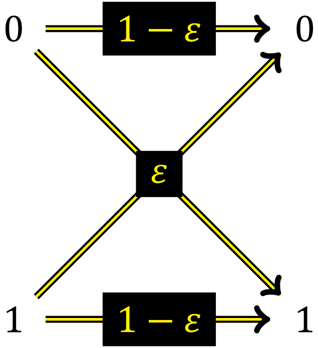
Factors of g-measures

Evgeny Verbitskiy
Univ. Leiden / Groningen
The Netherlands

Warwick, April 17, 2012

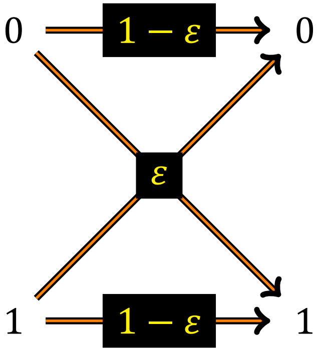
Binary symmetric channel

Input process $\{X_n\} \mapsto$ output process $\{Y_n\}$



$$\mathbb{P}(Y_n = 0 | X_n = 0) = \mathbb{P}(Y_n = 1 | X_n = 1) = 1 - \varepsilon$$
$$\mathbb{P}(Y_n = 0 | X_n = 1) = \mathbb{P}(Y_n = 1 | X_n = 0) = \varepsilon$$

Input process $\{X_n\} \mapsto$ Output process $\{Y_n\}$



Question: Take you favorite process (measure), what are the properties/entropy/... of the output process (measure).

Binary Symmetric Markov Process under BSC

- $\{X_n\}$ – Markov chain, $X_n \in \{-1, 1\}$,

$$\mathbf{P} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}.$$

- $\{Z_n\}$ – Bernoulli sequence, $Z_n = \{-1, 1\}$,

$$\mathbb{P}(Z_n = -1) = \varepsilon, \quad \mathbb{P}(Z_n = 1) = 1 - \varepsilon.$$

- $Y_n = X_n \cdot Z_n \quad \forall n \in \mathbb{Z}$

Binary Symmetric Markov Process under BSC

- $\{X_n\}$ – Markov chain, $X_n \in \{-1, 1\}$,

$$\mathbf{P} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}.$$

- $\{Z_n\}$ – Bernoulli sequence, $Z_n = \{-1, 1\}$,

$$\mathbb{P}(Z_n = -1) = \varepsilon, \quad \mathbb{P}(Z_n = 1) = 1 - \varepsilon.$$

- $Y_n = X_n \cdot Z_n \quad \forall n \in \mathbb{Z}$

If $\{Z_n\}$ is Markov, we have a **Gilbert-Elliot** channel.

Equivalently,

$$Y_n = \pi\left(X_n^{\text{ext}}\right),$$

for the Markov chain $\left\{X_n^{\text{ext}}\right\}$ with values in

$$\mathrm{A} = \{(1,1),(1,-1),(-1,1),(-1,-1)\},$$

with

$$\mathbf{P}^{\text{ext}} = \begin{pmatrix} (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \\ (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \\ p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \\ p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \end{pmatrix},$$

and an obvious deterministic function

$$\pi : \mathrm{A} \to \{-1, 1\}$$

Equivalently,

$$Y_n = \pi\left(X_n^{\text{ext}}\right),$$

for the Markov chain $\{X_n^{\text{ext}}\}$ with values in

$$\mathrm{A} = \{(1,1),(1,-1),(-1,1),(-1,-1)\},$$

with

$$\mathbf{P}^{\text{ext}} = \begin{pmatrix} (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \\ (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \\ p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \\ p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \end{pmatrix},$$

and an obvious deterministic function

$$\pi : \mathrm{A} \rightarrow \{-1,1\}$$

BSC is a 1-block factor of a Markov process

Information theory

BSC is the simplest textbook channel

Statistical mechanics

BSC on lattices $\mathbb{Z}^d \iff$ *time* $t = t(\varepsilon)$ *map*
infinite temperature Glauber dynamics

Probability/Statistics

BSC –> hidden Markov chains

Dynamical Systems

BSC –> 1-block factor

Information theory

BSC is the simplest textbook channel

Statistical mechanics

BSC on lattices $\mathbb{Z}^d$ $\iff$ *time* $t = t(\varepsilon)$ *map*

infinite temperature Glauber dynamics

Probability/Statistics

BSC –> hidden Markov chains

Dynamical Systems

BSC –> 1-block factor

What happens if you start with a nice process?

