Central limit theorem for the measure of balls in non-conformal dynamics

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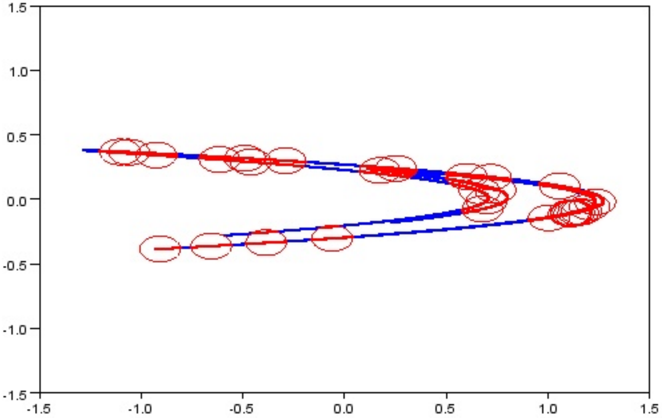
Warwick

Outline

How the measure behaves at small scales ?



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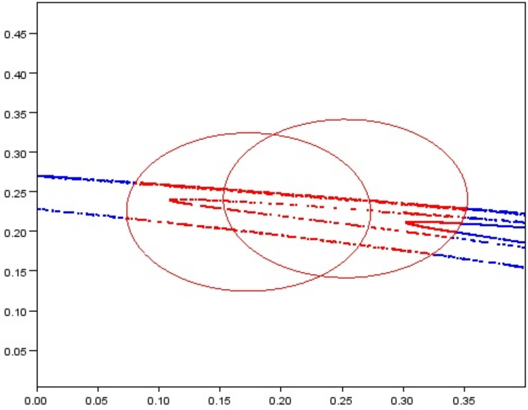

Motivation

For a smooth dynamical system and under suitable conditions, the pointwise dimension of an ergodic measure μ exists and is related to

- Hausdorff dimension
- entropy
- Lyapunov exponents

The existence of the pointwise dimension may be viewed as a Law of Large Number, it makes sense to ask for a Central Limit Theorem associated to it.

Some question related to this have been studied before, but in the conformal case: Law of Iterated Logarithm (Przytycki, Urbanski & Zdunik, Bhouri & Heurteaux)

Here we will work with non-conformal maps: $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

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Dimensions of a measure

Hausdorff dimension of a set A denoted by $\dim_H A$

Definition

Hausdorff dimension of a measure μ (Borel probability measure)

$$\dim_H \mu = \inf\{\dim_H A \colon \mu(A) = 1\}.$$

Definition

Pointwise dimension of a measure μ

$$\underline{d}_\mu(x) = \liminf_{\varepsilon\to0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}, \quad \overline{d}_\mu(x) = \limsup_{\varepsilon\to0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}$$

Proposition

For any Radon measure μ we have $\dim_H \mu = \operatorname{esssup} \underline{d}_\mu$.

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Existence of the pointwise dimension

Theorem (Ledrappier-Young 85)

Let f be a C^2 diffeomorphism of a Riemaniann manifold M and μ be an invariant measure. Then the stable and unstable pointwise dimensions $d_\mu^u(x)$ and $d_\mu^s(x)$ exists for μ-a.e. $x \in M$.

Theorem (Barreira-Pesin-Schmeling 99)

Assume additionnaly that the measure is hyperbolic (no zero Lyapunov exponents). Then the pointwise dimension $d_\mu(x)$ exists for μ-a.e. x and $d_\mu(x) = d_\mu^u(x) + d_\mu^s(x)$.

Remark

If in addition μ is ergodic then the pointwise dimension d_μ is equal to $\dim_H \mu$ μ-a.e.:

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Statement of the result

Theorem

Let $T : \mathbb{T}^d \circlearrowleft$ be a $C^{1+\alpha}$ expanding map and μ_φ be an equilibrium state of a Hölder potential φ. Suppose that T has skew product structure

$$T(x_1, \dots, x_d) = (f_1(x_1), f_2(x_1, x_2), \dots, f_d(x_1, \dots, x_d)).$$

and that the sequence of Lyapunov exponents

$$\lambda_{\mu,i} := \int \log \left| \frac{\partial f_i}{\partial x_i} \right| \circ \pi_i d\mu_\varphi, \quad i = 1, \dots, d$$

is increasing. Then there exists $\sigma \geq 0$ such that

$$\frac{\log \mu_\varphi(B(x,\varepsilon)) - \dim_H \mu_\varphi \log \varepsilon}{\sqrt{-\log \varepsilon}}$$

converges as $\varepsilon \to 0$, in distribution, to a random variable $\mathcal{N}(0, \sigma^2)$. The variance $\sigma^2 = 0$ iff μ_φ is absolutely continuous.

