EPSRC Symposium Workshop on Dynamical Systems and Dimension problem sessions

April 2011

1 K. Falconer (St. Andrews)

Let $A \subseteq \mathbb{R}^2$ and define the distance set $D(A)$ to be

$$D(A) := \{|x-y| \,:\, x, y \in A\}.$$

A well known result of Erdös [7] states that in the case that A is finite then

$$c^{-1}\#A^{1/2} \leq \#D(A) \leq \frac{c\#A}{(\log \#A)^{1/2}}.$$

for some constant $c > 1$. The lower bound was recently improved by Katz and Tardos [16] with the exponent being pushed up to ≈ 0.86. Recently an improvement on the lower bound came from Guth-Katz [15] who applied the 'Polynomial ham sandwich theorem' to obtain

$$\#D(A) \geq \frac{c\#A}{\log \#A}.$$

For more details on this we refer the reader to Terry Tao's blog

For A of infinite cardinality it is conjectured that $\dim_H(A) > 1$ implies that $D(A)$ has positive Lebesgue measure. Results in this direction have been obtained by Falconer and later Wolf for the bounds $\dim_H(A) > 3/2$ and $\dim_H(A) > 4/3$ respectively.

Question 1.1 *Let $A \subseteq \mathbb{R}^2$ and suppose that $\dim_H(A) > 1$. Does it follow that $\underline{\dim}_B(D(A)) = 1$?*

Question 1.2 *Let $A \subseteq \mathbb{R}^n$ and suppose that $\dim_H(A) > n/2$. Does it follow that $\mathcal{L}(D(A)) > 0$?*

A further question asked by Stratmann (Bremen):

Question 1.3 *Is there an application to limit sets of Kleinian groups?*

2 M. Urbanski (UNT)

Let (X, d) denote a compact metric space and let $A \subseteq X$. Let $f : A \to \mathbb{R}^n$ be Lipschitz. By a theorem of E. Marczewski we have that

$$\dim_{top}(f(A)) \leq [\dim_H(f(A))]$$

here $[\cdot]$ denotes the integer part of a real number. On the other hand, as Lipschitz maps cannot increase dimension we immediately have $\dim_H(f(A)) \leq \dim_H(A)$. Combining these observations we have

$$\sup\{\dim_{top}(f(A)) \,:\, f \text{ is Lipschitz}\} \leq [\dim_H(A)].$$

It should be noted that this is not necessarily an equality: Let k be a positive integer and $A \subset X$ be such that $k = \dim_H(A)$ and $\mathcal{H}^k(A) = 0$. This implies that $\mathcal{H}^k(f(A)) = 0$ and so $\dim_{top}(f(A)) \leq k - 1$.

Question 2.1 *Is it true that*

$$\sup\{\dim_{top}(f(A)) \,:\, f : A \to \mathbb{R}^n \textit{ is Lipschitz}\} \geq [\dim_H(A)] - 1?$$

3 J. Robinson (Warwick)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map and let $X \subset \mathbb{R}^n$ be such that $f(X) = X$. Douady and Oesterlé [6] gave an upper bound on the Hausdorff dimension of the set X.

Let $\mathcal{H}$ is a Hilbert space. If $X \subset \mathcal{H}$ is such that $\dim_B(X) < \infty$ then there exists a (smooth?) embedding $L : X \to \mathbb{R}^k$ with $k = 2\dim_B(X)$. The map $L^{-1} : L(X) \to X$ is only Hölder in general.

The Assouad dimension of a set X, denoted by $\dim_A(X)$, is defined to be the infemum over all positive real numbers s such that there exists a constant $M > 0$ such that for any $\rho > 0$ we have

$$N(X \cap B(x, \rho), r) \leq M(\rho/r)^s$$

for all $x \in X$ and $0 < r \leq \rho$, where $N(X, \delta)$ denotes the minimum number of balls of radius δ required to cover X.

If $\dim_A(X - X) < \infty$ then there exists a smooth embedding $L : X \to \mathbb{R}^N$ for $N \geq s$. Moreover L^{-1} is Lipschitz with a logarithmic correction.

