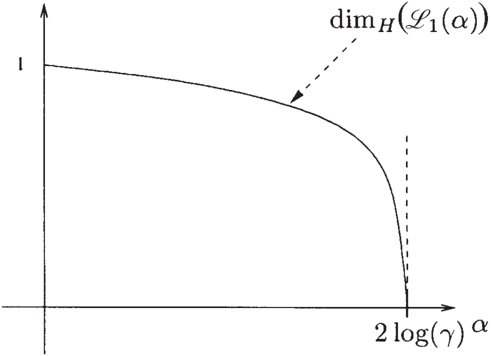
Observe $F^{-n+1}([1/2,1]) = \mathscr{C}_n$ and use **Infinite Ergodic Theory** for the transfer operator $\widehat{F}$ of F on $\left([0,1], \mathscr{B}, x^{-1} d\lambda(x)\right)$. □

- $S : [0,1] \to [0,1]$ diff'able, $x \in [0,1]$,

$$\Lambda(S,x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| S'(S^k(x)) \right|.$$

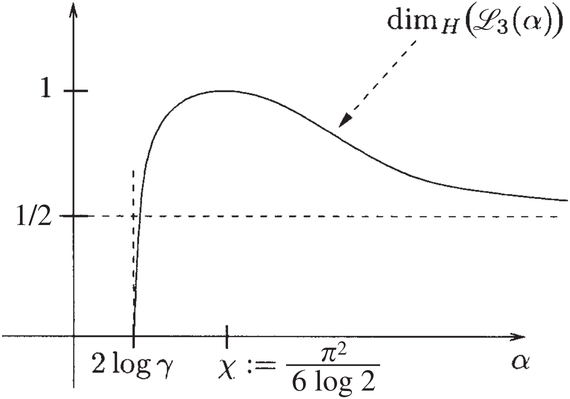
- Lyapunov spectra (K./Stratmann '07)

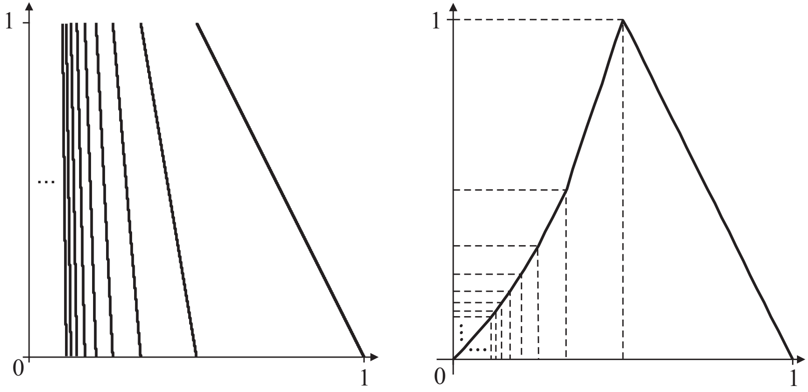
$$\mathscr{L}_1(\alpha) := \{x \in [0,1] : \Lambda(F,x) = \alpha\}.$$



- Lyapunov spectrum for the Gauss map (Pollicott/Weiss '99, K./Stratmann '07, Fan/Liao/Wang/Wu '09)

$$\mathscr{L}_3(\alpha) := \{x \in [0,1] : \Lambda(G,x) = s\}.$$





- α-Lüroth map
- α-Farey map
- J. Lüroth. Über eine eindeutige Entwicklung von Zahlen in eine unendliche Reihe. *Math. Ann.* **21**:411–423, 1883.

- countable partition $\alpha := \{A_n : n \in \mathbb{N}\}$ of $[0,1]$ consisting of left open, right closed intervals; ordered from right to left, starting with A_1.
- $a_n := \lambda(A_n)$; $t_n := \sum_{k=n}^{\infty} a_k$.
- α-Lüroth map $L_\alpha(x) := \begin{cases} (t_n - x)/a_n & \text{for } x \in A_n, n \in \mathbb{N}, \\ 0 & \text{for } x = 0. \end{cases}$
- α-Farey map

$$F_\alpha(x) := \begin{cases} (1-x)/a_1 & \text{for } x \in A_1, \\ a_{n-1}(x - t_{n+1})/a_1 + t_n & \text{for } x \in A_n, n \geq 2, \\ 0 & \text{for } x = 0. \end{cases}$$