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Functional CLT and byproducts

Corollary (Median)

If μ_φ is not absolutely continuous then

$$\mu_\varphi(x\colon \mu_\varphi(B(x,\varepsilon)) \le \varepsilon^{\dim_H \mu_\varphi}) \to 1/2.$$

Theorem (Functional CLT or WIP)

The all process converges in distribution in the Skorohod topology:

$$N_\varepsilon(t) := \frac{\log \mu_\varphi(B(x,\varepsilon^t)) - t \dim_H \mu_\varphi \log \varepsilon}{\sqrt{-\log \varepsilon}} \to \sigma W(t)$$

where W is the standard Brownian process.

Several corollaries follow (applying continuous functions of Brownian motion paths): Arc-sine law, Maximum, minimum, etc.

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Finer structure of the invariant measure

Following Przytycki, Urbanski & Zdunik we get (using functional CLT rather than Law of iterated logarithm)

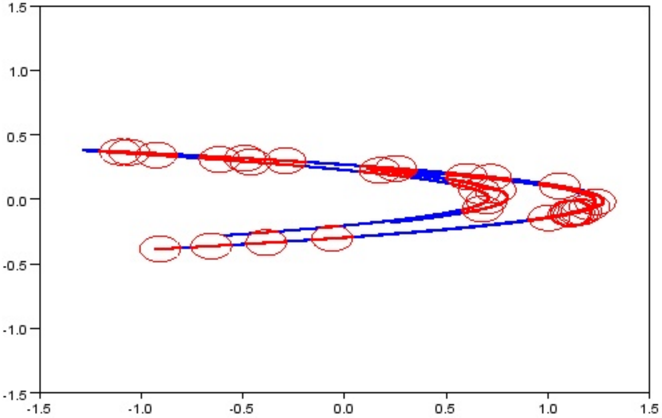
Corollary

Under the assumptions of the main theorem :

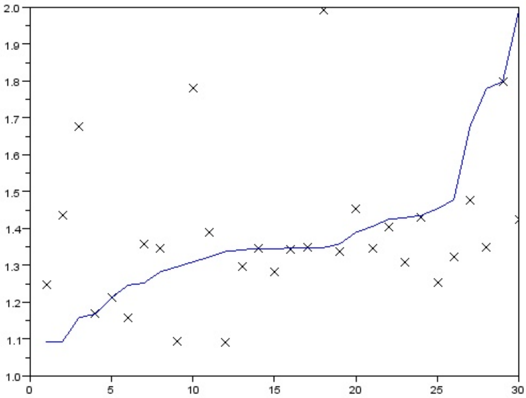
- *the measure* μ_φ
- *the Hausdorff measure in dimension* $\dim_H \mu_\varphi$

are mutually singular iff μ_φ *is not absolutely continuous wrt Lebesgue.*

Numerical (non-rigorous) illustration for Hénon map I

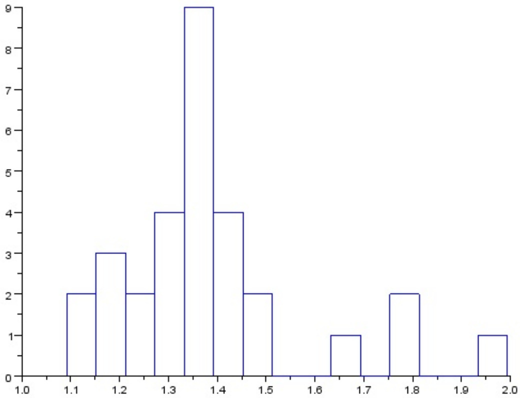


Numerical (non-rigorous) illustration for Hénon map II



$\log(\mu_\varphi(B(x_i,\varepsilon)))/\log(\varepsilon)$ for (30) randomly chosen centers x_i