Question 3.1 *Is it possible to obtain bounds on the Assouad dimension similar to Douady and Oesterlé?*

4 A. Mathé (Warwick)

Fix $a \in (0, 1/4)$ and let C_a denote the middle $(1-2a)$-Cantor set. For $x, y \in \mathbb{R}^2$ we let $l_{(x,y)}$ denote the unique line segment connecting these two points. We define $E \subset \mathbb{R}^2$ to be the set

$$E = \bigcup_{x \in C_a \times \{0\}} \bigcup_{y \in C_a \times \{1\}} l_{(x,y)}.$$

It is easy to show that $\dim_H(E) = 2\dim_H(C_a) + 1$. We define a map $\pi : C_a \times C_a \times [0,1] \to E$ by

$$\pi(x, y, \lambda) = \lambda(x, 0) + (1 - \lambda)(y, 1).$$

Let ν denote the push-forward under the map π of the measure $\mu \times \mu \times \mathrm{Leb}$, where μ denotes the (normalised) restriction of the $\log(2)/\log(3)$-dimensional Hausdorff measure. For $\alpha \geq 0$ we let

$$E(\alpha) = \{x \in E \,:\, \text{local dimension of } \nu \text{ at } x \text{ is } \alpha\}.$$

Question 4.1 *What is* $\dim_H(E(\alpha))$*?*

This is related to the following problem: for $\theta \in [0, \pi)$ we let p_θ denote the orthogonal projection of $\mathbb{R}^2$ on the the line through the origin making angle θ with the x-axis. We are interested in quantifying the dimension of points $x \in p_\theta(C_a \times C_a)$ such that $\dim_H\{y \in C_a \times C_a \,:\, p_\theta(y) = x\}$ is large.

5 Z. Buczolich (Budapest)

Let $f : \mathbb{S}^1 \to \mathbb{R}$ be measurable and for $\alpha \in (0, 2\pi)$ we let

$$M_n^\alpha f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha).$$

Setting

$$\Gamma_f = \{\alpha \in [0, 2\pi) \,:\, M_n^\alpha f(x) \text{converges for Lebesgue almost all } x\}.$$

In [2] it is shown that if Γ_f has positive Lebesgue measure then $f \in L^1$.

Question 5.1 *Let G be a compact abelian topological group with m denoting the Haar measure. If $m(\Gamma_f) >$ does it follow that $f \in L^1(m)$?*

Question 5.2 *Do there exist small sets A, for example of Hausdorff dimension zero, such that if $A \subset \Gamma_f$ then this implies $f \in L^1$?*

In [4] a non-L^1 function is constructed for which Γ_f is of Hausdorff dimension 1, but according to the above remarks, it is of zero Lebesgue measure. It would be interesting to determine the size of Γ_f for some functions with some kind of "standard singularities". For some related results concerning non L^1 functions see also the papers [26] and [27].

For $0 < t < 1$ we let

$$f(x) = \frac{1}{x|\log|x||^t}$$

for $0 < |x| < \frac{1}{2}$, setting $f(0) = 0$.

Question 5.3 *What is $\dim_H(\Gamma_f)$?*

If for the above function we still have $\dim_H(\Gamma_f) = 0$ then to obtain a Γ_f with larger Hausdorff dimension one can try to consider other functions such as

$$f(x) = \frac{1}{x|\log|x|||\log|\log|x|||^t}.$$

Let α, η irrational then by [3] there exists a non-integrable measurable function $f : \mathbb{S}^1 \to \mathbb{R}$ such that

$$M_n^\alpha f(x) \to 1$$
$$M_n^\eta f(x) \to 0$$

for Lebesgue almost all x.

Question 5.4 *Is it true that there exists a non-integrable measurable function* $f : \mathbb{S}^1 \to \mathbb{R}$ *such that for any* $c \in \mathbb{R}$ *there exists* α_c *such that the set*

$$\{x \in \mathbb{S}^1 \,:\, M_n^{\alpha_c} f(x) \to c\}$$

has full Lebesgue measure.