Side remark: RG method in Stat. Mech.

is used to compute some interesting quantities (critical exponents).

Side remark: RG method in Stat. Mech.

is used to compute some interesting quantities (critical exponents).

Spin system $\{\sigma_n : n \in \mathbb{Z}^d\}$ governed by Gibbs μ with potential H_0.

Side remark: RG method in Stat. Mech.

is used to compute some interesting quantities (critical exponents).

Spin system $\{\sigma_n : n \in \mathbb{Z}^d\}$ governed by Gibbs μ with potential H_0.

Apply renormalization: e.g., decimation

New spin system $\{\widetilde{\sigma}_n : n \in \mathbb{Z}^d\}$:

$$\widetilde{\sigma}_n = \sigma_{bn}, \quad b \in \mathbb{N}, n \in \mathbb{Z}^d.$$

Law($\{\widetilde{\sigma}_n\}$) is Gibbs (with potential H_1).

$$H_0 \mapsto H_1$$

Side remark: RG method in Stat. Mech.

is used to compute some interesting quantities (critical exponents).

Spin system $\{\sigma_n : n \in \mathbb{Z}^d\}$ governed by Gibbs μ with potential H_0.

Apply renormalization: e.g., decimation

New spin system $\{\widetilde{\sigma}_n : n \in \mathbb{Z}^d\}$:

$$\widetilde{\sigma}_n = \sigma_{bn}, \quad b \in \mathbb{N}, n \in \mathbb{Z}^d.$$

Law($\{\widetilde{\sigma}_n\}$) is Gibbs (with potential H_1).

$$H_0 \mapsto H_1$$

Repeat many times... [Kadanoff (66), Wilson (75),...]

Nice measures in 1D

Nice measures in 1D

are **regular (=GIBBS)**, and might be

- ☞ Gibbs in Stat. Mechanics sense (DLR)
- ☞ Gibbs in Dyn. Systems sense (Bowen)
- ☞ g-measure (Keane, one-sided DLR)

Nice measures in 1D

are **regular (=GIBBS)**, and might be

- ☞ Gibbs in Stat. Mechanics sense (DLR)
- ☞ Gibbs in Dyn. Systems sense (Bowen)
- ☞ g-measure (Keane, one-sided DLR)

... equivalent under **strong uniqueness conditions**.

GIBBS notions

- Gibbs/Statistical Mechanics/ two-sided

$$\mu(x_0|x_{\neq 0}) = \frac{1}{Z}\exp\Big(-\sum_{\Lambda \ni 0} U_\Lambda(x_\Lambda)\Big).$$

- g-measures/Dynamical Systems/ one-sided

$$\mu(x_0|x_{>0}) = g(x_0, x_1, \ldots), \quad g \in C(A^{\mathbb{Z}_+}), g > 0.$$

- Gibbs/Dyn. Systems (Bowen):
$\exists c, P \in \mathbb{R}, \phi \in C(A^{\mathbb{Z}_+})$

$$\frac{1}{c} \leq \frac{\mu\big([x_0 \ldots x_n]\big)}{\exp\Big(\sum_{j=0}^{n} \phi(\sigma^j x) - (n+1)P\Big)} \leq c.$$

Relation between different GIBBS notions

- R. Fernandez, S. Gallo, G. Maillard (2011): **unique** g-measure which is **not two-sided Gibbs**
- P. Walters (2005): example of μ on $A^{\mathbb{Z}}$ such that

$$\mu^+ \text{ on } A^{\mathbb{Z}_{\geq 0}}, \quad \mu^- \text{ on } A^{\mathbb{Z}_{\leq 0}},$$

  μ^+ is a g-measure, μ^- is **not**.
- unknown for the Dyson model

$$H_0(\sigma) = J \sum_{k \in \mathbb{Z}} \frac{\sigma_0 \sigma_k}{1 + |k|^\alpha}, \quad \alpha \in (1,2).$$

Relation between different GIBBS notions

- R. Fernandez, S. Gallo, G. Maillard (2011): **unique** g-measure which is **not two-sided Gibbs**
- P. Walters (2005): example of μ on $A^{\mathbb{Z}}$ such that

$$\mu^+ \text{ on } A^{\mathbb{Z}_{\geq 0}}, \quad \mu^- \text{ on } A^{\mathbb{Z}_{\leq 0}},$$

  μ^+ is a g-measure, μ^- is **not**.
- unknown for the Dyson model

$$H_0(\sigma) = J \sum_{k \in \mathbb{Z}} \frac{\sigma_0 \sigma_k}{1 + |k|^\alpha}, \quad \alpha \in (1,2).$$

