Rigidity Conjecture for C^3 Critical Circle Maps

joint with Pablo Guarino

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Definition

Critical circle map: orientation-preserving C^3 circle homeomorphism, with exactly one critical point of odd type.

We will focus on the case of **irrational** rotation number (no periodic orbits).

Topological Rigidity (Yoccoz 1984)

Any C^3 critical circle map f with $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$ is **minimal**.

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Rigidity Conjecture

Any two C^3 critical circle maps with the same irrational rotation number of **bounded type** are conjugate by a $C^{1+\alpha}$ circle diffeomorphism.

Recall that θ in $[0,1]$ is of *bounded type* if $\exists\ \varepsilon > 0$:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{\varepsilon}{q^2},$$

for any positive coprime integers p and q.

The set $\mathcal{BT} \subset [0,1]$ of bounded type numbers has Hausdorff dimension equal to 1, but Lebesgue measure equal to zero.

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Theorem (de Faria-de Melo, Yampolsky, Khanin-Teplinsky)

Let f and g be two critical circle maps such that:

- f and g are **real-analytic**.
- $\rho(f) = \rho(g) \in \mathbb{R} \setminus \mathbb{Q}$.

Let h be the conjugacy between f and g that maps the critical point of f to the critical point of g. Then:

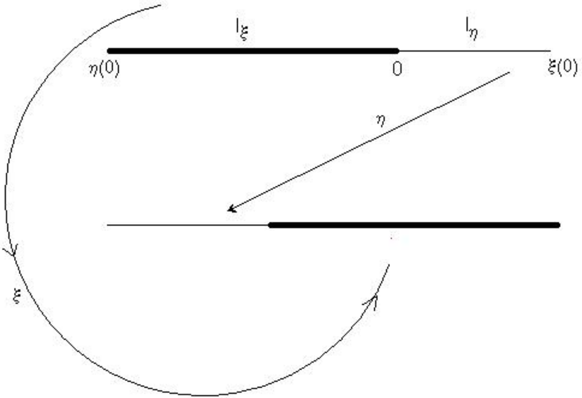
- h is a C^1 diffeomorphism.
- h is $C^{1+\alpha}$ in the critical point of f for a universal $\alpha > 0$.
- For a full Lebesgue measure set of rotation numbers (that contains all bounded type numbers), h is globally $C^{1+\alpha}$.

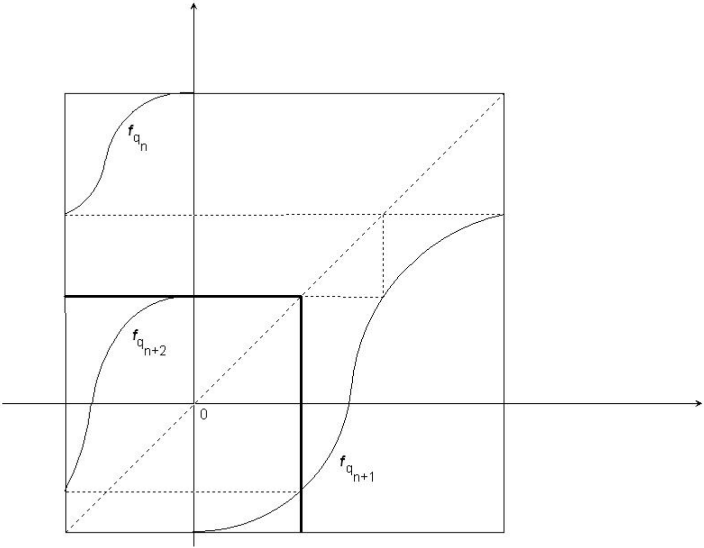
Main Theorem: Rigidity Conjecture for C^3 Critical Circle Maps.

Any two C^3 critical circle maps with the same irrational rotation number of **bounded type** are conjugate by a $C^{1+\alpha}$ circle diffeomorphism, for some $\alpha > 0$ that only depends on the rotation number.

Main tools: Renormalization operator and asymptotically holomorphic maps.