- λ is invariant with respect to L_α.
- L_α is the jump transformation of F_α with respect to A_1.
- α is said to be of finite type if $\sum_{n=1}^{\infty} t_n < \infty$
- α is said to be of infinite type if $\sum_{n=1}^{\infty} t_n = \infty$
- α is called expansive of exponent $\theta \geq 0$ if $t_n = \psi(n) n^{-\theta}$, for all $n \in \mathbb{N}$ and some slowly varying function ψ. Then:

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1 \text{ and } F'_\alpha(0+) = 1$$

- α is said to be expanding if $\lim_{n\to\infty} t_n / t_{n+1} = \rho > 1$. Then:

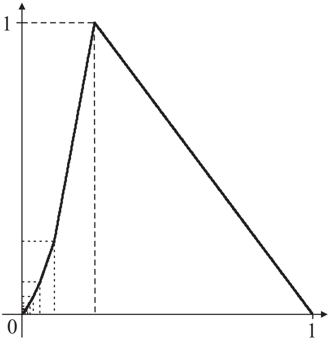
$$F'_\alpha(0+) = \rho.$$

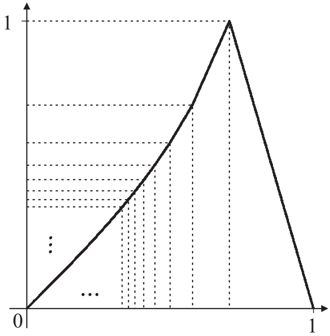
- $\exists \nu_\alpha \ll \lambda$ invariant with respect to F_α and density $\sum_{n=1}^{\infty} t_n / a_n \cdot \mathbf{1}_{A_n}$.
- $\nu_\alpha([0,1]) = +\infty \iff \alpha$ of infinite type.
- F_α and the tend map are topologically conjugate with conjugating homeomorphism given by (the α-Minkowski-? function)

$$\theta_\alpha(x) := -2 \sum (-1)^k \, 2^{-\sum_{i=1}^{k} \ell_i}$$

for $x = [\ell_1, \ell_2, \ldots]_\alpha = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\prod_{i<n} a_{\ell_i}\right) t_{\ell_n}$ (α-Lüroth Expansion).

Examples for different expansive α





- $t_n = 1/n^2$ – finite type.
- $t_n = 1/\sqrt{n}$ – infinite type.

- α-sum-level sets

$$\mathscr{L}_n^{(\alpha)} := \left\{ x \in C_\alpha(\ell_1, \ell_2, \dots, \ell_k) : \sum_{i=1}^{k} \ell_i = n, \text{ for some } k \in \mathbb{N} \right\},$$

where

$$C_\alpha(\ell_1, \dots, \ell_k) := \{x \in [0,1] : L_\alpha^{i-1}(x) \in A_{l_i}, \forall i = 1, \dots, k\}.$$

- Important fact: $\mathscr{L}_n^{(\alpha)} = F_\alpha^{-(n-1)}(A_1)$, for all $n \in \mathbb{N}$.

Theorem (K./Munday/Stratmann '11)

1. *We have that* $\sum_{n=1}^{\infty} \lambda(\mathscr{L}_n^{(\alpha)})$ *diverges, and that*

$$\lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) = \begin{cases} 0, & \text{if } \alpha \text{ is of infinite type;} \\ \left(\sum_{k=1}^{\infty} t_k\right)^{-1}, & \text{if } \alpha \text{ is of finite type.} \end{cases}$$

2. *Let* α *be either expansive of exponent* $\theta \in [0,1]$ *(*$K_\alpha := \frac{1}{\Gamma(2-\theta)\Gamma(1+\theta)}, k_\alpha := \frac{1}{\Gamma(2-\theta)\Gamma(\theta)}$*), or of finite type* $K_\alpha := k_\alpha := 1$.