- CMMC's, CCC's, VLMC's, uniform martingales, abs.reg. processes

Markov measures

Markov measures

Theorem. Suppose $\{X_n\}_{n\geq 0}$ is a Markov chain with $\mathbf{P} > 0$ and $\mathbb{P}$ is the invariant measure. Then the measure $\mathbb{Q} = \pi_*\mathbb{P} = \mathbb{P} \circ \pi^{-1}$ of the factor process

$$Y_n = \pi(X_n)$$

is **regular** (Gibbs, g-, CMMC's, ...):

$$\beta_n := \sup_{y_0^{n+1},\xi,\zeta} \left| \mathbb{Q}(y_0 | y_1^n, \xi_{n+1}^{\infty}) - \mathbb{Q}(y_0 | y_1^n, \zeta_{n+1}^{\infty}) \right| \to 0.$$

Moreover, there exist $C > 0$ and $\theta \in (0,1)$ such that

$$\beta_n \leq C\theta^n.$$

Markov measures

Theorem. Suppose $\{X_n\}_{n\geq 0}$ is a Markov chain with $\mathbf{P} > 0$ and $\mathbb{P}$ is the invariant measure. Then the measure $\mathbb{Q} = \pi_*\mathbb{P} = \mathbb{P} \circ \pi^{-1}$ of the factor process

$$Y_n = \pi(X_n)$$

is **regular** (Gibbs, g-, CMMC's, ...):

$$\beta_n := \sup_{y_0^{n+1},\xi,\zeta} \left| \mathbb{Q}(y_0 | y_1^n, \xi_{n+1}^{\infty}) - \mathbb{Q}(y_0 | y_1^n, \zeta_{n+1}^{\infty}) \right| \to 0.$$

Moreover, there exist $C > 0$ and $\theta \in (0,1)$ such that

$$\beta_n \leq C\theta^n.$$

At least 10 proofs $\Rightarrow$ various estimates of θ

Decay rate for $p = 0.4$, $\varepsilon = 0.1$

Birch	θ_B	≈ 0.99998
Harris	θ_H	≈ 0.97223
Baum and Petrie	θ_{BP}	$= 0.94$
Han and Marcus	θ_{HM}	$= 0.58$
Hochwald and Jelenković	θ_{HJ}	$= 0.2$
Fernández, Ferrari, and Galves	θ_{FFG}	$= 0.2$
Peres	θ_P	$= 0.2$

$$\theta_{HJ} = \theta_{FFG} = \theta_P = |1 - 2p| = \lambda_2(\mathbf{P}) =$$

$$\overline{\lim_{n\to\infty}} \left\| \mathbb{P}(X_n = \cdot | X_0 = 1) - \mathbb{P}(X_n = \cdot | X_0 = -1) \right\|^{\frac{1}{n}}$$

Decay rate for $p = 0.4$, $\varepsilon = 0.1$

Birch	θ_B	≈ 0.99998
Harris	θ_H	≈ 0.97223
Baum and Petrie	θ_{BP}	$= 0.94$
Han and Marcus	θ_{HM}	$= 0.58$
Hochwald and Jelenković	θ_{HJ}	$= 0.2$
Fernández, Ferrari, and Galves	θ_{FFG}	$= 0.2$
Peres	θ_P	$= 0.2$

$$\theta_{HJ} = \theta_{FFG} = \theta_P = |1 - 2p| = \lambda_2(\mathbf{P}) =$$
$$\overline{\lim_{n\to\infty}} \left\| \mathbb{P}(X_n = \cdot | X_0 = 1) - \mathbb{P}(X_n = \cdot | X_0 = -1) \right\|^{\frac{1}{n}}$$

independent of ε

In fact ...

Process $\{Y_n\}$ is **more random** than $\{X_n\}$.

In fact ...

Process $\{Y_n\}$ is **more random** than $\{X_n\}$. Memory decay rate should be **strictly smaller** than $|1 - 2p|$.

In fact ...

Process $\{Y_n\}$ is **more random** than $\{X_n\}$. Memory decay rate should be **strictly smaller** than $|1-2p|$.

Note that for $\epsilon = 0$, $\mathbb{Q} = \mathbb{P}$, and decay rate is zero.

In fact …

Process $\{Y_n\}$ is **more random** than $\{X_n\}$. Memory decay rate should be **strictly smaller** than $|1-2p|$.

