Ergodic theorems and Diophantine approximation on homogeneous varieties

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Based on joint work with

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Motivation

- Classical Diophantine approximation attempts to quantify the density of the set $\mathbb{Q}^d$ of rational vectors in affine space $\mathbb{R}^d$ and, more generally, the density of $\mathbb{Q}^d$ in any one of its completion $\mathbb{Q}_v^d$.

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- and a metric dist_v on the variety $\mathrm{X}(\mathbb{Q}_v)$ itself.

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- This function is a natural generalization of the uniform irrationality exponent of a real number ξ.
- $\omega_v(x,\epsilon)$ is non-increasing, bounded as $\epsilon \to 0^+$ if and only if $x \in \mathrm{X}(\mathbb{Q})$, and finite if and only if $x \in \overline{\mathrm{X}(\mathbb{Q})}$. For $x \in \overline{\mathrm{X}(\mathbb{Q})} \setminus \mathrm{X}(\mathbb{Q})$, the growth rate of $\omega_v(x,\epsilon)$ as $\epsilon \to 0^+$ provides a quantitative measure of the Diophantine properties of x with respect to $\mathbb{Q}$.

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We proceed to describe some examples.

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Theorem 1a. Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$:

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- The same results holds for $\mathbb{S}^3 \subset \mathbb{R}^4$, with almost sure exponent $3/2$ which is best possible, and uniform exponent 3.

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- Similar results hold for odd-dimensional spheres.

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- with the approximation rate being given as an explicit exponent.

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We will demonstrate some of the techniques in the case of principal homogeneous space, namely the group variety itself, but before that let us comment on some prior relevant results and methods.

Relevant results

- Using elementary methods based on rational parametrisations of spheres, it was shown by Schmutz 2008 that for every $x \in \mathrm{S}^d(\mathbb{R})$ and $\epsilon \in (0,1)$, there exists $z \in \mathrm{S}^d(\mathbb{Q})$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \text{const}\ \epsilon^{-2\lceil \log_2(d+1) \rceil}.$$

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- It is possible to derive a rate of equidistribution for the group orbits using this estimate, and then derive an exponent for the **uniform** rate of diophantine approximation on spheres.

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- In terms of our notation, he proved upper estimates on the function $\omega_\infty(x, \epsilon)$ and conjectured that for every $\delta > 0$, $\epsilon \in (0, \epsilon_0(\delta))$, and $x \in \overline{\mathrm{X}(K)} \subset \mathrm{X}(\mathbb{R})$,

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- This conjecture is remarkably strong as one can show that the exponent in this estimate is the best possible. A similar conjecture has been formulated in the case of algebraic tori.

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- The formulation of Diophantine approximation as a "shrinking target" problem for one-parameter group orbits was developed by the previous authors and also by Hersonsky-Paulin in the context of hyperbolic geometry. More recent further work along these lines was developed by Athreya, Ghosh and Prasad.

- The duality principle - namely the reduction of dynamical questions about the Γ-orbits in G/H to questions about the dynamics of H-orbits in $\Gamma \setminus G$ - was expanded significantly over the last decade by Ledrappier, Ledrappier-Pollicott, and Gorodnik. A definitive analysis of equidistribution results via duality is due to Gorodnik-Weiss. Effective forms of the duality principle have been developed by Gorodnik+N.

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- The LPS construction of dense subgroups of $SO_3(\mathbb{R})$ produces the best possible spectral gap estimate. This underlies the best possible rate of Diophantine approximation for $\mathbb{S}^2$, $\mathbb{S}^3$.

- A more general spectral estimate for averaging on Hecke points in the automorphic representation was established by Clozel-Oh-Ullmo 2002, and used to obtain quantitative equidistribution of Hecke points for smooth functions in the automorphic representation. This can be used to establish a **uniform** rate of diophantine approximation on the group variety.

General set-up

- The abstract set-up consists of an lcsc group G which is a product of two closed subgroups $G = G_{S_1} \times G_{S_2}$, and a lattice $\Gamma \subset G$, which is irreducible and embedded diagonally, namely its (injective) projection to each of the factors G_{S_1} and G_{S_2} is dense.

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- Typically, each of the groups G_{S_i} decomposes further, with $G_{S_1} = \prod_{v \in S_1} G_v$, and $G_{S_2} = \prod_{v \in S_2} G_v$, where S_1 and S_2 are two disjoint non-empty index sets.

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- We fix a distance function (called height) on each G_v, namely a proper continuous submultiplicative function $H_v : G_v \to \mathbb{R}_+$, with $H_v(xy) \leq H_v(x)H_v(y)$, and the associated left-invariant metrics d_v on the component groups G_v.

