# Mixing in a model of heat conduction

**Domokos Szász**
**Budapest University of Technology**
**(joint with Alex Grigo and Kostya Khanin)**

**Ergodic Theory and Dynamical Systems: Perspectives and Prospects**

**Warwick, April 17, 2012**

# Contents

# Non-equilibrium statistical physics

Goal: macroscopic laws from microscopic dynamics.
Optimally: from Newtonian (Hamiltonian) ones (classical statphys!)
Strong candidates:

- billiard models (quite realistic)
- (non-linear) oscillators

Spectacular successes for billiards:

- planar diffusion (or super-diffusion); Bunimovich, Chernov, Sinai '81, '91; Young '98; Sz.-Varjú '04, '07; Bálint-Gouëzel '06; Chernov-Dolgopyat 09 — , Rey-Bellet-Young '08, Melbourne-Nicol '09, Gouëzel '10, etc., etc.
- linear Boltzmann equation for the Lorentz gas (Boldrighini, Bunimovich, Sinai, '83)
- convergence to equilibrium of Lorentz gas (Krámli-Sz., '83)

# Derivation of heat equ.
# Fourier law of heat conduction

Oscillating interest:

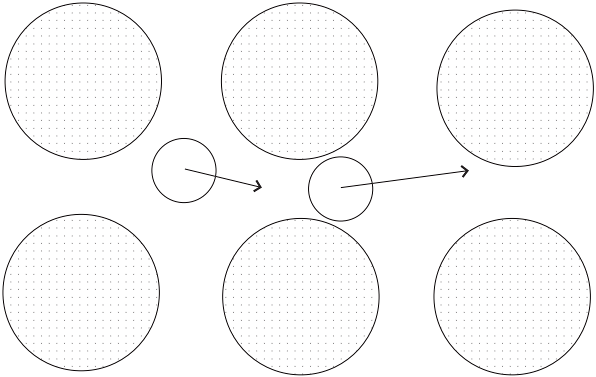
survey until 2000: Bonetto–Lebowitz–Rey-Bellet '00

Recent wave:

- Eckmann-Young '06: equilibrium measures under phenomenological assumptions
- **Gaspard-Gilbert '08–:** model of localized hard disks (balls), two step approach:
  1. 
     - derive a mesoscopic master equ. from the microscopic kinetic equ. of the Hamiltonian model
     - in the rare (but strong) interaction limit
     - it is a Markov jump process
  2. 
     - derive the macroscopic heat equ. $\partial_t u = \partial_x(\kappa(u)\partial_x u)$ from the mesoscopic master equ.
     - and determine $\kappa(u)$

# Quasi-1D model: 2 cells of the $N$ chain

## Periodic scatterers (shaded disks), confined moving disks (white circles)



# Parameter choice of G-G,'08.

- box size: $b$; periodic b. c.'s along $y$-axis
- chain length $= N$;
- radius of fixed scatterers (shaded circles) $= \rho_f$
- radius of moving disks (empty circles) $= \rho_m$
- condition of confinement: $\rho_f + \rho_m > b/2$
- condition of conductivity: $\rho_m > \rho_{\text{crit}} = \sqrt{(\rho_f + \rho_m)^2 - (l/2)^2}$
- small parameter $\varepsilon = \rho_m - \rho_{\text{crit}}$

G-G's trick:

- Keep $\rho_f + \rho_m =: \rho$ fixed
- If $\rho_m = \rho_{crit}$, then we have $N$ non-interacting billiards. Moreover, their phase spaces only depend on $\rho$!