- $f^{q_n}(c)$ closest approach to c, $\mathcal{R}^n$ first return to the interval $[f^{q_n}(c), f^{q_{n+1}}(c)] \ni c$, normalized: critical commuting pair.
- Renormalization operator acting on the space of normalized critical commuting pairs.
- Lanford: critical commuting pairs $\mapsto$ smooth conjugacy class of critical circle maps.


I_ξ
I_η
$\eta(0)$
0
$\xi(0)$
η
ξ


f_{q_n}
$f_{q_{n+2}}$
0
$f_{q_{n+1}}$

Theorem (de Faria-de Melo 1999)

There exists $\mathbb{A} \subset [0,1]$ with:

- $\text{Leb}(\mathbb{A}) = 1$
- $\mathcal{BT} \subset \mathbb{A}$

such that for any two C^3 critical circle maps f and g with $\rho(f) = \rho(g) \in \mathbb{A}$ we have that if:

$$d_0\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \to 0 \quad \text{when} \quad n \to +\infty$$

exponentially fast, then f and g are $C^{1+\alpha}$ conjugate, for some $\alpha > 0$ that only depends on the rotation number.

The remaining cases: exponential convergence in the C^2-metric implies C^1-rigidity for any irrational rotation number.(Khanin-Teplinsky 2007).

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Theorem (de Faria-de Melo 2000, Yampolsky 2003)

There exists λ in $(0,1)$ such that given critical circle maps f and g such that:

- f and g are **real-analytic**, and
- $\rho(f) = \rho(g) \in \mathbb{R} \setminus \mathbb{Q}$,

there exists $C > 0$ such that for all $n \in \mathbb{N}$:

$$d_r\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \leq C\lambda^n$$

for any $r \in \{0, 1, ..., \infty\}$. The constant is uniform for f and g in a compact set.

Theorem A

Given f and g two C^3 critical circle maps with:

$$\rho(f) = \rho(g) \in \mathcal{BT},$$

there exist $C > 0$ and $\lambda \in (0, 1)$ such that for all $n \in \mathbb{N}$:

$$d_0\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \leq C\lambda^n.$$

Theorem B

There exists a C^ω-compact set $\mathcal{K}$ of real-analytic critical commuting pairs such that:

Given a C^3 critical circle map f with **any** irrational rotation number θ there exist:

- $C > 0$ and $\lambda \in (0,1)$ with $\lambda = \lambda(\theta)$, and
- $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$,

such that for all $n \in \mathbb{N}$:

- $d_0\big(\mathcal{R}^n(f), f_n\big) \leq C\lambda^n$, and
- $\rho(f_n) = \rho\big(\mathcal{R}^n(f)\big)$.

Theorem B + defaria-demelo2000 $\Longrightarrow$ Theorem A
Theorem A+ defaria-demelo1999 $\Longrightarrow$ Main Theorem

Asymptotically holomorphic maps

Let $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ be the canonical lift of a C^3 critical circle map f. There exist $R > 0$ and a C^3 map F defined in $\{|\Im(z)| < R\}$, which is an extension of $\widetilde{f}$, such that:

- $$\frac{\partial F}{\partial \bar{z}}(x) = 0 \quad \text{for every } x \in \mathbb{R},$$
- $$\frac{\frac{\partial F}{\partial \bar{z}}(z)}{\big(\Im(z)\big)^2} \to 0 \quad \text{uniformly as } \Im(z) \to 0,$$
- F commutes with unitary horizontal translation in A_R, and
- the critical points of F in A_R are the integers, and they are of cubic type.

We are able to control the iterates of the extension, by controlling the distortion of **Poincaré disks**:

Almost Schwarz inclusion (Graczyk-Sands-Świątek 2005)

Let $h : I \to J$ be a C^3 diffeomorphism between compact intervals, and let H be any C^3 extension of h to a complex neighborhood of I, which is asymptotically holomorphic of order 3 on I.