   (a) ***Weak renewal law.*** $\displaystyle\sum_{k=1}^{n} \lambda\left(\mathscr{L}_k^{(\alpha)}\right) \sim K_\alpha \cdot n \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$

   (b) ***Strong renewal law.*** $\displaystyle\lambda\left(\mathscr{L}_n^{(\alpha)}\right) \sim k_\alpha \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$

Fact (Renewal Equation)

For each $n \in \mathbb{N}$, we have that

$$\lambda\left(\mathscr{L}_n^{(\alpha)}\right) = \sum_{m=1}^{n} a_m \lambda\left(\mathscr{L}_{n-m}^{(\alpha)}\right).$$

Proof.

Proved by induction using linearity. □

Proof of $\sum_{n=0}^{\infty} \lambda(\mathscr{L}_n^{(\alpha)})$ diverges.

Define $a(s) := \sum_{n=1}^{\infty} a_n s^n$ and $\ell(s) := \sum_{m=0}^{\infty} \lambda\left(\mathscr{L}_m^{(\alpha)}\right) s^m$. Then for $s \in (0,1)$ we have that $\ell(s) - 1 = \ell(s)a(s)$, and hence, $\ell(s) = 1/(1 - a(s))$. Since $a(1) = 1$ we have $\lim_{s \nearrow 1} \ell(s) = \infty$ □

Proof Part (1).

Classical *Renewal Theorem* by Erdős, Pollard and Feller gives

$$\lim_{n\to\infty} \lambda(\mathscr{L}_n^{(\alpha)}) = \frac{1}{\sum_{m=1}^{\infty} m \cdot a_m} = \frac{1}{\sum_{k=1}^{\infty} t_k}.$$

(P. Erdős, H. Pollard, W. Feller. A property of power series with positive coefficients. *Bull. Amer. Math. Soc.* **55**:201-204, 1949) □

Proof Part (2).

For the finite case consider part (1). For the expansive case apply a strong renewal theorems obtained in [**K. B. Erickson**. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151**, 1970], [**A. Garsia, J. Lamperti**. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37**, 1963]. □

α-Farey Free Energy Function

- $S : [0,1] \to [0,1]$ diff'able, $x \in [0,1]$,

$$\Lambda(S,x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| S'(S^k(x)) \right| .$$

- α-Farey Lyapunov spectrum, $s \in \mathbb{R}$,

$$\sigma_\alpha(s) := \dim_H \left(\{ x \in [0,1] : \Lambda(F_\alpha, x) = s \} \right).$$

- α-Farey free energy function $v : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$v(u) := \inf \left\{ r \in \mathbb{R} : \sum_{n=1}^{\infty} a_n^u \exp(-rn) \leq 1 \right\}.$$

- We say that F_α exhibits no phase transition if and only if v is diff'able everywhere.

Theorem (K./Munday/Stratmann '11)

Let α either expanding, or expansive and eventually decreasing. For $s_- := \inf\{-(\log a_n)/n : n \in \mathbb{N}\}$ and $s_+ := \sup\{-(\log a_n)/n : n \in \mathbb{N}\}$, we have that $\sigma_\alpha(s)$ vanishes outside the interval $[s_-, s_+]$ and for each $s \in (s_-, s_+)$, we have

$$\sigma_\alpha(s) = \inf_{u \in \mathbb{R}} \left(u + s^{-1} v(u)\right).$$

1. *α **expanding**: F_α exhibits no phase transition. In particular, v is strictly decreasing and bijective.*
2. *α **expansive of exponent θ and eventually decreasing**: F_α exhibits no phase transition $\iff$ α is of infinite type. In particular, $v \geq 0$ and $v|_{[1,\infty)} = 0$.*

- α-Lüroth Lyapunov spectrum, $s \in \mathbb{R}$

$$\tau_\alpha(s) := \dim_H(\{x \in \mathscr{U} : \Lambda(L_\alpha, x) = s\}).$$

- α-Lüroth pressure function $p : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$p : u \mapsto \log \sum_{n=1}^{\infty} a_n^u.$$

- We say that L_α exhibits no phase transition if and only if the pressure function p is differentiable everywhere (that is, the right and left derivatives of p coincide everywhere, with the convention that $p'(u) = \infty$ if $p(u) = \infty$).