6 P. Shmerkin (Surrey)

Let μ be a Radon measure on $\mathbb{R}^m$. For $s \geq 0$ we define the *Riesz* s*-energy* of μ to be the quantity

$$I_s(\mu) = \int\int |x-y|^{-s} d\mu(x) d\mu(y).$$

We define the correlation dimension of μ to be

$$\dim_{cor}(\mu) = \sup\{s \,:\, I_s(\mu) < \infty\}.$$

For variants of Marstrand's projection theorem we are interested in modifying the definitions. For $x \in \mathbb{R}^m$ and $k \leq s < k+1$ we let $\phi^s(x) = z_1 z_2 \cdots z_k z_k^{s-k}$, where the $\{z_i\}_{i=1}^n$ are the absolute values of the coordinates of x written in decreasing order. We define

$$I'_s(\mu) = \int\int \phi^s(x-y)^{-1} d\mu(x) d\mu(y)$$

and let

$$\dim'(\mu) = \sup\{s \,:\, I'_s(\mu) < \infty\}.$$

It can be shown that there exists $c > 0$ such that $\phi^s(x) \leq c|x|^s$ for all $x \in \mathbb{R}^m$ and so $\dim'(\mu) \leq \dim_{cor}(\mu)$.

Question 6.1 *Does there exists a Radon measure* μ *such that* $\dim'(\mu) < \dim_{cor}(\mu)$*?*

Falconer: Might this be related to the dimension prints of C.A Rogers? (cf. [23][Appendix A])

The motivation for this question comes from the following generalisation of Marstrand's projection theorem due to Lopez-Velazquez and Moreira [18]. Let μ be a Radon measure on $\mathbb{R}^d$ and let $s = \dim'(\mu)$. Let $\pi : \mathbb{R}^d \to \mathbb{R}^k$ be a 'nice' linear map of rank k then for Lesbesgue almost all $t \in \mathbb{R}^d$ we have

$$\dim(\pi D_t(\mu)) \geq \min\{s, k\}$$

here D_t denotes the linear map $\mathrm{diag}(t_1, t_2, \ldots, t_d)$.

7 M. Rams (Warsaw)

Let $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be contractions such that f_1 and f_2 share a common fixed point while f_3 has a different fixed point.

Question 7.1 *Under the assumptions above is it true that for the associated attractor E we have that $\dim_H(E) < \dim_{sim}(E)$?*

Note that if f_1 and f_2 are both linear then they commute and the conclusion above holds.

8 M. Hochmann (Princeton)

A result due to Furstenberg [12] states that if $X \subset [0,1]$ is such that $f_2 X = f_3 X = X$ then X is either finite or $X = [0,1]$. Thus if $f_2 X_2 = X_2$ and $f_3 X_3 = X_3$ and X_2, X_3 are infinite and not $[0,1]$ then $X_2 \neq X_3$.

Question 8.1 *Let $X_2, X_3 \subset [0,1]$ be such that $f_2 X_2 = X_2$ and $f_3 X_3 = X_3$. If $\emptyset \neq I \subset [0,1]$ is such that $\# I \cap X_2 = \infty$ then does it follow that $I \cap X_2 \neq I \cap X_3$?*

We know that this in the case that one supports an ergodic measure μ of positive entropy with $\mu(I) > 0$, this follows from the Rudolph-Johnson [14, 24] theorem. If one of X_2 or X_3 is minimal then the problem reduces to Furstenberg's result.

Next, suppose that μ is a probability measure on $[0,1]$. We let μ_t denote a scaled copy of μ. As an example we could take $\mu = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}$ and $\mu_t = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{t\}}$. We set $\nu = \int_0^1 \mu_t dt$. Then $\nu = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\mathcal{L}|_{[0,1]}$.

Question 8.2 *What conditions can be placed on μ or the translated part $t \mapsto \mu_t$ so that ν is large (e.g. $\dim(\nu) = 1$).*

9 K. Nair (Liverpool)

Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence with $a_n \to \infty$ and set

$$D(N,x) = \sup_{I \subset [0,1]} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_I(\{a_n x\}) - \mathrm{Leb}(I) \right|.$$

It can be shown that for Lebesgue almost all x we have

$$D(N,x) = o(N^{-1/2}(\log(N))^{3/2}(\log\log(N))^{-1+\epsilon}).$$

For the sequence $a_n = o(n^p)$, $p > 1$ if we set

$$E_q = \{x : \limsup_{N\to\infty} N^q D(N,x) > 0\}$$

then for $q \in (0, 1/2)$ we have

$$\dim_H(E_q) \leq 1 - \frac{1-2q}{p+q}.$$

Question 9.1 *Is this the best possible?*

Useful references are Baker [1] and Nair [21].