Note that for $\epsilon = 0$, $\mathbb{Q} = \mathbb{P}$, and decay rate is zero.

Theorem. For all $p, \varepsilon \in (0,1)$, memory decay rate

$$\theta^* = \overline{\lim_n}\left(\beta_n\right)^{\frac{1}{n}}$$

$$\beta_n = \sup_{y_0^{n+1},\xi,\zeta}\left|\mathbb{Q}(y_0|y_1^n,\xi_{n+1}^{\infty}) - \mathbb{Q}(y_0|y_1^n,\zeta_{n+1}^{\infty})\right| \to 0.$$

satisfies

$$\theta^* < |1-2p|.$$

Theorem.

$$2 \cdot \mathbb{Q}(y_0|y_1, y_2, \ldots) = a_0 - \cfrac{b_0}{a_1 - \cfrac{b_1}{a_2 - \cfrac{b_2}{a_3 - \ldots}}}$$

where for $i \geq 0$

$$a_i = 1 + q_i, \quad b_i = 4\varepsilon(1-\varepsilon)q_i$$
$$q_i = (1-2p)y_i y_{i+1}$$

Theorem.

$$2 \cdot \mathbb{Q}(y_0|y_1, y_2, \ldots) = a_0 - \cfrac{b_0}{a_1 - \cfrac{b_1}{a_2 - \cfrac{b_2}{a_3 - \ldots}}}$$

where for $i \geq 0$

$$a_i = 1 + q_i, \quad b_i = 4\varepsilon(1-\varepsilon)q_i$$
$$q_i = (1-2p)y_i y_{i+1}$$

PROOF: BSM over BSC = RFIM in 1D.

Identification of potential

all thermodynamic quantities are real-analytic in ε .

$$\mathbb{Q}(y_0|y_1^\infty) = \sum_{k=0}^{\infty} \psi_k(y_0^\infty)\varepsilon^k = \sum_{k=0}^{\infty} \psi_k(y_0^{k+1})\varepsilon^k$$

$$\log \mathbb{Q}(y_0|y_1^\infty) = \sum_{k=0}^{\infty} \phi_k(y_0^\infty)\varepsilon^k = \sum_{k=0}^{\infty} \phi_k(y_0^{k+1})\varepsilon^k$$

$$h(\mathbb{Q}) = h(\mathbb{P}) + \sum_{k=1}^{\infty} c_k \epsilon^k$$

Han-Marcus (2006), Zuk-Domany-Kanter-Aizenman (2006), Pollicott (2011)

g-measures on full shifts

Suppose $g : \mathrm{A}^{\mathbb{Z}_+} \to [0, 1]$ is **continuous** and **positive**.

Definition. A measure μ on $\mathrm{A}^{\mathbb{Z}_+}$ is a g-**measure** if

$$\mu(X_0 = x_0 | X_1 = x_1, \ldots, X_n = x_n, \ldots) = g(x),$$

for μ-a.a. $x = (x_0, x_1, \ldots, x_n, \ldots)$.

g-measures on full shifts

Suppose $g : \mathbb{A}^{\mathbb{Z}_+} \to [0,1]$ is **continuous** and **positive**.

Definition. A measure μ on $\mathbb{A}^{\mathbb{Z}_+}$ is a g-**measure** if

$$\mu(X_0 = x_0 | X_1 = x_1, \dots, X_n = x_n, \dots) = g(x),$$

for μ-a.a. $x = (x_0, x_1, \dots, x_n, \dots)$.

Equivalently, for all $f \in C(\mathbb{A}^{\mathbb{Z}_+}, \mathbb{R})$, one has

$$\int f(x)\mu(dx) = \int \Big[\sum_{a \in \mathbb{A}} f(ax) g(ax)\Big] \mu(dx)$$

g-measures

Positive and continuous function g,

$$\text{var}_n(g) = \sup_{x_0^n = \bar{x}_0^n} \left| g(x) - g(\bar{x}) \right| \to 0 \quad \text{as } n \to \infty.$$

Theorem (Walters 1975). Continuous positive normalized function g with **summable variation**

$$\sum_{n=0}^{\infty} \text{var}_n(g) < \infty,$$

admits a unique g-measure.

