Statistical stability for Lorenz-like attractors

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Ergodic Theory and Dynamical Systems:
Perspectives and Prospects

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Lorenz (1963) proposed a system of differential equations $X$ in $\mathbb{R}^3$ having an attractor with sensitive dependence on the initial conditions.
A geometric model for Lorenz equations was introduced in the seventies by Guckenheimer and Williams.
The vector field $X$ is linear in a neighborhood of the singularity $(0, 0, 0)$ whose eigenvalues satisfy

$$0 < -\lambda_3 < \lambda_1 < -\lambda_2,$$
There is a cross-section $\Sigma$ intersecting the stable manifold of the singularity along a curve $\Gamma$. 
Local behaviour

\[ \tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x| \]
\[ \tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x| + T_0, \]
The geometric model admits a **Lorenz-like attractor** $\Lambda$:

- $\Lambda$ is an invariant set under the flow;
- there is an open neighborhood $U$ of $\Lambda$ such that
  
  $$\Lambda = \bigcap_{t>0} X_t(U);$$

- $\Lambda$ contains a dense orbit;
- sensitive dependence on the initial conditions in $U$;
- $\Lambda$ contains the singularity $O$.

$\Lambda$ is a **singular-hyperbolic** attractor.
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The attractor

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$\Lambda$ is a **singular-hyperbolic** attractor.
The return map $P$ admits a stable foliation $\mathcal{F}$ on $\Sigma$ with the properties:

- **invariant**: the image by $P$ of a leaf $\xi$ in $\Sigma$ distinct from $\Gamma$ is contained in another stable leaf;

- **contracting**: the diameter of $P^n(\xi)$ goes to zero when $n \to \infty$, uniformly over all leaves;

- it induces a map $f$ on the quotient space $\Sigma/\mathcal{F} \sim [-1, 1] = I$.

The foliation $\mathcal{F}$ is $C^1$-Hölder when the vector field $X$ is $C^2$. Assuming the strong dissipative condition at the equilibrium

$$\frac{-\lambda_2}{\lambda_1} > \frac{-\lambda_3}{\lambda_1} + 2,$$

then $\mathcal{F}$ is $C^2$, and the one-dimensional quotient map $f$ is $C^2$ smooth away from the singularity.
Poincaré return map

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Lorenz map

- $f$ is discontinuous at $x = 0$ and
  \[ \lim_{x \to 0^+} f(x) = -1, \quad \lim_{x \to 0^-} f(x) = 1; \]

- $f$ is differentiable on $I \setminus \{0\}$ and
  \[ f'(x) > \sqrt{2}, \quad \text{for all } x \in I \setminus \{0\}; \]

- the derivative tends to infinity near 0
  \[ \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) = +\infty. \]
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There is a $C^2$ neighborhood $U$ of $X$ such that for each $Y \in U$
- $U$ is a trapping region containing the cross-section $\Sigma$ of $Y$;
- the maximal positively invariant subset $\Lambda_Y = \bigcap_{t \geq 0} Y^t(U)$ inside $U$ is a Lorenz-like attractor;
- the first return Poincaré map $P_Y$ on $\Sigma$ admits a $C^2$ uniformly contracting foliation $\mathcal{F}_Y$.
- the induced one-dimensional quotient map $f_Y = P_Y / \mathcal{F}_Y$ is a $C^2$ Lorenz map;
- there exist (unique) SRB measures for the Lorenz map $f_Y$, the Poincaré map $P_Y$ and the flow $Y$ on $U$.

**Theorem (Tucker)**

For the classical parameter values, the Lorenz equations support a robust strange attractor.
Robustness

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*For the classical parameter values, the Lorenz equations support a robust strange attractor.*
SRB measures

Theorem

Each Lorenz map $f_Y$ has a unique ergodic acip $\bar{\mu}_Y$ whose density wrt Lebesgue has bounded variation.

$\bar{\mu}$ is an SRB measure: for Lebesgue almost every $x \in I$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\bar{\mu},$$

for any continuous function $\varphi : I \to \mathbb{R}$.

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Each Lorenz-like attractor \( Y \) supports a unique SRB measure \( \mu_Y \)
Statistical stability: continuous variation of the SRB measures with weak* topology as a function of the dynamical system.

Strong statistical stability: continuous variation of the densities of the SRB measures in the $L^1$-norm.

Theorem (Keller)
Lorenz maps are strongly statistically stable.

Theorem (A., Soufi)
Lorenz-like flows are statistically stable.
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SRB measures for the Poincaré return maps

Given a bounded function $\phi : \Sigma \to \mathbb{R}$, define

$$\phi_+(x) := \sup_{y \in \xi(x)} \phi(y) \quad \text{and} \quad \phi_-(x) := \inf_{y \in \xi(x)} \phi(y),$$

where $\xi(x)$ is the leaf in foliation $\mathcal{F}$ which contains $x$.