# Liouville equation

Ernst-Dorfman, '89: The kinetic equ. for the $N$-particle density $p_N(q_1, v_1, \ldots, q_N, v_N; t)$ is

$$\partial_t p_N = \sum_{j=1}^{N} \left(-v_j \partial_{q_j} + K_{wall,j} + C_{j,j+1}\right) p_N$$

- the first two terms on the RHS describe the billiard dynamics of each disk within its cell (denote wall collision rate by $\nu_{\mathrm{wall},\varepsilon}$)
- the third one: the interaction of neighboring disks provides energy transfer (denote binary collision rate by $\nu_{\mathrm{bin},\varepsilon}$)

# Scale separation

G-G '08-: Scale separation at

$\varepsilon \to 0$, i. e. $\nu_{\mathrm{wall},\varepsilon}(\sim \nu_{wall,crit} > 0) \gg \nu_{\mathrm{bin},\varepsilon} \to 0$

1. **they derive a master equation for the density** $P_N(E_1, \ldots, E_N; t)$ $\quad (E_j = v_j^2 : 1 \leq j \leq N)$
2. from the master equation they obtain the **coefficient of heat conductivity**: $\kappa(T) = \sqrt{T}$ ($T$ being the temperature) i. e. the equation $\partial_t u = C \,.\, \Delta u^{3/2}$.

**Our aim: Rigorous theory**

# Dynamical approach to step 1

By Hirata-Saussol-Vaienti, '98 (also Collet-Eckmann, '06, Chazotte-Collet '10): *If*

- *a dynamical system* $(M, T, \mu)$ *is* **mixing in a controlled way** *(e. g.* $\alpha$*-mixing)*
- *and* $A_\varepsilon$ **is a sequence of nice subsets** *(to avoid e. g. neighborhoods of periodic points) with* $\lim_{\varepsilon \to 0} \mu(A_\varepsilon) = 0$

*then the successive entrance times of the dynamics into* $A_\varepsilon$ *form a Poisson process on the time scale of* $\mu(A_\varepsilon)^{-1}$.

For simplicity let $N = 2$ with free boundary conditions along $x$-axis.
The model is isomorphic to a 4D semi-dispersing billiard.
It is K-mixing, but no mixing rate is known. (exponential mixing: Bálint-Tóth, '08 is for dispersing billiards, only, and, moreover, it is hypothetical).

# Expected result for 2-disk chain

## similarly for $N$-disks, too

(joint with IP Tóth, work in slow progress)

$N = 2$, free boundary condition along $x$-axis. Dynamics:
$(M_\varepsilon = \{q_1, v_1; q_2, v_2 | dist(q_1, q_2) \geq 2\rho_m, v_1^2 + v_2^2 = 1\}, S^{\mathbb{R}}, \mu_\varepsilon)$.
Denote by $0 < \tau_{1,\varepsilon} < \tau_{2,\varepsilon} < \dots$ successive binary collision times of the two disks. Then, as $\varepsilon \to 0$

- $(\nu_{\mathrm{bin},\varepsilon}\tau_{1,\varepsilon}, \nu_{\mathrm{bin},\varepsilon}\tau_{2,\varepsilon}, \dots)$ converges to a Poisson process
- $E_1(\nu_{\mathrm{bin},\varepsilon} t), E_2(\nu_{\mathrm{bin},\varepsilon} t)$ converges to a jump Markov process on the state space $E_1 + E_2 = 1$ where $E_j(t) = \frac{1}{2} v_j^2(t); j = 1, 2$
- the transition kernel $k(E_1^+ | E_1^-)$ is calculated by verifying Boltzmann's 'microscopic chaos' property

Note: $\nu_{\mathrm{bin},\varepsilon} \sim \mathrm{const}.\varepsilon^3$.

# Expected result for 2-disk chain

## similarly for $N$-disks, too

(joint with IP Tóth, work in slow progress)

$N = 2$, free boundary condition along $x$-axis. Dynamics:

$(M_\varepsilon = \{q_1, v_1; q_2, v_2 | dist(q_1, q_2) \geq 2\rho_m, v_1^2 + v_2^2 = 1\}, S^{\mathbb{R}}, \mu_\varepsilon)$.

Denote by $0 < \tau_{1,\varepsilon} < \tau_{2,\varepsilon} < \dots$ successive binary collision times of the two disks. Then, as $\varepsilon \to 0$

- $(\nu_{\mathrm{bin},\varepsilon}\tau_{1,\varepsilon}, \nu_{\mathrm{bin},\varepsilon}\tau_{2,\varepsilon}, \dots)$ converges to a Poisson process
- $E_1(\nu_{\mathrm{bin},\varepsilon} t), E_2(\nu_{\mathrm{bin},\varepsilon} t)$ converges to a jump Markov process on the state space $E_1 + E_2 = 1$ where $E_j(t) = \frac{1}{2} v_j^2(t); j = 1, 2$
- the transition kernel $k(E_1^+ | E_1^-)$ is calculated by verifying Boltzmann's 'microscopic chaos' property

Note: $\nu_{\mathrm{bin},\varepsilon} \sim \mathrm{const.}\varepsilon^3$.