α-Lüroth Lyapunov Spectrum

Theorem (K./Munday/Stratmann '11)

For $t_- := \min\{-\log a_n : n \in \mathbb{N}\}$ *we have that* τ_α *vanishes on* $(-\infty, t_-)$, *and for each* $s \in (t_-, \infty)$ *we have*

$$\tau_\alpha(s) = \inf_{u \in \mathbb{R}} \left(u + s^{-1} p(u)\right).$$

Moreover, $\lim_{s\to\infty} \tau_\alpha(s) = t_\infty := \inf\{r > 0 : \sum_{k=1}^{\infty} a_n^r < \infty\} \leq 1$.

1. α ***expanding***: L_α *exhibits no phase transition and* $t_\infty = 0$.
2. α ***expansive of exponent*** $\theta > 0$ ***and eventually decreasing***: $t_\infty = 1/(1+\theta)$.
   L_α *exhibits no phase trans.* $\iff \sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)} \frac{\log n}{n} = \infty$.
3. α ***expansive of exponent*** $\theta = 0$ ***and eventually decreasing***: $t_\infty = 1$.
   L_α *exhibits no phase trans.* $\iff \sum_{n=1}^{\infty} a_n \log(a_n) = \infty$.

Theorem (Munday '10)

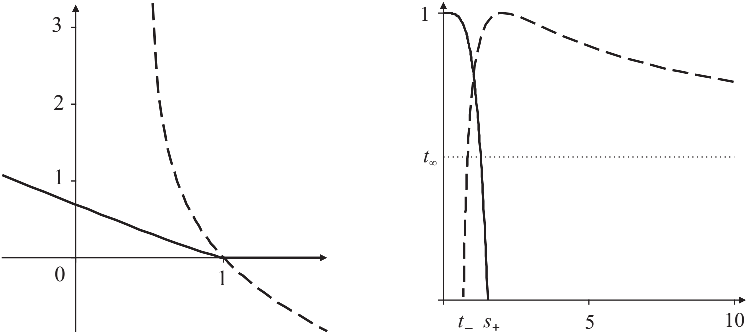
The critical value t_∞ is also equal to the Hausdorff dimension of the Good-type set $G_\infty^{(\alpha)}$ associated to L_α, given by

$$G_\infty^{(\alpha)} := \{[\ell_1, \ell_2, \ldots]_\alpha : \lim_{n\to\infty} \ell_n = \infty\}.$$

- If L_α exhibits a phase transition, that is $\sum a_n^{t_\infty} < +\infty$ with finite right derivative t_0 in t_∞, then for $t \in [t_0, +\infty)$,

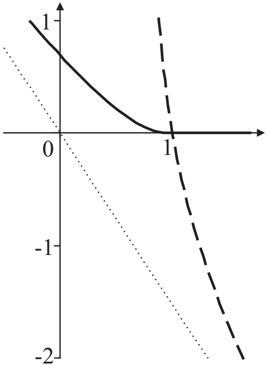
$$\tau_\alpha(t) = \frac{\log \sum_{n=1}^{\infty} a_n^{t_\infty}}{t} + t_\infty.$$

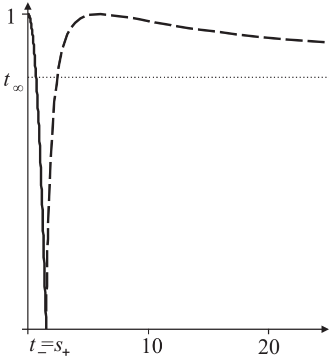
Expansive Example : The classical alternating Lüroth system



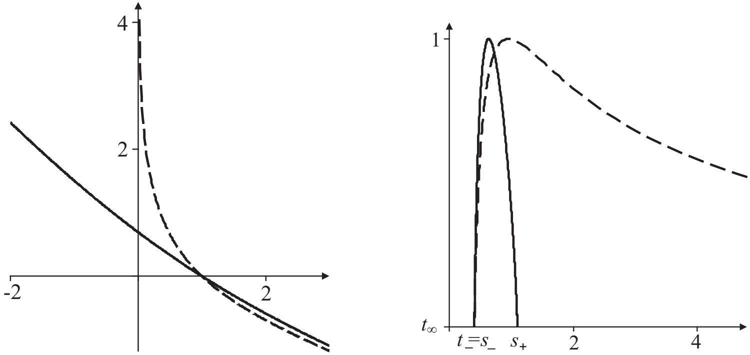
- For $\alpha_H := \{(1/(n+1), 1/n], n \in \mathbb{N}\}$ The figure shows the α_H-Farey free energy v (solid line), the α_H-Lüroth pressure function p (dashed line), and the associated dimension graphs σ_{α_H} and τ_{α_H}. Here, $t_- = \log 2, t_\infty = 1/2$ and $s_+ = (\log 6)/2$. **We have $p(t_\infty) = \infty$, no phase transition for the α_H-Farey free energy function and the α_H-Lüroth pressure function.**