g-measures

Finite range

Markov chains with $\mathbf{P} > 0$,

$\mathbf{P} = (p_{ij})$ with some $p_{ij} = 0$ **excluded**

Exponential decay

Hölder continuous functions g

Review: renormalization of g-measures

Theorem. If $\beta_n = \mathrm{var}_n(g) \to 0$ **sufficiently fast**, then $\nu = \mu \circ \pi^{-1}$ is a $\widetilde{g}$-measure:

$$\nu(y_0 | y_1^{\infty}) = \widetilde{g}(y)$$

with

$$\widetilde{\beta_n} = \mathrm{var}_n(\widetilde{g}) \to 0.$$

Review: renormalization of g-measures

Theorem. If $\beta_n = \mathrm{var}_n(g) \to 0$ **sufficiently fast**, then $\nu = \mu \circ \pi^{-1}$ is a $\tilde{g}$-measure:

$$\nu(y_0 | y_1^\infty) = \tilde{g}(y)$$

with

$$\widetilde{\beta}_n = \mathrm{var}_n(\tilde{g}) \to 0.$$

Denker & Gordin (2000)	$\beta_n = \mathcal{O}(e^{-\alpha n})$
Chazottes & Ugalde (2011)	$\sum_n n^2 \beta_n < \infty$
Kempton & Pollicott (2011)	$\sum_n n\beta_n < \infty$
Redig & Wang (2010)	$\sum_n \beta_n < \infty$
V. (2011)	$\sum_n \beta_n < \infty$

Decay rates

☞ If $\beta_n = \mathcal{O}(e^{-\alpha n})$, then

$$\widetilde{\beta}_n = \begin{cases} \mathcal{O}(e^{-\widetilde{\alpha}\sqrt{n}}), & [CU, KP] \\ \mathcal{O}(e^{-\widetilde{\alpha} n}), & [DG, RW]. \end{cases}$$

Decay rates

- If $\beta_n = \mathcal{O}(e^{-\alpha n})$, then

$$\widetilde{\beta}_n = \begin{cases} \mathcal{O}(e^{-\widetilde{\alpha}\sqrt{n}}), & [CU, KP] \\ \mathcal{O}(e^{-\widetilde{\alpha} n}), & [DG, RW]. \end{cases}$$

- If $\beta_n = \mathcal{O}(n^{-\alpha})$, then

$$\widetilde{\beta}_n = \begin{cases} \mathcal{O}(n^{-\alpha+2}), & [CU, RW] \\ \mathcal{O}(n^{-\alpha+1}), & [KP] \end{cases}$$

Decay rates

- ☞ If $\beta_n = \mathcal{O}(e^{-\alpha n})$, then

$$\widetilde{\beta}_n = \begin{cases} \mathcal{O}(e^{-\widetilde{\alpha}\sqrt{n}}), & [CU, KP] \\ \mathcal{O}(e^{-\widetilde{\alpha}n}), & [DG, RW]. \end{cases}$$

- ☞ If $\beta_n = \mathcal{O}(n^{-\alpha})$, then

$$\widetilde{\beta}_n = \begin{cases} \mathcal{O}(n^{-\alpha+2}), & [CU, RW] \\ \mathcal{O}(n^{-\alpha+1}), & [KP] \end{cases}$$

Problem: Repeated application only for exp. decay

Fibres

$$X = \mathrm{A}^{\mathbb{Z}_+}, \quad Y = \mathrm{B}^{\mathbb{Z}_+}, \quad \pi : X \to Y, \quad \nu = \mu \circ \pi^{-1}$$

Fibres

$$X = \mathrm{A}^{\mathbb{Z}_+}, \quad Y = \mathrm{B}^{\mathbb{Z}_+}, \quad \pi : X \to Y, \quad \nu = \mu \circ \pi^{-1}$$

For $y \in Y$, the **fibre over y** is

$$X_y = \{x \in X : \pi(x) = y\}.$$

Fibres

$$X = \mathrm{A}^{\mathbb{Z}_+}, \quad Y = \mathrm{B}^{\mathbb{Z}_+}, \quad \pi : X \to Y, \quad \nu = \mu \circ \pi^{-1}$$

For $y \in Y$, the **fibre over y** is

$$X_y = \{x \in X : \ \pi(x) = y\}.$$

Definition. A family of measures $\mu_Y = \{\mu_y\}_{y \in Y}$ is called a family of **conditional measures** for μ on fibres X_y if

(a) μ_y is a Borel probability measure on X_y

(b) for all $f \in L^1(X, \mu)$, the map $y \to \int_{X_y} f(x)\mu_y(dx)$ is measurable and

$$\int_X f(x)\mu(dx) = \int_Y \int_{X_y} f(x)\mu_y(dx)\nu(dy).$$

Disintegration Theorem, John von Neumann (1932): conditional measures $\mu_Y = \{\mu_y\}_{y \in Y}$ on fibres X_y **exist**