# Idea of proof

- since binary collisions are rare, most of the time the two disks evolve independently
- between two binary collisions - with an overwhelming probability - there is averaging in each of the in-cell, 2D billiard dynamics
- for these typically long time intervals it is natural to apply Chernov-Dolgopyat averaging
- for that purpose
  - ??? one has to check that for an incoming proper family of stable pairs, so is the outgoing family ???
  - one applies martingale approximation for jump processes (á la Dolgopyat-Sz.-Varjú, Duke '09 )

# Our approach (with Grigo and Khanin)

1. Introduce a (mesoscopic) stochastic model close to that of GG;
2. Find lower bounds for the spectral gap of its generator (*appropriately* depending on system size $N$).
3. Establish hydrodynamics limit transition to obtain heat equ.

*Appropriate* dependence $= O(\frac{1}{N^2})$ for continuous time dynamics.

Existing gap bounds almost exclusively for models with a finite state-space (like exclusion-like processes).
Continuous state space model: for Kac-model from '56 spectral gap estimate by Janvresse in '01, only.

# A (mesoscopic) stochastic model of energies

State space: $x = (x_1, \dots, x_N) \in \mathbb{R}_+^N$
Generator $\mathcal{L}$ of the continuous time Markov jump process $X(t)$ (given on $\mathbb{R}_+^N$) acting on bounded functions $A : \mathbb{R}_+^N \to \mathbb{R}$ is

$$\mathcal{L}A(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) \left[A(T_{i,\alpha}x) - A(x)\right]$$

where $P(x_i, x_{i+1}, d\alpha)$ is a probability measure on $[0, 1]$.
The maps $T_{i,\alpha}$ model the energy exchange between the neighboring sites $i$ and $i+1$, and are defined by

$$T_{i,\alpha}(x_i) = \alpha(x_i + x_{i+1})$$

$$T_{i,\alpha}(x_{i+1}) = (1 - \alpha)(x_i + x_{i+1})$$

# Remarks

- Total energy is invariant, i. e.

$$\mathcal{S}_{\epsilon,N} = \left\{ x \in \mathbb{R}_+^N \middle| \sum_{i=1}^{N} \frac{1}{N} x_i = \epsilon \right\}$$

  is invariant wrt dynamics;

- Standing assumptions: for any $E, E'$ the kernel $P(E, E', d\alpha)$
  1. is symmetric wrt $1/2$ ;
  2. is never equal to $\frac{1}{2}(\delta_0 + \delta_1)$ (i. e. $\{E_1^+, E_2^+\} \neq \{E_1, E_2\}$)
  3. plus an appropriate condition for $\Lambda$.

# Mesoscopic generator in the GG model, case $d = 3$

$$\Lambda(E_1, E_2) = \Lambda_{tot}(E_1 + E_2)\ \Lambda_{part}(\frac{E_1}{E_1 + E_2})$$

(product!) where

$$\Lambda_{tot}(s) = \sqrt{s} \qquad\qquad \Lambda_{part}(\beta) = \frac{2\pi}{6}\ \frac{\frac{1}{2} + \beta \vee (1-\beta)}{\sqrt{\beta \vee (1-\beta)}}$$

and

$$P(x_1, x_2, d\alpha) = P(\frac{x_1}{x_1 + x_2}, d\alpha) = P(\beta, d\alpha)$$

with $\beta = \frac{x_1}{x_1+x_2}$ (simple dependence!), where

$$\frac{P(\beta, d\alpha)}{d\alpha} = \frac{3}{2}\ \frac{1 \wedge \sqrt{\frac{\alpha\wedge(1-\alpha)}{\beta\wedge(1-\beta)}}}{\frac{1}{2} + \beta \vee (1-\beta)}.$$