Expansive Example: $a_n := \zeta(5/4)^{-1} n^{-5/4}$



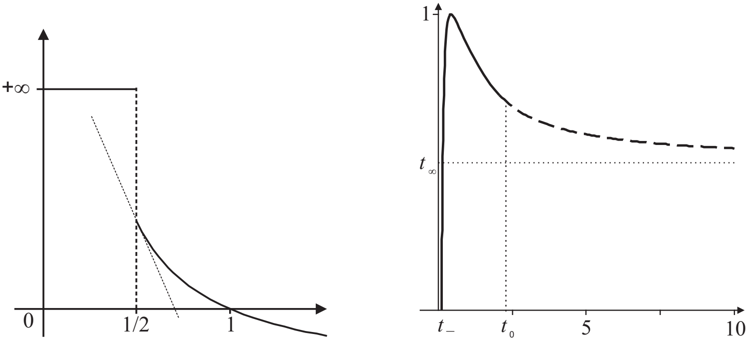


- **The Farey spectrum and the Lüroth spectrum intersect in a single point, for α expansive.** The α-Farey free energy v (solid line), the α-Lüroth pressure function p (dashed line), and the associated dimension graphs for $a_n := \zeta(5/4)^{-1} n^{-5/4}$. Here, F_α exhibits no phase transition.



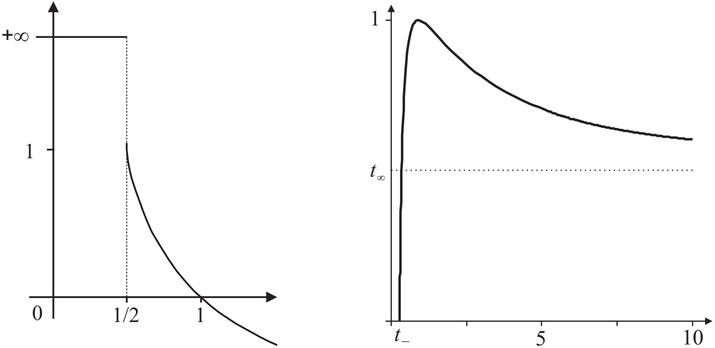
- **The Farey spectrum is completely contained in the Lüroth spectrum, for α expanding.** The α-Farey free energy v (solid line), the α-Lüroth pressure function p (dashed line), and the associated dimension graphs. The α-Farey system is given in this situation by the tent map with slopes 3 and $-3/2$.

Example for Lüroth Phase Transition $a_n := \frac{C}{n^2 \cdot (\log(n+5))^{12}}$



- **Finite critical value $p(t_\infty) < \infty$ with phase transition for the α-Lüroth pressure function and α expansive.** The α-Lüroth pressure function p, and the associated dimension graphs. In this case $t_\infty = 1/2$ and $p(1/2) < \infty$ and L_α has a phase transition.

Examples: No Lüroth Phase Transition $a_n := \frac{C}{n^2 \cdot (\log(n+5))^4}$



- **Finite critical value $p(t_\infty) < \infty$ and no phase transition for the α-Lüroth pressure function and α expansive.** The α-Lüroth pressure function p, and the associated dimension graphs for the α-Lüroth system. In this case $t_\infty = 1/2$ and $p(1/2) < \infty$, but L_α exhibits no phase transition.