Disintegration Theorem, John von Neumann (1932): conditional measures $\mu_Y = \{\mu_y\}_{y\in Y}$ on fibres X_y **exist**

In modern terms,

$$\int_{X_y} f(x)\mu_y(dx) = \mathbb{E}\left(f \,\middle|\, \pi^{-1}\mathfrak{B}_Y \right)$$

Disintegration Theorem, John von Neumann (1932): conditional measures $\mu_Y = \{\mu_y\}_{y \in Y}$ on fibres X_y **exist**

In modern terms,

$$\int_{X_y} f(x)\mu_y(dx) = \mathbb{E}\left(f \,\middle|\, \pi^{-1}\mathfrak{B}_Y \right)$$

$$\mathbb{E}f = \mathbb{E}\left[\mathbb{E}\left(f \,\middle|\, \pi^{-1}\mathfrak{B}_Y \right)\right]$$

Continuity of conditional probabilities

Theorem. Suppose μ is a g-measure for some continuous positive function g. Suppose also that $\pi : X \to Y$ is such that μ admits a family of conditional measures $\mu_Y = \{\mu_y\}_{y\in Y}$ on fibres $\{X_y\}_{y\in Y}$ such that for every $f \in C(X, \mathbb{R})$ the map

$$y \mapsto \int_{X_y} f(x)\mu_y(dx)$$

is continuous on Y (in the product topology). Then $\nu = \mu \circ \pi^{-1}$ is a $\tilde{g}$-measure on Y with

$$\begin{aligned}\tilde{g}(y) &= \tilde{g}((y_0, y_1, y_2, \ldots)) \\ &= \int_{X_y} \Big[\sum_{\bar{x}_0 \in \pi^{-1}(y_0)} g((\bar{x}_0, x_1, x_2, \ldots)) \Big] \mu_y(dx).\end{aligned}$$

Guiding principle

∃ continuous family of conditional measures $\mu_Y = \{\mu_y\}$ then $\nu = \mu \circ \pi^{-1}$ is **GIBBS**

Guiding principle

$\exists$ continuous family of conditional measures $\mu_Y = \{\mu_y\}$ then $\nu = \mu \circ \pi^{-1}$ is **GIBBS**

Remarks:

- ☞ **at most one continuous** family $\{\mu_y\}$
- ☞ for GIBBS μ, the measure μ_y must be GIBBS on X_y for the **same** potential
- ☞ **Hidden Phase Transitions scenario**
  $\nu = \mu \circ \pi^{-1}$ is GIBBS if and only if

$$\left|\mathcal{G}(X_y, \Phi)\right| = 1 \quad \forall y.$$

- ☞ HPT's form an obstruction to continuity of $\{\mu_y\}$?

Summable variation

Fibres are nice lattice systems

$$X_y = \prod_{i=0}^{\infty} \pi^{-1}(y_i) = \prod_{i=0}^{\infty} A_i$$

Summable variation

Fibres are nice lattice systems

$$X_y = \prod_{i=0}^{\infty} \pi^{-1}(y_i) = \prod_{i=0}^{\infty} A_i$$

For g-functions of **summable variation**, there exists a **unique GIBBS** state (=non-homogeneous equilibrium state) μ_y for $\log g$ on X_y.
[Fan-Pollicott (2000)]

Summable variation

Fibres are nice lattice systems

$$X_y = \prod_{i=0}^{\infty} \pi^{-1}(y_i) = \prod_{i=0}^{\infty} A_i$$

For g-functions of **summable variation**, there exists a **unique GIBBS** state (=non-homogeneous equilibrium state) μ_y for $\log g$ on X_y.
[Fan-Pollicott (2000)]

Continuity of $\{\mu_y\}$: uniform convergence of fibrewise shifted Ruelle-Perron-Frobenius operators

$$P_y^n h(x) \to \int_{X_y} h\, \mu_y(dx) \quad \text{as } n \to \infty$$

Construction of $\mu_Y = \{\mu_y\}$

μ is g-measure, $\mu(x_0|x_1^\infty) = g(x)$.