This uses the Erdös-Turan sequence

$$ND(x_1, x_2, \ldots, x_N) \leq c(\frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} \sum_{n=1}^{N} e^{2\pi i h x_i}).$$

Other ingredients are Frostman's lemma, the large sieve inequality and the Koksma norm inequality.

For a positive integer n we let $d(n) = \#\{m \in \mathbb{N} : m|n\}$ we want to estimate the sum $\sum_{n\leq x} d(n)$. Works of A.G. Abercombie, W.D. Banks and I. Shaparlinski are relevant to this. Let $B_\alpha = \{[n\alpha] : n \in \mathbb{N}\} \subset \mathbb{N}$ and we set

$$\begin{aligned} S_\alpha(f,x) &= \sum_{n\in B_\alpha\cap\{1,2,\ldots,x\}} f(n) \\ S(f,x) &= \sum_{n\leq x} f(n) \\ \Delta_\alpha(f,x) &= \left|S_\alpha(f,x) - \alpha^{-1}S(f,x)\right| \\ M(f,x) &= 1 + \max\{|f(n) : n \leq x\}. \end{aligned}$$

Then $\Delta_\alpha(f,x) = O(M(f,x)x^{2/3+\epsilon})$ for almost all α. If f satisfies $f(mn) = f(m)f(n)$ whenever $\gcd(m,n) = 1$. For example $f(n) = F(T^n(x))$ for $F \in L^\infty(X, \mathcal{B}, \mu, T)$.

10 A.H. Fan (Picardie)

Fan posed a question related to "Coupon collections".

11 L. Liao (LAMA)

For $x \in \mathbb{Q}$, $\mu > 1$ a result of Schmeling and Troubetzkoy [28] states that

$$\dim_H\{y \in [0,1] : \|nx - y\| < n^{-\mu} \text{ infinitely often}\} = \mu^{-1}.$$

Question 11.1 *Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be such that $\phi \searrow 0$ and* $\limsup_{n\to\infty} n\phi(n) < 1/2$. *What is the dimension of the set*

$$\{y \in [0,1] : \|nx - y\| < \phi(n) \textit{ infinitely often}\}?$$

Now suppose that x is of bounded type, i.e. if we write

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

then the sequence $(a_n)_n$ is bounded. A result of Fan and Wu [11] states that

$$\dim_H\{y \in [0,1] : \|nx - y\| < \phi(n) \text{ infinitely often}\} = \lim_{n\to\infty} \frac{\log(n)}{-\log(\phi(n))}.$$

12 F. Przytycki (Warsaw)

Consider $f_0 : z \mapsto z^2$ on $\mathbb{S}^1$. Let f be a small smooth perturbation of f_0 and let $J(f)$ denote the associated f-invariant Jordan curve close to the unit circle.

Question 12.1 *Is it possible that*

$$\sup\{\dim_H(\mu) : \mu \textit{ is } f - \textit{invariant probability measure on } J(f)\} < 1?$$

13 M. Urbanski (UNT)

Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2 + c$. Let J_c denote the Julia set of f.

Question 13.1 *Is it true that*

$$\sup\{\dim_H(\mu) : \mu \textit{ is } f\textit{-invariant}\} = \dim_H(J_c)?$$

14 S. Van Strien (Warwick)

Let M denote a smooth Riemannian manifold and m a volume form.

Question 14.1 *How exceptional is it for a diffeomorphism $f : M \to M$ to have a hyperbolic fixed point p such that*

$$m\left\{x \in M \,:\, \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^i(x)} \to \delta_{\{p\}}\right\} > 0?$$

15 A.H. Fan (Picardie)

Let (X, d) denote a compact metric space and μ a Borel probability measure with support in X. Recall the definition of the Riesz s-energy of μ

$$I_s(\mu) = \int\int \frac{1}{d(x,y)^s} d\mu(x) d\mu(y).$$

For $x \in X$ we define the s-potential at x to be

$$U_s(x) = \int \frac{1}{d(x,y)^s} d\mu(y).$$

Question 15.1 *Compute*

$$\dim_H\{x \,:\, U_s(x) = \infty\}.$$

The s-dimensional capacity of this set is zero (perhaps this follows from Matilla [19][Theorem 8.7] ?). Kaufmann showed that $\mathcal{H}^{s+\epsilon}\{x \,:\, U_s(x) = \infty\} = 0$ for all $\epsilon > 0$ which implies that $\dim_H\{x \,:\, U_s(x) = \infty\} \leq s$.