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- We consider the associated distance (height) function on G, given by the product of the local factors.

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- Formulated quantitatively, we would like to realize the simultaneous approximation $d_v(g_v, \gamma) \leq \epsilon_v$ for $v \in S_1$ by an element $\gamma \in \Gamma$ with height bounded by $H(\gamma) \leq \left(\prod_{v \in S_1} \epsilon_v\right)^{\kappa}$ with $0 < \kappa < \infty$ fixed.

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- We are interested in two kinds of results : the first states that a certain rate κ is valid for almost all points in G_{S_1}, and the second that a certain rate $\kappa' \geq \kappa$ is valid for all points in G_{S_1}.

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- We fix a semisimple algebraic matrix group G defined over K, and our goal is simultaneous approximation of elements in the group $G_{S_1} = \prod_{v \in S_1} G_v$ where $G_v = \mathrm{G}(K_v)$.

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- The set of matrices whose entries are K-rational and w-integral for every valuation $w \notin (S_1 \cup S_2)$ is a lattice which is embedded diagonally in $\Gamma \subset G = G_{S_1} \times G_{S_2}$, whose elements satisfy integrality constrains.

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- The projection of the lattice to G_{S_1} is a dense subgroup of G_{S_1}, consisting of the elements allowed in the approximation process.

The approximation process

- To approximate a point $g_1 \in G_{S_1}$ by an element $\gamma \in \Gamma$ we must insure that $\gamma^{-1} g_1$ lies in a small neighborhood $\mathcal{O}^1_\varepsilon$ of the identity element in G_{S_1}. Equivalently, we can look at the Γ-orbit of $g_1 G_{S_2}$ in the homogeneous space $G/G_{S_2} \cong G_{S_1}$ and its approach to the identity coset $[G_{S_2}]$. In G, the condition is $(\gamma^{-1}, \gamma^{-1})(g_1, e) \in \mathcal{O}^1_\varepsilon \times G_{S_2}$.

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- The application of the duality principle in this case consists of the observation that to find such an element γ, it suffices to find a point x in a small neighborhood $(g_1 \mathcal{O}^1_\varepsilon, \mathcal{O}^2_\varepsilon)$ of g_1 in G, whose orbit under the complementary group G_{S_2} is close to a point $(\gamma^{-1}, \gamma^{-1}) \in \Gamma$.

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- Equivalently, in the homogeneous space $\Gamma \setminus G$ the orbit of the small neighborhood $\Gamma g_1 \mathcal{O}_\varepsilon$ of the coset Γg_1 under G_{S_2} must return to small neighborhoods of the identity coset $[\Gamma]$.

The approximation process

- To approximate a point $g_1 \in G_{S_1}$ by an element $\gamma \in \Gamma$ we must insure that $\gamma^{-1} g_1$ lies in a small neighborhood $\mathcal{O}_\varepsilon^1$ of the identity element in G_{S_1}. Equivalently, we can look at the Γ-orbit of $g_1 G_{S_2}$ in the homogeneous space $G/G_{S_2} \cong G_{S_1}$ and its approach to the identity coset $[G_{S_2}]$. In G, the condition is $(\gamma^{-1}, \gamma^{-1})(g_1, e) \in \mathcal{O}_\varepsilon^1 \times G_{S_2}$.

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- Thus the rate at which the Γ-orbit of $[g_1 G_{S_2}]$ in G/G_{S_2} visits neighborhoods of the coset $[G_{S_2}]$ is translated to the rate in which the G_{S_2}-orbit of Γg_1 in $\Gamma \setminus G$ visits neighborhoods of the coset $[\Gamma]$.

Ergodic theorem for G-actions

- Very briefly, to obtain a quantitative gauge for recurrence to neighborhoods of the identity coset in $\Gamma \setminus G$ we consider height balls of increasing size in the acting group G_{S_2}, and we seek an estimate saying that in a ball of height h in G_{S_2} we can find an element that yields an approximation to the coset $[\Gamma] \in \Gamma \setminus G$ up to distance bounded by $h^{-\kappa}$.

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- The underlying method to derive such an estimate is spectral, and we use the estimate of the operator norms of normalized height ball averages on G_{S_2} acting by convolution on $L^2(\Gamma \setminus G)$.

- These estimates rely of the spectral estimates for spherical functions in the automorphic representation and their quality is determined by the bounds towards the generalized Ramanujan-Petersson conjectures.