# Stick process scaling to the porous medium equ.

Feng-Iscoe-Seppalainen, '96

$$\mathcal{K}A(x) = \Sigma_{i=1}^{N-1} \frac{1}{2} \int_0^{x_i} u^{\alpha-2} \left( \sum_{j=\pm 1} [A(x^{u,i,j}) - A(x)] \right) du$$

where $\alpha > 1$ and

$$x_k^{u,i,j} = \begin{cases} x_k & \text{if } k \neq i, i+j \\ x_i - u & \text{if } k = i \\ x_{i+j} + u & \text{if } k = i+j \end{cases}$$

This model can be understood as a zero-range energy model
Then the expected limiting equ. is $\partial_t u = \mathrm{const.}\ \Delta(u^\alpha)$, the nonlinear heat equ. (porous medium equ.) if $\alpha \neq 1$.

# Goal: Limiting heat equ. in GG model

In the limit as $N \to \infty$ and $\xi = i/N$, $t = N^2 \tau$ the empirical process

$$\sum_{i=1}^{N} \frac{1}{N} \delta_{\mathsf{X}_i(t)}$$

should converge to a process with density $u(\xi, \tau)$ solving

$$\partial_\tau u(\xi,\tau) = \partial_\xi(\text{const} \sqrt{u(\xi,\tau)}\, \partial_\xi u(\xi,\tau))$$

# $L^2_{\pi_{\epsilon,N}}$–spectral gap for reversible $\pi_{\epsilon,N}$: simple model!

**Assume: for every $E_1, E_2$ we have $\Lambda(E_1, E_2) = \Lambda^*$ and $P(E_1, E_2, d\alpha) = P^*(d\alpha)$.**

**Theorem**

*If the stationary distribution $\pi_{\epsilon,N}$ of $\mathrm{X}(t)$ on $\mathcal{S}_{\epsilon,N}$ is reversible, then*

$$\sigma(\mathcal{L}^*) \subset \Big( -\infty, -\frac{1}{2}\Lambda^*\,[1 - 4\,\sigma_P^2]\,\sin^2\Big[\frac{\pi}{N+2}\Big]\Big] \cup \{0\}\ ,$$

*where 0 is a simple eigenvalue corresponding to the constant eigenfunction.*

# Spectral gap in the general case: Assumptions

Let $\pi_{\epsilon,N}$ be a reversible stationary distribution of $\mathcal{L}$ on $\mathcal{S}_{\epsilon,N}$. Suppose that there exist a constant $\Lambda^\star > 0$ and a probability measure $P^\star$ such that the following are satisfied:

**(i)** Rate function $\Lambda$ satisfies $\Lambda(E_1, E_2) \geq \Lambda^\star$

**(ii)** (Doeblin-type) There exists a constant $\beta > 0$ such that $P$ satisfies the minorization condition $P(E_1, E_2, .) \geq \beta\, P^\star(.)$

**(iii)** The unique stationary distribution $\pi^\star_{\epsilon,N}$ of $\mathcal{L}^\star$ on $\mathcal{S}_{\epsilon,N}$ (corresponding to $\Lambda^\star$ and $P^\star$) is reversible.

**(iv)** The measures $\pi_{\epsilon,N}$ and $\pi^\star_{\epsilon,N}$ are uniformly equivalent, i.e. there exist two constants $0 < C^-_{\epsilon,N} \leq C^+_{\epsilon,N} < \infty$ such that their Radon-Nikodym derivative satisfies $C^-_{\epsilon,N} \leq \frac{\pi_{\epsilon,N}(dx)}{\pi^\star_{\epsilon,N}(dx)} \leq C^+_{\epsilon,N}$.