Lemma

Let α be a partition such that $\lim_{n\to\infty} t_n/t_{n+1} = \rho \geq 1$ and such that α is either expanding, or expansive of exponent θ and eventually decreasing. Then:

1. $\lim_{n\to\infty} \frac{\log a_n}{n} = \lim_{n\to\infty} \frac{\log t_n}{n} = -\log \rho$. α expansive with $\theta > 0$, then $a_n \sim \theta n^{-1} t_n$.
2. If α expansive with $\theta = 0$, then we have $t_\infty = 1$.
3. If α is expanding, or expansive with $\theta > 0$, then $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \rho$.
4. There exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, with $\lim_{k\to\infty} \varepsilon_k = 0$, such that for all $n \in \mathbb{N}$ and $x \in \bigcup_{k\geq n} A_k$ we have that

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \log \left| F'_\alpha(F^k_\alpha(x)) \right| - \log \rho \right| < \varepsilon_n.$$

Theorem (K./Jaerisch)

Consider the two potential functions $\varphi, \psi : \mathscr{U} \to \mathbb{R}$ given for $x \in A_n$, $n \in \mathbb{N}$, by $\varphi(x) := \log a_n$ and $\psi(x) := z_n$, for some fixed sequence $(z_n)_{n\in\mathbb{N}}$ of negative real numbers. For all $s \in \mathbb{R}$ we then have that

$$\dim_H \left\{ x \in \mathscr{U} : \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \psi(L_\alpha^k(x))}{\sum_{k=0}^{n-1} \varphi(L_\alpha^k(x))} = s \right\} \leq \max\{0, -t^*(-s)\}.$$

The function $t : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is given by

$$t(v) := \inf \left\{ u \in \mathbb{R} : \sum_{n=1}^{\infty} a_n^u \exp(v z_n) \leq 1 \right\}$$

and t^* is the Legendre transform of t.

Theorem (K./Jaerisch)

With

$$r_- := \inf\left\{-t^+(v) : v \in Int(dom(t))\right\},$$
$$r_+ := \sup\left\{-t^+(v) : v \in Int(dom(t))\right\},$$

we have for each $s \in (r_-, r_+)$,

$$\dim_H \left\{ x \in \mathscr{U} : \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \psi(L_\alpha^k(x))}{\sum_{k=0}^{n-1} \varphi(L_\alpha^k(x))} = s \right\} = -t^*(-s).$$

where t^+ denotes the right derivative of t, $Int(A)$ denotes the interior of the set A, and $dom(t) := \{v \in \mathbb{R} : t(v) < +\infty\}$.

- Set $z_n := -n$, then $v : u \mapsto \inf\left\{r \in \mathbb{R} : \sum_{n=1}^{\infty} a_n^u \exp(-rn) \leq 1\right\}$ is the inverse of t.
- $s_- = 1/r_+$ and $s_+ = 1/r_-$.
- For $s \in (s_-, s_+)$, it follows that

$$\begin{aligned} \sigma_\alpha(s) &= -t^*(-1/s) = \inf_{v \in \mathbb{R}} \left(t(v) + s^{-1}v\right) \\ &= \inf_{u \in \mathbb{R}} \left(u + s^{-1} \log \sum_{n=1}^{\infty} a_n^u\right) \end{aligned}$$

and $\sigma(s)$ vanishes outside of (s_-, s_+).

- Consider $Z(u,v) := \sum_{n=1}^{\infty} \exp\left(n\left(\frac{u \log a_n}{n} - v\right)\right)$.
- α expanding $\implies \forall u_0 \in \mathbb{R}\{Z(u_0,v) : v \in \mathbb{R}\} = (0,\infty) \implies \exists f(u_0)$ is unique solution of $Z(u_0, f(u_0)) = 1$. By the implicit function theorem there is no phase transition.
- α expansive
    - For $u < 1$ argue as above
    - For $u \geq 1$ we have $\sum_{n=1}^{\infty} a_n^u e^{-wn} \begin{cases} < 1 & \text{for } w \geq 0 \\ = \infty & \text{for } w < 0 \end{cases} \implies$ $v(u) = 0$
- Consider $f'(u) = \frac{\sum_{n=1}^{\infty} a_n^u e^{-f(u)n} \log a_n}{\sum_{n=1}^{\infty} n a_n^u e^{-f(u)n}}$ for $u \nearrow 1$.
    - Infinite type: Denominator tends to ∞.
    - $\lim_{u \nearrow 1} f'(u) = \lim_{u \nearrow 1} \sum_{n=1}^{\infty} \frac{\log a_n}{n} \frac{n a_n e^{-f(u)n}}{\sum_{k=1}^{\infty} k a_k e^{-f(u)k}} = 0$.