Numerical (non-rigorous) illustration for Hénon map III



Histogram of $\log(\mu_\varphi(B(x,\varepsilon)))/\log(\varepsilon)$ (for $\varepsilon = 0.1$)

Notations and steps of the proof

We will do the proof in dimension $d = 2$. The map is denoted

$$T(x,y) = (f(x), g(x,y)), \quad (x,y) \in \mathbb{T}^2.$$

Projection $\pi(x,y) = x$. Lyapunov exponents

$$\lambda^u_{\mu_\varphi} = \int \log|f'| \circ \pi d\mu_\varphi < \lambda^{uu}_{\mu_\varphi} = \int \log|\frac{\partial g}{\partial y}| d\mu_\varphi.$$

Denote the dimension by $\delta := \dim_H \mu_\varphi$. Set Pressure $P(\varphi) = 0$.

Steps of the proof:

- Replace $N_\varepsilon(t)$ by $N'_\varepsilon(t)$ defined symbolically: balls $\to$ cylinders
- Replace $N'_\varepsilon(t)$ by a non-homogeneous Birkhoff sum $N''_\varepsilon(t)$
- Abstract probabilistic arguments

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A fibered partition

Lemma

There is an invariant splitting $E^u \oplus E^{uu}$ defined μ-a.e. with Lyapunov exponents λ^u and λ^{uu}.

Choose x_0 and y_0 such that $S_0 = \{x_0\} \times \mathbb{T} \cup \mathbb{T} \times \{y_0\}$ is small.
Let $\mathcal{R}_n$ be the partition into connected components of $T^{-n}(\mathbb{T}^2 \setminus S_0)$.
Let $\mathcal{P}_n = \pi\mathcal{R}_n$.
Let $F_k = \prod_{j=0}^{k-1} f' \circ f^j$ and $G_k = \prod_{j=0}^{k-1} \frac{\partial g}{\partial y} \circ T^k$.

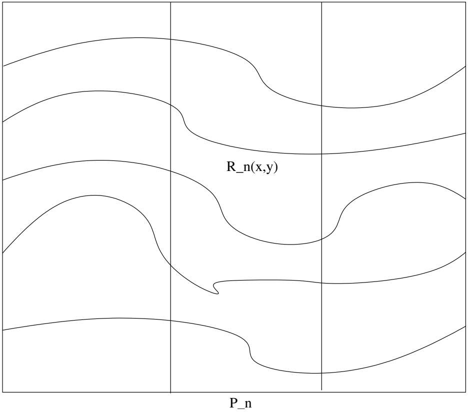
Definition

Let $\varepsilon > 0$. Define

- $n_\varepsilon(x, y)$ the largest integer n s.t. $|G_n(x, y)|\varepsilon \leq 1$
- $m_\varepsilon(x)$ the largest integer m s.t. $|F_m(x)|\varepsilon \leq 1$
- multi-temporal approximation of the ball

$$C_\varepsilon(x, y) = \mathcal{R}_{n_\varepsilon(x,y)}(x, y) \cap \mathcal{P}_{m_\varepsilon(x)}(x) \times \mathbb{T}.$$

The fibered partition $\mathcal{R}_n$



Approximation of the ball

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There exists a constant $\underline{c} < 1$, positive a.e., and a function $\overline{c}_\varepsilon > 1$, satisfying $\overline{c}_\varepsilon = O(|\log \varepsilon|)$ a.e. such that

$$C_{\underline{c}\varepsilon}(x,y) \subset B((x,y),\varepsilon) \subset C_{\overline{c}_\varepsilon \varepsilon}.$$

Step 1

If the process $N'_\varepsilon(t) := \dfrac{\log \mu_\varphi(C_{\varepsilon^t}(x,y)) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}}$ *converges in distribution to* σW *then* $N_\varepsilon(t)$ *also.*

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Gibbs measure and projections