- The defining feature of this analysis is that the estimate of the operator norm of a normalized ball of height h decays like $h^{-\kappa}$. This of course determines a rate in which the ball averages distribute the mass of a neighborhood of a point in the space $\Gamma \setminus G$ as the height increases, and thus the rate in which this convolution must enter shrinking small neighborhoods of the identity coset, measures by the decay of the volume of these shrinking targets.

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- Matching the decay of the volume of the shrinking targets to the rate of decay of the norm ball averages we obtain an upper bound on the rate of approximation via a Borel Cantelli argument.

Upper bounds on operator norms

- The fundamental quantity we should now estimate is the operator norm of a family of averaging operators on height balls, acting on $L^2(\Gamma \setminus G)$.

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- The averaging operators on $L^2(\Gamma \setminus G)$ are defined by

$$\pi(\beta_h)\phi(\zeta) = \frac{1}{m_G(B_h)} \int_{g \in B_h} \phi(g^{-1}\zeta) dm_G(g)$$

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  where m_G is Haar measure on G.
- The lattice Γ being irreducible in G, the action of G_{S_2} on $\Gamma \setminus G$ is ergodic.

The spectral gap property

- Furthermore, β_h satisfy the quantitative mean ergodic theorem with parameter q, namely for every $\phi \in L^2(\Gamma \setminus G)$,

$$\left\| \pi(\beta_h)\phi - \int_{G/\Gamma} \phi \, d\mu \right\|_2 \ll_\delta m_G(B_h)^{-\frac{1}{q}+\delta}$$

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- Equivalently this estimate is a quantitative version of the spectral gap property of the automorphic representation.

- The parameter q is determined by the integrability parameter of the spherical functions appearing in the automorphic representation and is subject to the Ramanujan conjectures.

- Caveat : We have not mentioned yet the fact that there may appear automorphic characters in the automorphic representation, which imply that the limit in the ergodic theorem stated above is valid only in the space orthogonal to their span. The span is finite dimensional so this issue can easily be resolved separately.

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- To conclude the estimate of operator norms we note the fact that the volume growth of height balls is estimated by

$$m_{G_{S_2}}(B_h) \gg h^{\mathfrak{a}_{S_2}(G)}$$

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- Thus both the spectral parameter and the volume growth can be given explicitly, and this yields an explicit upper bound for the rate of approximation.

- finally, to establish that the rate is best possible, the matching lower bound is estimated essentially by a pigeon hole argument, which estimates the rate in which the set of elements in Γ of bounded height can become ε-dense in a variety of a given dimension.

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- the rate of growth of lattice elements of bounded height is of course the basic problem of counting lattice point in a semisimple groups which has a very general solution.

- Going back to conclude the proof of Theorem 1, in order to deduce the exponents for S^2 and S^3, consider the algebraic group G of norm one elements of Hamilton's quaternion algebra, which can be identified with the variety S^3.

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- Since $p = 1 \mod 4$, the quaternion algebra splits over p and ramifies at ∞. In this case, we have $\mathfrak{q}_{V_{\mathbb{Q}} \setminus \{p\}}(\mathrm{G}) = 2$, namely the local representations appearing in the automorphic representation are all tempered, as noted by Lubotzky-Phillips-Sarnak.

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- S^2 can be identified with the homogeneous space consisting of pure quaternions of norm 1, and similar considerations give the exponent 2.

Quadrics

Let Q be a non-degenerate quadratic form in three variables defined over a number field $K \subset \mathbb{R}$, $a \in K$, and

$$\mathrm{X} = \{Q(x) = a\}.$$

For a finite set of non-Archimedean places of K, we denote by O_S the ring of S-integers. We suppose that Q is isotropic over S and $\mathrm{X}(O_S) \neq \emptyset$. Then assuming the Ramanujan–Petersson conjecture for PGL_2 over K, our main results imply that (w.r.t. the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^3$, the completion at $v = \infty$)

(i) for almost every $x \in \mathrm{X}(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x, \delta))$, there exists $z \in \mathrm{X}(O_S)$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-2-\delta},$$

where the exponent 2 is the best possible.

(ii) for every $x \in \mathrm{X}(\mathbb{R})$ with $\|x\| \le r$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(r, \delta))$, there exists $z \in X(O_S)$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-4-\delta}.$$

Using the best currently known estimates towards the Ramanujan–Petersson conjecture our method gives unconditional solutions to (i) and (ii) with

$$\mathrm{H}(z) \leq \epsilon^{-\frac{18}{7}-\delta} \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-\frac{36}{7}-\delta}$$

respectively. Moreover, when $K = \mathbb{Q}$, (i) and (ii) give unconditional solutions to the problem of diophantine approximation on the hyperboloid $X(\mathbb{R})$ (when Q is isotropic over $\mathbb{R}$), with

$$\mathrm{H}(z) \leq \epsilon^{-\frac{64}{25}-\delta} \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-\frac{128}{25}-\delta}$$

respectively.
We also mention that a positive proportion of all places satisfy the bound predicted by the Ramanujan–Petersson conjecture. For such S, results (i) and (ii) hold unconditionally.