A result of Fan and Wu states that

$$\sup_{r>00} \frac{\mu(B_r(x))}{r^s} \leq U_s(x) \leq c_1 \sup_{r>0} \frac{\mu(B_r(x))}{r^s} + c_2.$$

If μ denotes a Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ then one can show that

$$U_s(x) \approx \sum_{k=1}^{\infty} 2^{sk} \mu(I_k(x)).$$

16 K. Falconer (St. Andrews)

Let $f_1, f_2, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}^n$ be a family of contracting similarities and $E \subset \mathbb{R}^n$ the associated attractor. In [9] it is proved that $\dim_H(E) = \dim_B(E)$.

We say that a compact set $F \subset \mathbb{R}^n$ is *sub self-similar* if

$$F \subset \bigcup_{i=1}^{m} f_i(F).$$

If the iterated function system satisfies the open set condition then $\dim_H(F) = \dim_B(F)$ (cf. [10]).

Question 16.1 *Is the open set condition a necessary assumption for* $\dim_H(F) = \dim_B(F)$*?*

17 K. Simon (Budapest)

Fix a positive integer $m \geq 2$ and let $0 < p_0, p_1, \ldots, p_{m-1} < 1$. Starting with the unit interval, we remove the interval $[i/m, (i+1)/m]$ with probability p_i. We repeat this process to the remaining intervals repeating ad infinitum. Providing $\sum_{i=0}^{m-1} p_i > 1$ the random construction will be almost surely non-empty.

For such a construction Dekking and Simon [5] and Mora, Simon and Solomyak [20] proved that we have the following dichotomy almost surely we have either $C_1 - C_2$ is either

- small, i.e. $\mathrm{Leb}(C_1 - C_2) = 0$.
- large, i.e. $C_1 - C_2$ contains an interval.

Question 17.1 *Can one do the same with three random constructions? i.e.* $C_1 + C_2 + C_3$ *is either small or large.*

This is presumably related to the Palis conjecture?

Define the cross-correlations:

$$\gamma_k := \sum_{i=0}^{m-1} p_i p_{(i+k)(\mod m)}.$$

Then if $\gamma_i > 0$ for $i = 0, 1, \ldots, m-1$ we have that $C_2 - C_1$ contains an interval. If there exists i such that $\gamma_i, \gamma_{i+1} < 1$ then $C_2 - C_1$ does not contain an interval.

Shmerkin: if all the probabilities are equal then it follows from the work of Peres et al.

Fraser: What happens if we define it with different measures?

Now let $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be a family of contracting similarities and Λ the associated attractor. We suppose further that for each $i \neq j$ we have that $f_i(\Lambda) \cap f_j(\Lambda)$.

Question 17.2 *Does the open set condition hold?*

18 A. Ferguson (Warwick)

Let $\{f_i(x) = r_i x + a_i\}_{i=1}^m$ denote a family of contracting similarities of $\mathbb{R}$ with contraction ratios $r_1, r_2, \ldots, r_m$. We assume that $f_i \neq f_j$ for all $i \neq j$ and that $\sum_{i=1}^m r_i \leq 1$.

If the iterated function system $\{f_i\}_{i=1}^m$ satisfies the open set condition then a result of Hutchinson [13] states that the Hausdorff dimension of the associated attractor Λ is given by Moran's formula, i.e. $\dim_H(\Lambda) = s$ where s is the unique real number satisfying

$$\sum_{i=1}^{m} |r_i|^s = 1.$$

We shall refer to the unique real number satisfying Moran's equation as the similarity dimension and denote it by $\dim_{sim}(\Lambda)$.