The measure μ_φ is $e^{-\varphi}$-conformal thus $\mu_\varphi(T^{n_\varepsilon} C_\varepsilon) = \int_{C_\varepsilon} e^{-S_{n_\varepsilon}\varphi} d\mu_\varphi$.
Hence $\log \mu_\varphi(C_\varepsilon) \approx S_{n_\varepsilon}\varphi + \log \mu_\varphi(T^{n_\varepsilon} C_\varepsilon)$.
But $T^{n_\varepsilon} C_\varepsilon(x,y) = \pi^{-1} f^{n_\varepsilon} \mathcal{P}_{m_\varepsilon}(x) = \pi^{-1}\mathcal{P}_{m_\varepsilon - n_\varepsilon}(f^{n_\varepsilon}(x))$.

Theorem (Chazottes-Ugalde 09, Kempton-Pollicott, Verbitsky, ...)

The projection $\pi_*\mu_\varphi$ is a Gibbs measure for f, for a potential ψ regular (stretched exponential variations).

Set pressure $P_f(\psi) = 0$. Since $\log \mu_\varphi(T^{n_\varepsilon} C_\varepsilon) \approx S_{m_\varepsilon - n_\varepsilon}\psi \circ f^{n_\varepsilon}$ we obtain

Key lemma

We have $\log \mu_\varphi(C_\varepsilon(x,y)) \approx S_{n_\varepsilon(x,y)}(\varphi - \psi\circ\pi)(x,y) + S_{m_\varepsilon(x,y)}\psi\circ\pi(x,y)$

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Non-homogeneous Birkhoff sum

The intermediate entropies $h^u = h_{\pi_*\mu_\varphi}(f)$ and $h^{uu} = h_{\mu_\varphi}(T) - h_{\pi_*\mu_\varphi}(f)$ satisfy $h^u = -\int \psi\circ\pi d\mu_\varphi$, $h^{uu} = -\int(\varphi - \psi\circ\pi)d\mu_\varphi$.

Lemma

Setting $\delta^u = h^u/\lambda^u$ and $\delta^{uu} = h^{uu}/\lambda^{uu}$, the dimension is $\delta = \delta^{uu} + \delta^u$.

Lemma

We have $-\delta\log\varepsilon \approx \delta^{uu} S_{n_\varepsilon}\log\left|\frac{\partial g}{\partial y}\right| + \delta^u S_{m_\varepsilon}\log|f'|$.

Set $\phi_1 = \varphi - \psi\circ\pi + \delta^{uu}\log\left|\frac{\partial g}{\partial y}\right|$ and $\phi_2 = \psi\circ\pi + \delta^u\log|f'|$.

Step 2

If the process $N''_\varepsilon(t) := \dfrac{S_{n_{\varepsilon t}}\phi_1 + S_{m_{\varepsilon t}}\phi_2}{\sqrt{-\log\varepsilon}}$ converges in distribution to σW then $N'_\varepsilon(t)$ also.

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We have $-\delta\log\varepsilon \approx \delta^{uu} S_{n_\varepsilon}\log\left|\frac{\partial g}{\partial y}\right| + \delta^u S_{m_\varepsilon}\log|f'|$.

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If the process $N''_\varepsilon(t) := \dfrac{S_{n_{\varepsilon^t}}\phi_1 + S_{m_{\varepsilon^t}}\phi_2}{\sqrt{-\log\varepsilon}}$ converges in distribution to σW then $N'_\varepsilon(t)$ also.

Non-homogeneous Birkhoff sum

The intermediate entropies $h^u = h_{\pi_*\mu_\varphi}(f)$ and $h^{uu} = h_{\mu_\varphi}(T) - h_{\pi_*\mu_\varphi}(f)$ satisfy $h^u = -\int \psi \circ \pi d\mu_\varphi$, $h^{uu} = -\int (\varphi - \psi \circ \pi) d\mu_\varphi$.

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Weak invariance principle

Set $\phi = (\phi_1, \phi_2)$. We have $\int \phi d\mu_\varphi = 0$. Let

$$\mathcal{Y}_k(t) = \frac{1}{\sqrt{k}} S_{\lfloor kt \rfloor}\phi + \textit{Interpolation}.$$

Let Q be the limiting covariance matrix of $\frac{1}{\sqrt{k}} S_k\phi$.