Lower bound for the rate of approximation

The basic gauge of the quality of approximation of an element $g_v \in G_v$ by elements by elements in the projection of Γ to G_v is given by

$$\omega_v(g, \epsilon) := \min\{\mathrm{H}(\gamma) : \gamma \in \Gamma,\ d_v(g_v, \gamma_v) \le \epsilon\}\,.$$

For a subset Y of G_v, we set

$$\omega_v(Y, \epsilon) := \sup_{y \in Y} \omega_v(y, \epsilon).$$

Assuming that Y is not contained in the projection of Γ to G_v, one can give a very general lower bound on $\omega_v(Y, \epsilon)$ that depends only on two fundamental metric properties of Y.

The first is the Minkowski dimension $d_v(Y)$, which measures the growth of the number of ϵ-balls in G_v needed to cover Y, and the second is the exponent $\mathfrak{a}_v(Y)$, which measures the growth of the set of approximating points of height at most h in a neighborhood of Y. The precise definition are as follows.
The *Minkowski dimension* of a subset Y of G_v is defined by

$$d(Y) := \liminf_{\epsilon \to 0^+} \frac{\log D(Y, \epsilon)}{\log(1/\epsilon)},$$

where $D(Y, \epsilon)$ denotes the least number of balls of radius ϵ (w.r.t. the distance d_v) needed to cover Y.

The *exponent* of a subset Y of G_v is defined by

$$\mathfrak{a}_v(Y) := \inf_{\mathcal{O} \supset Y} \limsup_{h \to \infty} \frac{\log A_v(\mathcal{O}, h)}{\log h},$$

where $\mathcal{O}$ runs over open neighborhoods of Y in G_v, and

$$A_v(\mathcal{O}, h) := |\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leq h, \gamma_v \in \mathcal{O}\}|.$$

Since Y is not contained in the projection of Γ to G_v, $\omega_v(Y, \epsilon) \to \infty$ as $\epsilon \to 0^+$.

Let us fix a sufficiently small neighborhood $\mathcal{O}$ of Y, two positive constants $\delta_1, \delta_2 > 0$ and $0 < \epsilon < \epsilon_0(\mathcal{O}, \delta_1, \delta_2)$. By definition of $A_v(\mathcal{O}, h)$, we can assume that $A_v(\mathcal{O}, h) \leq h^{\mathfrak{a}_v(Y)+\delta_1}$ for all sufficiently large h. In particular, this holds for $h = \omega_v(Y, \epsilon)$ for sufficiently small ϵ, since $\omega_v(Y, \epsilon) \to \infty$ as $\epsilon \to 0^+$. Hence $A_v(\mathcal{O}, \omega_v(Y, \epsilon)) \leq \omega_v(Y, \epsilon)^{\mathfrak{a}_v(Y)+\delta_1}$.
Now by definition of $\omega_v(Y, \epsilon)$, for each point $y \in Y$ we can find $\gamma \in \Gamma$ such that $d_v(y_v, \gamma_v) \leq \epsilon$, and $H(\gamma) \leq \omega_v(Y, \epsilon)$. Draw a ball of radius ϵ around each of the $A_v(\mathcal{O}, \omega_v(Y, \epsilon))$ points γ_v which satisfy that their height is bounded by $h = \omega_v(Y, \epsilon)$ and their v-component is in $\mathcal{O}$. Clearly their union cover Y with balls of radius ϵ, so that by definition $D(Y, \epsilon) \leq A_v(\mathcal{O}, \omega_v(Y, \epsilon))$. Finally, by definition of the Minkowsky dimension, for sufficiently small ϵ, $\epsilon^{-d(Y)+\delta_2} \leq D(Y, \epsilon)$.

Clearly, the inequalities obtained

$$\epsilon^{-d(Y)+\delta_1} \leq D(Y,\epsilon) \leq A_v(\mathcal{O},\omega_v(Y,\epsilon)) \leq \omega_v(Y,\epsilon)^{\mathfrak{a}_v(Y)+\delta_2}$$

imply that for every $\delta > 0$ and sufficiently small $\epsilon > 0$ depending on δ,

$$\omega_v(Y,\epsilon) \geq \epsilon^{-\frac{d(Y)}{\mathfrak{a}_v(Y)}+\delta}.$$