**Lemma**

Given any continuous function $\phi : \Sigma \to \mathbb{R}$ both limits

$$\lim_{n \to \infty} \int (\phi \circ P^n)_- d\bar{\mu} \quad \text{and} \quad \lim_{n \to \infty} \int (\phi \circ P^n)_+ d\bar{\mu}$$

exist and they coincide.
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exist and they coincide.
**Corollary**

There is a (unique) probability $P$-invariant measure $\tilde{\mu}$ on $\Sigma$ such that

$$\int \phi \ d\tilde{\mu} = \lim_{n \to \infty} \int (\phi \circ P^n)_- d\tilde{\mu} = \lim_{n \to \infty} \int (\phi \circ P^n)_+ d\tilde{\mu},$$

for every continuous function $\phi : \Sigma \to \mathbb{R}$.

**Theorem**

The Lorenz-like attractor supports a unique SRB measure $\mu$ defined for any continuous function $\varphi : \tilde{U} \to \mathbb{R}$ as

$$\int \varphi \ d\mu = \frac{1}{\int \tau d\tilde{\mu}} \int \int_0^{\tau(x)} \varphi(X(x, t)) dt d\tilde{\mu}(x)$$
**Corollary**

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Proposition

If $X_n$ is a sequence converging to $X_0$ in $C^2$ topology, then

$$\tilde{\mu}_n \rightarrow \tilde{\mu}_0 \quad \text{in weak}^* \text{ topology},$$

where $\tilde{\mu}_n = \tilde{\mu}_{X_n}$ for all $n \geq 0$.

We need to show that for any continuous $\varphi : \Sigma \rightarrow \mathbb{R}$ we have

$$\lim_{n \to \infty} \int \varphi d\tilde{\mu}_n = \int \varphi d\tilde{\mu}_0.$$

By definition

$$\lim_{n \to \infty} \int \varphi d\tilde{\mu}_n = \lim_{n \to \infty} \lim_{m \to \infty} \int \inf(\varphi \circ P^m_n) \, d\tilde{\mu}_n.$$
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We have

\[
\left| \int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_0 \right| \leq \\
\left| \int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n \right| \\
+ \left| \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_0 \right|.
\]

The second term tends to zero because

\[
\bar{\mu}_n \overset{\text{weak}^*}{\longrightarrow} \bar{\mu}_0.
\]

We are left to prove that the first term converges to zero when \( n \to \infty \).
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Letting $\lambda = \text{Lebesgue}$

\[
\left| \int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n \right|
\]

\[
= \left| \int \inf(\varphi \circ P_n^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda - \int \inf(\varphi \circ P_0^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda \right|
\]

\[
\leq \int \left| \inf(\varphi \circ P_n^m) - \inf(\varphi \circ P_0^m) \right| \left| \frac{d\bar{\mu}_n}{d\lambda} \right| d\lambda
\]

\[
\leq C \int \left| \inf(\varphi \circ P_n^m) - \inf(\varphi \circ P_0^m) \right| d\lambda
\]

The rate of the contraction of the stable foliation on the cross-section can be taken the same for all vector fields. So, the last expression can be made uniformly small.
Letting $\lambda = \text{Lebesgue}$

$$ | \int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n | $$

$$ = \left| \int \inf(\varphi \circ P_n^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda - \int \inf(\varphi \circ P_0^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda \right| $$

$$ \leq \int |\inf(\varphi \circ P_n^m) - \inf(\varphi \circ P_0^m)| \left| \frac{d\bar{\mu}_n}{d\lambda} \right| d\lambda $$

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Let $X_n$ be any sequence converging to $X_0$ in $C^2$ topology. Then

$$\mu_n \longrightarrow \mu_0,$$

in the weak* topology.

\[
\left| \int \varphi \ d\mu_n - \int \varphi \ d\mu_0 \right| \text{ is bounded by the sum of the terms}
\]

\[
\left| \frac{1}{\int \tau_n d\tilde{\mu}_n} - \frac{1}{\int \tau_0 d\tilde{\mu}_0} \right| \int_0^{\tau_0(x)} \int_0^{\tau_0(x)} |\varphi(X_0(x, t))| \, dt \, d\tilde{\mu}_0(x),
\]

and

\[
\left| \frac{1}{\int \tau_n d\tilde{\mu}_n} \int_0^{\tau_n(x)} \varphi(X_n(x, t)) \, dt \, d\tilde{\mu}_n - \int_0^{\tau_0(x)} \varphi(X_0(x, t)) \, dt \, d\tilde{\mu}_0 \right|.\]
Statistical stability for the flow