Fix $y \in Y$; for $n \in \mathbb{Z}_+$, define $g_n^y : X_y \to \mathbb{R}$ by

$$g_n^y(x) = g(x_n, x_{n+1}, \dots) \frac{\sum_{\bar{x}_0^{n-1} \in \pi^{-1} y_0^{n-1}} \prod_{k=0}^{n-1} g(\bar{x}_k^{n-1} x_n x_{n+1}^{+\infty})}{\sum_{\bar{x}_0^{n} \in \pi^{-1} y_0^{n}} \prod_{k=0}^{n} g(\bar{x}_k^{n} x_{n+1}^{+\infty})}$$

$$= \frac{\mu(x_n | y_0^{n-1}, x_{n+1}^\infty)}{\mu(y_n | y_0^{n-1}, x_{n+1}^\infty)}$$

The more **natural choice**

$$g_n^y(x) = \frac{g(x_n, x_{n+1}, x_{n+2}, \dots)}{\sum_{\bar{x}_n \in \pi^{-1} y_n} g(\bar{x}_n, x_{n+1}, x_{n+2}, \dots)} = \frac{\mu(x_n | x_{n+1}^\infty)}{\mu(y_n | x_{n+1}^\infty)}$$

Define a sequence of **averaging operators** P_n^y

$$P_n^y f(x) = \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0 \dots a_n x_{n+1} \dots) f(a_0 \dots a_n x_{n+1} \dots),$$

$$G_n^y(x) = \prod_{k=0}^{n} g_n^y(x)$$

Operators P_n^y are positive and satisfy $P_n^y \mathbf{1} = \mathbf{1}$.

A probability measure ρ on X_y is called a **non-homogeneous equilibrium state associated to** $G^y = \{g_n^y\}$ if

$$(P_n^y)^* \rho = \rho$$

Define a sequence of *averaging operators* P_n^y on $C(X_y, \mathbb{R})$

$$P_n^y f(x) = \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0 \dots a_n x_{n+1} \dots) f(a_0 \dots a_n x_{n+1} \dots),$$

$$G_n^y(x) = \prod_{k=0}^{n} g_n^y(x)$$

Operators P_n^y are positive and satisfy $P_n^y \mathbf{1} = \mathbf{1}$.

A probability measure ρ on X_y is called a **non-homogeneous equilibrium state associated to** $G^y = \{g_n^y\}$ if

$$(P_n^y)^* \rho = \rho,$$

Uniqueness of g-measures

Uniqueness of g-measures

Berbee (1987)

$$\sum_n \exp\Big(-\sum_{k=0}^{n} \mathrm{var}_k(\log g)\Big) < \infty$$

Johansson & Öberg (2003): square summability (ℓ^2)

$$\sum_n \big(\mathrm{var}_n(\log g)\big)^2 < \infty$$

suffices.

Uniqueness of g-measures

Berbee (1987)

$$\sum_n \exp\Big(-\sum_{k=0}^{n} \mathrm{var}_k(\log g)\Big) < \infty$$

Johansson & Öberg (2003): square summability (ℓ^2)

$$\sum_n \big(\mathrm{var}_n(\log g)\big)^2 < \infty$$

suffices.

Berger, Hoffman & Sidoravicius: $\ell^{2+\varepsilon}$ is not enough
In ℓ^2-case: unknown speed of convergence
$P_n f \to \int f \, d\mu$

Johansson–Öberg–Pollicott (2010)

- ☞ Generalizes previous results
- ☞ speed of convergence

non-homogeneous version?

Berbee (1987): unique μ_g if

$$\sum_n \exp\Big(-\sum_{k=0}^{n} \mathrm{var}_k(\log g)\Big) < \infty$$

Berbee (1987): unique μ_g if

$$\sum_n \exp\Big(-\sum_{k=0}^{n} \mathrm{var}_k(\log g)\Big) < \infty$$

Moreover, μ_g =Law $\big(\{X_n\}\big)$, then

$$X_n = f(Z_n)$$

for some Markov process $\{Z_n\}$.

functions of Markov chains with $\mathbf{P} \geq 0$

... are not necessarily GIBBS!

Walters-van den Berg example

$$X_n = \pm 1, \quad X_n \sim B(p, 1-p), \quad p \neq \frac{1}{2},$$

Process $Y_n = X_n \cdot X_{n+1}$ is **really bad**

functions of Markov chains with $\mathbf{P} \geq 0$

... are not necessarily GIBBS!

Walters-van den Berg example

$$X_n = \pm 1, \quad X_n \sim B(p, 1-p), \quad p \neq \frac{1}{2},$$

Process $Y_n = X_n \cdot X_{n+1}$ is **really bad**

$Y_n = \phi(X_n^*)$, where $\{X_n^*\}$ is a Markov chain

$$\mathbf{P} = \begin{pmatrix} p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \\ p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \end{pmatrix}.$$

☞ **Dynamical Systems Approach** (Walters, 1986)

- π is finite-to-one: $|X_y| = 2$
- $\nu = B(p, 1-p) \circ \pi^{-1} = B(1-p, p) \circ \pi^{-1}$
- ν is not GIBBS for any **nice** ψ.