# Spectral gap for $\mathcal{L}$

**Theorem**

*Then the spectrum of $\mathcal{L}$ in $L^2_{\pi_{\epsilon,N}}$ satisfies*

$$\sigma(\mathcal{L}) \subset \Big( -\infty, -\beta \, \frac{C^-_{\epsilon,N}}{C^+_{\epsilon,N}} \, \Lambda^\star \, \frac{1}{2} \, [1 - 4\,\sigma^2_{P^\star}] \, \sin^2 \left[ \frac{\pi}{N+2} \right] \Big] \cup \{0\} \,,$$

*where 0 is a simple eigenvalue.*

Michiko SASADA, '11 (work in progress):

- $\frac{C}{N^2}$ spectral gap for stick models
- hope to extend methods to our energy exchange model

# Description of reversible product measures

**Lemma (Reversible product measures and system size)**

*Let $\nu$ be a probability measure on $\mathbb{R}_+$. Then the product (probability) measure $\mu(dx) = \nu(dx_1) \cdots \nu(dx_N)$ on $\mathbb{R}_+^N$ is reversible for $\mathrm{X}(t)$ (with generator) for some $N$ if and only if it is reversible for $N = 2$.*

# Particular case

Assume that the rate function $\Lambda$ and the transition kernel $P$ are of the form

$$\Lambda(x_i, x_{i+1}) = \Lambda_{tot}(x_i + x_{i+1})\,\Lambda_{part}\left(\frac{x_i}{x_i + x_{i+1}}\right)$$

$$P(x_i, x_{i+1}, d\alpha) = P\left(\frac{x_i}{x_i + x_{i+1}}, d\alpha\right) = P\Big(\beta, d\alpha\Big)$$

where

$$\beta = \frac{x_i}{x_i + x_{i+1}}$$

As seen, they are satisfied in the GG model!

# Characterization of rev. product meas., $N \geq 2$

**Theorem (Reversible product measures)**

*Suppose: Markov chain on $[0,1]$ with kernel $P(\beta, d\alpha)$ (the energy exchange!) has a unique invariant distribution $p(.)$. Suppose also that $\forall s > 0$ $\Lambda_{tot}(s) > 0$ and $\forall$ $0 < \beta < 1$ $\Lambda_{part}(\beta) > 0$. Then the product measure $\mu(dx) = \nu(dx_1) \cdots \nu(dx_N)$ is reversible for $X(t)$ if and only if either of the following two holds:*

1. *(degenerate) There exists $\epsilon > 0$ such that $\nu(dx_1) = \delta(\epsilon, dx_1)$.*
2. *(gamma) There exists $\epsilon > 0$ and $d > 0$ such that*

$$\nu(dx_1) = \frac{dx_1}{\epsilon} \left[\frac{x_1}{\epsilon}\right]^{\frac{d}{2}-1} \frac{e^{-\frac{x_1}{\epsilon}}}{\Gamma(\frac{d}{2})}$$

$$p(d\beta) = d\beta \left[\beta\,(1-\beta)\right]^{\frac{d}{2}-1} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})^2} \Lambda_{part} \frac{1}{Z}$$

# GG model, $d = 3$ revisited

In previous theorem $Z$ is the normalizing constant, and

$$\int p(d\beta) \int P(\beta, d\alpha)\, \psi(\alpha, \beta) = \int p(d\beta) \int P(\beta, d\alpha)\, \psi(\beta, \alpha)$$

for all bounded $\psi : [0, 1]^2 \to \mathbb{R}$.

GG-model, $d = 3$

$$\nu(dx_1) = \frac{dx_1}{\epsilon} \sqrt{\frac{x_1}{\epsilon}}\, \frac{2\, e^{-\frac{x_1}{\epsilon}}}{\sqrt{\pi}}$$

$$\nu_{tot}(ds) = \frac{ds}{\epsilon} \left[\frac{s}{\epsilon}\right]^2 \frac{e^{-\frac{s}{\epsilon}}}{2}, \qquad \nu_{part}(d\beta) = d\beta \sqrt{\beta\,(1-\beta)}\, \frac{8}{\pi}$$

$$p(d\alpha) = d\alpha \sqrt{\alpha\,(1-\alpha)}\, \frac{8}{\pi}\, \Lambda_{part}(\alpha)\, \frac{1}{Z}$$

# Main result for GG, $d = 3$

## Corollary

*If $\Lambda_s(s)$ is replaced by any non-negative continuous function, which is bounded away from zero, then the following hold for any $N$ and $\epsilon$.*