Theorem

Let $X_n$ be any sequence converging to $X_0$ in $C^2$ topology. Then

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\mu_n \longrightarrow \mu_0, \quad \text{in the weak}^* \text{ topology}.
$$

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The statistical stability of the Poincaré return map and the fact that there are uniform constants \( a, a_0, b \) and \( b_0 \) such that

\[
b_0 - a_0 \log |x - c_n| \leq \tau_n(x, y, 1) \leq b - a \log |x - c_n|,
\]

where the \( c_n \) is the discontinuity point of the map \( f_{X_n} \), yield

**Lemma**

\[
\lim_{n \to +\infty} \int \tau_n \, d\tilde{\mu}_n = \int \tau_0 \, d\tilde{\mu}_0
\]

And defining

\[
h_n(x) = \int_0^{\tau_n(x)} \varphi(X_n(x, t)) \, dt, \quad \text{for } n \geq 0
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**Lemma**

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Replace the usual expanding condition $\lambda_3 + \lambda_1 > 0$ in the Lorenz-like attractor $X$ by the **contracting condition**

$$\lambda_3 + \lambda_1 < 0.$$ 

There is a trapping region $U$ for $X_0$ on which $\Lambda = \cap_{t \geq 0} X_0^t(U)$ is a singular-hyperbolic attractor attractor.

$\Lambda$ is 2-dimensionally almost persistent in the $C^3$ topology: $X$ is a 2-dimensional density point of the set of vector fields $Y$ for which $\Lambda_Y = \cap_{t \geq 0} Y^t(U)$ is an attractor.
Rovella flow

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The quotient map $f_0 : I \setminus \{0\} \to I$ satisfies

- $\lim_{x \to 0^\pm} f_0(x) = \mp 1$;
- $\pm 1$ are pre-periodic and repelling;
- $f_0$ is of class $C^3$ on $I \setminus \{0\}$ with negative Schwarzian derivative;

Figure: One-dimensional map
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**Figure:** One-dimensional map
Theorem (Rovella)

There is $E \subseteq [0, 1]$ with full density at 0 such that:

1. for all $a \in E$, $f_a$ is of class $C^3$ on $x \in I \setminus \{0\}$ and satisfies
   \[ K_2|x|^{s-1} \leq f_a'(x) \leq K_1|x|^{s-1}; \]

2. there exists $\lambda > 1$ such that for $a \in E$
   \[ (f_a^n)'(\pm 1) > \lambda^n \quad \text{for all } n \geq 0; \]

3. there is $\alpha > 0$ such that for all $a \in E$
   \[ |f_a^{n-1}(\pm 1)| > e^{-\alpha n} \quad \text{for all } n \geq 1 \]

Theorem (Metzger)

Each $f_a$ admits an absolutely continuous invariant probability $\mu_a$ which is unique and ergodic.
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Each $f_a$ admits an absolutely continuous invariant probability $\mu_a$ which is unique and ergodic.
Assume $f$ is nonuniformly expanding:

$$\exists c > 0 : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(f'(f^i(x))) > c, \quad \text{Lebesgue a.e. } x$$

with slow recurrence to the critical set:

$$\forall \epsilon > 0 \exists \delta > 0 : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log d_\delta(f^i(x), \mathcal{C}) \leq \epsilon, \quad \text{Lebesgue a.e. } x$$

where $d_\delta$ is the $\delta$-truncated distance is defined as

$$d_\delta(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leq \delta, \\ 1 & \text{if } |x - y| > \delta. \end{cases}$$
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This allows us to introduce the expansion time

\[ \mathcal{E}^f(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log f'(f^i(x)) > d, \forall n \geq N \right\} \]

the recurrence time

\[ \mathcal{R}^f(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(f^i(x), C) < \epsilon, \forall n \geq N \right\} \]

and the tail set at time \( n \)

\[ \Gamma^f_n = \left\{ x \in I : \mathcal{E}^f(x) > n \text{ or } \mathcal{R}^f(x) > n \right\}. \]

**Theorem (A.)**

Assume there are \( C > 1 \) and \( \gamma > 1 \) such that \( |\Gamma^f_n| \leq Cn^{-\gamma} \) for all \( n \geq 1 \) and \( f \in \mathcal{F} \). Then, each \( f \in \mathcal{F} \) is strongly statistically stable.
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\]

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\[
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**Theorem (A., Soufi)**

Rovella maps are nonuniformly expanding with slow recurrence to the critical set. Moreover, there are $C, \tau > 0$ such that for all $n \in \mathbb{N}$ and $a \in E$,

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Corollary

Rovella family is strongly statistically stable
Problems

1. Statistical (in)stability in the full set of parameters.
2. Statistical stability for Rovella flows.
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Thank you!