Lemma

Let α be a partition which is either expanding, or expansive of exponent θ and eventually decreasing. With

$$\Pi(L_\alpha, x) := \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \log \left| L_\alpha'(L_\alpha^k(x)) \right| \right) / \left(\sum_{k=0}^{n-1} N(L_\alpha^k(x)) \right),$$

we then have for each $s \geq 0$ that the sets

$$\{x \in \mathscr{U} : \Pi(L_\alpha, x) = s\} \text{ and } \{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s\}$$

coincide up to a countable set of points.

Proof of Theorem for α-Lüroth

- Set $z_n := -1$, $p : u \mapsto \log \sum_{n=1}^{\infty} a_n^u$ is the inverse of t.
- $t_- := 1/r_+ = \inf\{-\log a_n : n \in \mathbb{N}\}$ and $t_+ := +\infty$
- For $s \in (t_-, +\infty)$, it follows that

$$\begin{aligned} \tau_\alpha(s) &= -t^*(-1/s) = \inf_{v \in \mathbb{R}} \left(t(v) + s^{-1} v \right) \\ &= \inf_{u \in \mathbb{R}} \left(u + s^{-1} \log \sum_{n=1}^{\infty} a_n^u \right) \end{aligned}$$

and $\tau_\alpha(s)$ vanishes for $s < t_-$.

- For the right derivative of the pressure function p of L_α, we have that
$$p'(u) = \frac{\sum_{n=1}^{\infty} a_n^u \log a_n}{\sum_{n=1}^{\infty} a_n^u}.$$
- Clearly, p is real-analytic on (t_∞, ∞).
- Hence, we have that L_α exhibits no phase transition if and only if $\lim_{u \searrow t_\infty} -p'(u) = +\infty$.
- If α is expanding, then there is no phase transition. This follows, since, by the technical Lemma, we have that $p(u) < \infty$, for all $u > 0$. In particular, $t_\infty = 0$. If α is expansive with $\theta = 0$, we have by the Technical Lemma $t_\infty = 1$. Hence, $\lim_{u \searrow t_\infty} p'(u) = \infty$ if and only if $-\sum_{n=1}^{\infty} a_n \log(a_n) = \infty$.

- If α is expansive such that $t_n = \psi(n)n^{-\theta}$, then the Technical Lemma implies that there exists ψ_0 such that $\psi_0(n) \sim \theta\psi(n)$ and $a_n = \psi_0(n)n^{-(1+\theta)}$. Consequently, we have that $t_\infty = 1/(1+\theta)$. Hence, we now observe that

$$\lim_{u \searrow t_\infty} -p'(u) = t_\infty^{-1} \lim_{u \searrow t_\infty} \frac{\sum_{n=1}^{\infty} \left(n^{-1-\theta}\psi_0(n)\right)^u \log\left(n(\psi_0(n))^{-\frac{1}{1+\theta}}\right)}{\sum_{n=1}^{\infty} \left(n^{-1-\theta}\psi_0(n)\right)^u}.$$

  - $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)}(\log n)/n < \infty \implies$ numerator and denominator both converge $\implies \lim_{u \searrow t_\infty} -p'(u) < \infty \implies$ phase transition.
  - $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)}(\log n)/n = \infty$:
    - $\sum_{n=1}^{\infty} n^{-1}\psi_0(n)^{1/(1+\theta)} < \infty \implies \lim_{u \searrow t_\infty} -p'(u) = \infty$.
    - $\sum_{n=1}^{\infty} n^{-1}\psi_0(n)^{1/(1+\theta)} = \infty \implies$ $\forall k \in \mathbb{N} : \lim_{u \searrow t_\infty} (k^{-(1+\theta)}\psi_0(k))^u / \sum_{n=1}^{\infty} (n^{-(1+\theta)}\psi_0(n))^u = 0$ $\implies \lim_{u \searrow t_\infty} -p'(u) = \infty$.
  - $\implies$ no phase transition.

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