1. *The product measure $\mu(dx) = \nu(dx_1) \cdots \nu(dx_N)$ with $\nu(dx_1) = \frac{dx_1}{\epsilon} \sqrt{\frac{x_1}{\epsilon}} \frac{2\, e^{-\frac{x_1}{\epsilon}}}{\sqrt{\pi}}$ is the unique reversible product measure for $\mathrm{X}(t)$.*
2. *On every $\mathcal{S}_{\epsilon,N}$ there exists a unique stationary distribution $\pi_{\epsilon,N}$. This measure is obtained by conditioning $\mu(dx)$.*
3. *The spectrum $\sigma(\mathcal{L})$ of the generator $\mathcal{L}$ acting on $L^2_{\pi_{\epsilon,N}}$ satisfies*

$$\sigma(\mathcal{L}) \subset \Big( -\infty, -C \sin^2 \Big[ \frac{\pi}{N+2} \Big] \Big] \cup \{0\}$$

*for some constant $C$, which may depend on the choice of $\Lambda_{tot}$.*

# Main parts of proof

1. Comparison
2. Gap bound for simple model

# Comparison method 1

Then the associated Dirichlet form

$$\mathcal{D}_{\epsilon,N}(A) = \int \pi_{\epsilon,N}(dx)\, A(x)\, [-\mathcal{L}A](x)$$

is defined for all $A \in L^2_{\pi_{\epsilon,N}}$, and has the representation

$$= \frac{1}{2} \sum_{i=1}^{N-1} \int \pi_{\epsilon,N}(dx)\, \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha)\, [A(T_{i,\alpha}x) - A(x)]^2 \, .$$

# Comparison method 2

The basic idea to prove convergence rates for $X(t)$ is to compare the spectral gap of its generator $\mathcal{L}$ to that of a simple reference process. In order to distinguish these two generators we use a superscript $\star$

$$\mathcal{L}^\star A(x) = \Lambda^\star \sum_{i=1}^{N-1} \int P^\star(d\alpha)\left[A(T_{i,\alpha}x) - A(x)\right]$$

$$\mathcal{D}^\star_{\epsilon,N}(A) = \frac{1}{2}\int \pi^\star_{\epsilon,N}(dx) \sum_{i=1}^{N-1} \Lambda^\star \int P^\star(d\alpha)\left[A(T_{i,\alpha}x) - A(x)\right]^2$$

to denote the invariant measure, the generator and the corresponding Dirichlet form of the reference process.

# Spectral gap in the general case: Assumptions

Let $\pi_{\epsilon,N}$ be a reversible stationary distribution of $\mathcal{L}$ on $\mathcal{S}_{\epsilon,N}$. Suppose that there exist a constant $\Lambda^\star > 0$ and a probability measure $P^\star$ such that the following are satisfied:

**(i)** Rate function $\Lambda$ satisfies $\Lambda(E_1, E_2) \geq \Lambda^\star$

**(ii)** (Doeblin-type) There exists a constant $\beta > 0$ such that $P$ satisfies the minorization condition $P(E_1, E_2, .) \geq \beta\, P^\star(.)$

**(iii)** The unique stationary distribution $\pi^\star_{\epsilon,N}$ of $\mathcal{L}^\star$ on $\mathcal{S}_{\epsilon,N}$ (corresponding to $\Lambda^\star$ and $P^\star$) is reversible.

**(iv)** The measures $\pi_{\epsilon,N}$ and $\pi^\star_{\epsilon,N}$ are uniformly equivalent, i.e. there exist two constants $0 < C^-_{\epsilon,N} \leq C^+_{\epsilon,N} < \infty$ such that their Radon-Nikodym derivative satisfies $C^-_{\epsilon,N} \leq \frac{\pi_{\epsilon,N}(dx)}{\pi^\star_{\epsilon,N}(dx)} \leq C^+_{\epsilon,N}$.