☞ **Dynamical Systems Approach** (Walters, 1986)

- π is finite-to-one: $|X_y| = 2$
- $\nu = B(p, 1-p) \circ \pi^{-1} = B(1-p, p) \circ \pi^{-1}$
- ν is not GIBBS for any **nice** ψ.

☞ **Statistical Mechanics** (van den Berg)

$$\nu(1|y_1 \dots, y_n) = \frac{a\lambda^{S_n} + b}{c\lambda^{S_n} + d}, \quad \frac{a}{c} \neq \frac{b}{d},$$

and $|\lambda| < 1$ and

$$S_n = y_1 + y_1 y_2 + \dots + y_1 y_2 \dots y_n.$$

- Chazottes-Ugalde (2003) [MC]
- Han–Marcus (2006) [MC]
- Kempton (2011)
- Yoo (2010) [MC]
- Method based on uniqueness of non-homogeneous equilibrium states also works.
- Seemingly similar results in Statistics/Information Theory [MC]

Subshifts of finite type

$X \subseteq \mathrm{A}^{\mathbb{Z}_+}$ is a **subshift of finite type** (or, TMC) defined by $0/1$ matrix M of size $|A| \times |A|$

$$X = \left\{x \in A^{\mathbb{Z}_+} : \quad M(x_n, x_{n+1}) = 1 \quad \forall n \geq 0\right\}.$$

Non-homogeneous subshifts of finite type

- sequence of finite sets $\{S_n\}$
- sequence $\mathcal{M} = \{M_n\}$ of $0/1$ matrices of size $|S_n| \times |S_{n+1}|$
- non-homogeneous subshift of finite type

$$X_{\mathcal{M}} = \left\{x = (x_n) \in \prod S_n : \quad M_n(x_n, x_{n+1}) > 0\right\}$$

Irreducibility condition: There exists $k > 0$ such that

$$\prod_{i=n}^{n+k} M_i > 0 \quad \forall n.$$

Irreducible SFT X_M admits a unqiue g-measure for a positive continuous function $g : X_M \to \mathbb{R}$ of summable variation.

Irreducible SFT X_M admits a unqiue g-measure for a positive continuous function $g : X_M \to \mathbb{R}$ of summable variation.

Fan-Pollicott: true for irreducible non-homogeneous SFT's.

Irreducible SFT X_M admits a unqiue g-measure for a positive continuous function $g : X_M \to \mathbb{R}$ of summable variation.

Fan-Pollicott: true for irreducible non-homogeneous SFT's.

Require fibres to be irreducible non-homogeneous SFT's.

Prospects and perspectives

Preservation of GIBBS property in $d = 1$. Proofs -
rely on something which could work in $\mathbb{Z}^d$ as well,
- go in the the direction of HPT.

Preservation for specific potentials.

Theory of hidden GIBBS processes.

Practical implications of being non-GIBBS.

Not necessarily symbolic systems

No hidden phase transitions

van Enter, Fernandez, Sokal: 7 step plan

- ☞ if $\forall y, |\mathcal{G}_{X_y}(\Phi)| = 1 \quad \Rightarrow \quad \nu \in \mathcal{G}_Y;$
- ☞ if $\exists y, |\mathcal{G}_{X_y}(\Phi)| > 2 \quad \Rightarrow \quad \nu \notin \mathcal{G}_Y.$

No hidden phase transitions

van Enter, Fernandez, Sokal: 7 step plan

- ☞ if $\forall y$, $|\mathcal{G}_{X_y}(\Phi)| = 1 \quad \Rightarrow \quad \nu \in \mathcal{G}_Y$;
- ☞ if $\exists y$, $|\mathcal{G}_{X_y}(\Phi)| > 2 \quad \Rightarrow \quad \nu \notin \mathcal{G}_Y$.

True in all known cases!

$\mathbb{Z}^d$ vs $\mathbb{Z}$: easy to organize phase-transitions

Potential Φ, inverse temperature β ($\beta < \beta_c(\Phi)$):

$$|\mathcal{G}_X(\beta\Phi)| = 1$$

Conditioning on image spins can lower the temperature beyond the critical value,

$$|\mathcal{G}_{X_y}(\beta\Phi)| > 2$$