There exist $K > 0$ and $\delta > 0$ such that for any $a < b$ in I and $\theta \in (0, \pi)$:

$$\text{If} \quad diam\big(D_\theta(a,b)\big) < \delta \quad \text{then} \quad H\big(D_\theta(a,b)\big) \subseteq D_{\tilde{\theta}}\big(h(a), h(b)\big),$$

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By the **real bounds** (Herman, Świątek, 1988):

$$\sum_{j=1}^{q_{n+1}-1} \left|\widetilde{f}^j(I_n)\right|^2 < \max_{j \in \{1,\dots,q_{n+1}-1\}} \left|\widetilde{f}^j(I_n)\right|$$

goes to zero exponentially fast. For each $n \in \mathbb{N}$ we get an open interval J_n, with $\overline{I_n} \subset J_n$ and $|J_n| > (1+\varepsilon)|I_n|$, and $\{\theta_n\} \to 0$ exponentially fast such that:

$$F^{-j}\left(D_\theta((J_n))\right) \subset D_{\theta-\theta_n}\left(\widetilde{f}^{-j}(J_n)\right)$$

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Since F is C^3 we have for big n and $j \in \{0, 1, ..., q_{n+1} - 1\}$:

$$\left| \frac{\partial F}{\partial \bar{z}} (F^j(z)) \right| << \left| \widetilde{f}^j(I_n) \right|^2 \quad \text{in} \quad F^{-j}\left(D_\theta((J_n))\right).$$

and also the conformal distortion is bounded by a constant times $\left| \widetilde{f}^j(I_n) \right|^2$ By the chain rule for the $\frac{\partial}{\partial \bar{z}}$ derivative, and the control obtained via real bounds for the $\frac{\partial}{\partial z}$ derivative: The conformal distortion of $F^{q_{n+1}-1}$ is bounde by $C\lambda^n$ on the pre-image of the Poincaré disk $D_\theta((J_n))$. By controlling the distortion around the critical point, we pull-back this estimates to an $\mathbb{R}$-symmetric topological disk containing I_n.

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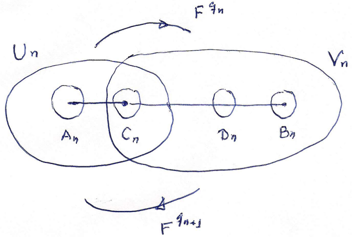
C^3 COMPLEX BOUNDS



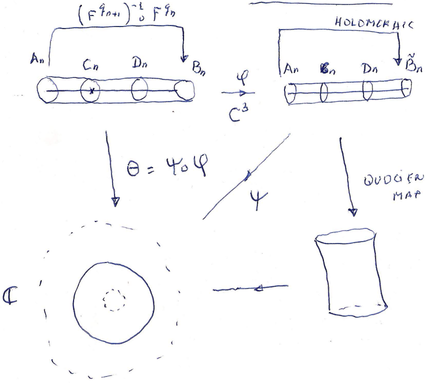
$F^{q_{n+1}}(C_n) = A_n$

$F^{q_n}(C_n) = B_n$ BRANCH COVERING

$F^{q_n}(A_n) = D_n = F^{q_{n+1}}(D_n)$ DIFFEOS

- GEOMETRIC BOUND
- EXPONENTIA SMALL CONFORMAL DISTORTION

HOLOMORPHIC APROXIMATION



$\left\{\begin{array}{l} \varphi \ C^3 \text{ EXP. SMALL CONF. DISTORTION} \\ \varphi \circ (F^{q_{n+1}})^{-1} \circ F^{q_n} \circ \varphi^{-1} |A_n : A_n \to \tilde{B}_n \end{array}\right.$

HOLOMORPHIC

$\Downarrow$

Θ EXP. SMALL CONF. DISTORTION

$F_n \stackrel{\text{def}}{=} \Theta \circ (F^{q_n}, F^{q_{n+1}}) \circ \Theta^{-1}$

ESTENSION TO THE ANULUS OF C^3 CRITICA CIRCLE MAP

$\left\{\begin{array}{l} \tilde{F}_n \text{ HOL. APP. OF } F_n \\ \tilde{\Theta} \text{ HOL. APP OF } \Theta \end{array}\right. \Rightarrow \tilde{F}_n = \tilde{\Theta} \circ (\xi_n, \eta_n) \circ \tilde{\Theta}^{-1}$

(ξ_n, η_n) HOL. APP COMM. PAIR OF $(F^{q_n}, F^{q_{n+1}})$

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