# Comparison method 3

Since we assume reversibility, the generator is self-adjoint, and hence we have the following variational characterization

$$\gamma = \inf\left\{ \frac{\mathcal{D}_{\epsilon,N}(A)}{\mathrm{Var}_{\epsilon,N}(A)} \;:\; A \in L^2_{\pi_{\epsilon,N}},\ \mathrm{Var}_{\epsilon,N}(A) \neq 0 \right\}$$

of the spectral gap $\gamma$ of $\mathcal{L}$ acting on $L^2_{\pi_{\epsilon,N}}$, where $\mathrm{Var}_{\epsilon,N}(A)$ denotes the variance of $A$ with respect to $\pi_{\epsilon,N}$.

$\mathcal{D}_{\epsilon,N}(A)$ can be bounded from below by using (i)-(iv), and $\mathrm{Var}_{\epsilon,N}(A)$ from above by using (iv).

# Hydrodynamical limit: simple cases

1. If $\Lambda \equiv const.$ and $P(\beta, d\alpha) = P(d\alpha)$
   (in fact, $P(d\alpha)$ need not be abs. cont.)
   then the limiting equation is

$$\partial_t u = C \quad \Delta u;$$

2. If $\Lambda(E_1, E_2) = E_1 + E_2$ and $P(\alpha) = \delta_{1/2}$, then the limiting equation is

$$\partial_\tau u(\xi,\tau) = \partial_\xi (C\, u(\xi,\tau)\, \partial_\xi u(\xi,\tau)) = C/2\ \Delta u^2(\xi,\tau)$$

# Summary

1. Introduced a mesoscopic stochastic model close to GG model

2. Formulated conditions ensuring appropriate lower bound ($\frac{1}{N^2}$) for spectral gap in terms of $N$

3. Now one can attack hydrodynamic limit (à la Varadhan). BUT: it is a non-gradient system! (except for toy models)

4. Tasks:
   - Prove hydrodynamic limit
   - Improve conditions, in particular, on bdedness away from 0 of $\Lambda$ (numerical evidence!)
   - Return to Sz.-Tóth-approach

# Vaserstein-distance

Recall that the definition of the Vaserstein-$p$ distance is

$$\rho_p(\mu,\nu) = \inf_{\substack{X\sim\mu \\ Y\sim\nu}} [\mathbb{E}D(X,Y)^p]^{\frac{1}{p}} \qquad \text{and set} \qquad \rho(\mu,\nu) \equiv \rho_1(\mu,\nu)$$

where $\mu$ and $\nu$ are two probability measures on a compact metric space $(\mathcal{S}, \mathrm{d})$.

We will be using $p = 2$.

Furthermore, for $p = 1$ the duality

$$\rho(\mu,\nu) = \inf_{\substack{X\sim\mu \\ Y\sim\nu}} \mathbb{E}d(X,Y) = \sup_{f:\ \ \mathrm{Lip}(f)\le 1} (\mu(f) - \nu(f))$$

follows by the Kantorovich-Rubinstein theorem.

# Proof of convergence in Vaserstein-2 distance cont'd

**Proposition (Rate of convergence in Vaserstein-2 distance)**

*Let* $\mathrm{U}(t)$ *and* $\mathrm{U}'(t)$ *be any two Markov chains generated by* $\hat{\mathcal{L}}$ *on* $\mathcal{S}_{\epsilon,N}$*. Then*

$$\begin{aligned}\rho_2(\mathrm{U}(t),\mathrm{U}'(t)) &\leq \rho_2(\mathrm{U}(0),\mathrm{U}'(0))\ \exp\left(-\frac{1}{2}\left[1-4\,\sigma_P^2\right]\sin^2\left[\frac{\pi}{N+2}\right]t\right)\\ &\leq \epsilon\, N\sqrt{N-1}\ \exp\left(-\frac{1}{2}\left[1-4\,\sigma_P^2\right]\sin^2\left[\frac{\pi}{N+2}\right]t\right)\end{aligned}$$

*holds for all* $t$*.*

- If $\sigma_P^2 < \frac{1}{4}$, then there exists a unique stationary distribution $\pi_{\epsilon,N}$ on each $\mathcal{S}_{\epsilon,N}$.
- This rate of convergence is again $O(N^{-2})$, and thus optimal.