For $\underline{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ we let $\Lambda_{\underline{a}}$ denote the attractor of the iterated function system $\{f_i(x) = r_i x + a_i\}_{i=1}^m$. Falconer [8] considered the typical behaviour of the dimension, proving that

$$\dim_H(\Lambda_{\underline{a}}) = \dim_{sim}(\Lambda_{\underline{a}})$$

for Lebesgue almost all $\underline{a} \in \mathbb{R}^m$. Another random construction that overcomes the limitations of the open set condition was considered by Simón and Pollicott [22] who used the method of transversality to solve the $\{0, 1, 3\}$-problem.

In another direction, a long standing conjecture in the field (cf. [25]) that

$$\dim_H(\Lambda) < \dim_{sim}(\Lambda)$$

implies that there exist $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l \in \{1, 2, \ldots, m\}$ such that

$$f_{i_1} f_{i_2} \cdots f_{i_k} = f_{j_1} f_{j_2} \cdots f_{j_l}.$$

Such a condition is easily seen to be sufficient to cause non coincidence of Hausdorff and similarity dimensions.

Question 18.1 *Let $f_{m+1} : \mathbb{R} \to \mathbb{R}$ be a contracting similarity so that the iterated function system $\{f_i\}_{i=1}^{m+1}$ satisfies the standing assumptions outlined at the beginning, let Lambda$'$ denote the associated attractor. Does $\dim_H(\Lambda) < \dim_{sim}(\Lambda)$ imply that $\dim_H(\Lambda') < \dim_{sim}(\Lambda')$?*

A related problem concerning the dimension of measures due to A. Mathé is as follows. For

$$\underline{p} \in \Delta := \{\underline{q} = (q_1, q_1, \ldots, q_m) \,:\, q_i \geq 0, \sum_{i=1}^{m} q_i = 1\}$$

we let $\mu_{\underline{p}}$ denote the unique probability measure satisfying

$$\mu_{\underline{p}} = \sum_{i=1}^{m} p_i (f_i)_* (\mu_{\underline{p}}).$$

If the iterated function system $\{f_i\}_{i=1}^{m}$ satisfies the open set condition then the measure $\mu_{\underline{p}}$ is easily shown to have dimension

$$s(\underline{p}) = \frac{\sum_{i=1}^{m} p_i \log(p_i)}{\sum_{i=1}^{m} p_i \log(r_i)}.$$

Question 18.2 *If there exists $\underline{p} \in \mathrm{int}(\Delta)$ such that $\dim_H(\mu_{\underline{p}}) = s(\underline{p})$ does it follow that $\dim_H(\mu_{\underline{q}}) = s(\underline{q})$ for all $\underline{q} \in \Delta$?*

Question 18.3 *Relate questions (18.1) and (18.2).*

19 M. Rams (Warsaw)

Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_1(x) = \frac{x}{2}$$
$$f_2(x) = \frac{3x+1}{2}.$$

And let μ denote the unique Borel probability measure satisfying

$$\mu = \frac{1}{2}\left((f_1)_*(\mu) + (f_2)_*(\mu)\right).$$

The iterated function system $\{f_1, f_2\}$ is contracting on average with respect to this measure.

Question 19.1 *Is the measure μ absolutely continuous?*

20 E. Järvenpää

Let M denote a smooth closed Riemannian manifold of dimension m. Let $\phi_t : T^1M \to T^1M$ denote the geodesic flow on the unit tangent bundle. Let μ denote a ϕ_t ergodic and invariant measure. Let $\pi : T_1M \to M$ denote the canonical projection, i.e. $\pi(x, v) = x$. A theorem of Ledrappier and Lindenstrauss [17] states that

$$\dim_H(\pi_*(\mu)) = \min\{\dim_H(\mu), m\}.$$

Moreover, if $\dim_H(\mu) > 2$ then $\pi_*(\mu)$ is absolutely continuous.

Question 20.1 *What happens when the dimension is equal to* 2*?*

Let $\Delta\psi_n = \lambda_n\psi_n$ denote the eigenvalues of the Laplacian.

Question 20.2 (Quantum Unique Ergodicity) *Is it true that*

$$|\psi_n|^2\text{Leb} \to \text{Leb}?$$

References

[1] R. C. Baker. Metric number theory and the large sieve. *J. London Math. Soc. (2)*, 24(1):34–40, 1981.

[2] Z. Buczolich. Arithmetic averages of rotations of measurable functions. *Ergodic Theory Dynam. Systems*, 16(6):1185–1196, 1996.