Theorem (WIP, Folklore*)

The process $\mathcal{Y}_k$ converges in distribution towards a bi-dimensional Brownian motion $\mathcal{B}(t)$ with covariance Q (in particular $\mathcal{B}(t) \sim \mathcal{N}(0, tQ)$).

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Random change of time

Take $a > 1/\lambda^u$.
Set $\mathcal{Z}_k(t_1, t_2) = (\mathcal{Y}_{k,1}(t_1), \mathcal{Y}_{k,2}(t_2))$ for $t_1, t_2 \in [0, a]$.
Set $\Gamma(t) = (t/\lambda^{uu}, t/\lambda^u)$.

Definition

Define the random change of time $\Gamma_k(t) = (n_{e^{-kt}}/k, m_{e^{-kt}}/k)$ if both arguments are less than a, $\Gamma_k(t) = \Gamma(t)$ otherwise.

Let $\beta\colon C([0,1], \mathbb{R}^2) \to C([0,1], \mathbb{R})$ defined by $\beta(u) = u_1 + u_2$

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- $N''_{e^{-k}}(t) \approx \beta(\mathcal{Z}_k \circ \Gamma_k)(t)$
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The method can be applied to

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- surface diffeomorphisms
- some non uniformly expanding maps

Some questions are left

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Application: log-normal fluctuations of return time

Let $\tau_\varepsilon(x) = \min\{k \geq 1 \colon d(T^k(x), x) < \varepsilon\}$ be the first ε-return time.

Corollary (log-normal fluctuations of first return time ($\sigma \neq 0$))

$$\frac{\log \tau_\varepsilon(x) + \dim_H \mu_\varphi \log \varepsilon}{\sqrt{-\log \varepsilon}} \to \mathcal{N}(0, \sigma^2).$$

Log-normal fluctuations for repetition time of first n-symbols known:
Collet, Galves and Schmitt : exponential law for hitting time + CLT for information function (Gibbsian source).
Kontoyannis : strong approximation + ASIP for information function.

Proof.

(1) If (T, μ) mixes rapidly Lipschitz observables and d_μ exists then $\tau_\varepsilon(x) \approx \varepsilon^{-\dim_H \mu}$ μ-a.e. [Rousseau-S 10]. Refine so that log-normal fluctuations are preserved: strong approximation.
(2) CLT for measure of balls (the main theorem)
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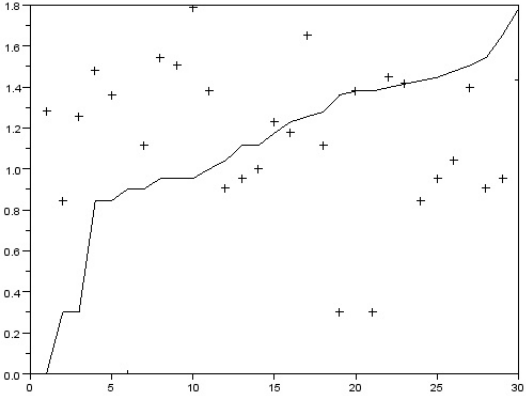
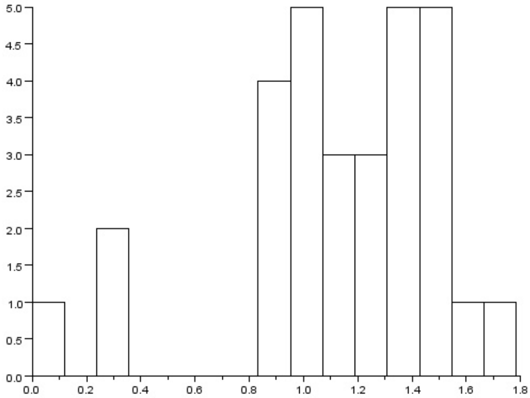
Numerical (non-rigorous) illustration for Hénon map IV



$\log(\tau_\varepsilon(x_i))/\log(\varepsilon)$ for (30) randomly chosen centers x_i

Numerical (non-rigorous) illustration for Hénon map V



Histogram of $\log(\tau_\varepsilon(x_i))/\log(\varepsilon)$ (for $\varepsilon = 0.1$)