[3] Z. Buczolich. Ergodic averages and free $\mathbf{Z}^2$ actions. *Fund. Math.*, 160(3):247–254, 1999.

[4] Z. Buczolich. Non-L^1 functions with rotation sets of Hausdorff dimension one. *Acta Math. Hungar.*, 126(1-2):23–50, 2010.

[5] M. Dekking and K. Simon. On the size of the algebraic difference of two random Cantor sets. *Random Structures Algorithms*, 32(2):205–222, 2008.

[6] A. Douady and J. Oesterlé. Dimension de Hausdorff des attracteurs. *C. R. Acad. Sci. Paris Sér. A-B*, 290(24):A1135–A1138, 1980.

[7] P. Erdös. On sets of distances of n points. *Amer. Math. Monthly*, 53:248–250, 1946.

[8] K. J. Falconer. The Hausdorff dimension of some fractals and attractors of overlapping construction. *J. Statist. Phys.*, 47(1-2):123–132, 1987.

[9] K. J. Falconer. Dimensions and measures of quasi self-similar sets. *Proc. Amer. Math. Soc.*, 106(2):543–554, 1989.

[10] K. J. Falconer. Sub-self-similar sets. *Trans. Amer. Math. Soc.*, 347(8):3121–3129, 1995.

[11] A.-H. Fan and J. Wu. A note on inhomogeneous Diophantine approximation with a general error function. *Glasg. Math. J.*, 48(2):187–191, 2006.

[12] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory*, 1:1–49, 1967.

[13] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[14] A. S. A. Johnson. Measures on the circle invariant under multiplication by a nonlacunary subsemigroup of the integers. *Israel J. Math.*, 77(1-2):211–240, 1992.

[15] N. H. Katz and L. Guth. On the erdos distinct distance problem in the plane. *preprint: http://arxiv.org/abs/1011.4105*, 2010.

[16] N. H. Katz and G. Tardos. A new entropy inequality for the Erdős distance problem. In *Towards a theory of geometric graphs*, volume 342 of *Contemp. Math.*, pages 119–126. Amer. Math. Soc., Providence, RI, 2004.

[17] F. Ledrappier and E. Lindenstrauss. On the projections of measures invariant under the geodesic flow. *Int. Math. Res. Not.*, (9):511–526, 2003.

[18] J. E. López Velázquez and C. G. Moreira. A variant of marstrand's theorem for projections of cartesian products. *preprint: http://arxiv.org/abs/1106.5776*, 2011.

[19] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[20] P. Móra, K. Simon, and B. Solomyak. The Lebesgue measure of the algebraic difference of two random Cantor sets. *Indag. Math. (N.S.)*, 20(1):131–149, 2009.

[21] R. Nair. Some theorems on metric uniform distribution using L^2 methods. *J. Number Theory*, 35(1):18–52, 1990.

[22] M. Pollicott and K. Simon. The Hausdorff dimension of λ-expansions with deleted digits. *Trans. Amer. Math. Soc.*, 347(3):967–983, 1995.

[23] C. A. Rogers. *Hausdorff measures*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1970 original, With a foreword by K. J. Falconer.

[24] D. J. Rudolph. $\times 2$ and $\times 3$ invariant measures and entropy. *Ergodic Theory Dynam. Systems*, 10(2):395–406, 1990.

[25] K. Simon. Overlapping cylinders: the size of a dynamically defined Cantor-set. In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 259–272. Cambridge Univ. Press, Cambridge, 1996.

[26] Y. G. Sinai and C. Ulcigrai. A limit theorem for Birkhoff sums of non-integrable functions over rotations. In *Geometric and probabilistic structures in dynamics*, volume 469 of *Contemp. Math.*, pages 317–340. Amer. Math. Soc., Providence, RI, 2008.

[27] Y. G. Sinai and C. Ulcigrai. Renewal-type limit theorem for the Gauss map and continued fractions. *Ergodic Theory Dynam. Systems*, 28(2):643–655, 2008.

[28] S. Trubetskoĭ and Ĭ. Shmeling. Inhomogeneous Diophantine approximations and angular recurrence for billiards in polygons. *Mat. Sb.*, 194(2):129–144, 2003.