Dynamical properties of the negative beta transformation

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Outline

1. The $(-\beta)$-transformation and $(-\beta)$-shift
2. Dynamical properties of $(-\beta)$-transformation
3. $(-\beta)$-number VS β-number
4. Questions

The $(-\beta)$-transformation and $(-\beta)$-shift

I. The β-transformation and the $(-\beta)$-transformation

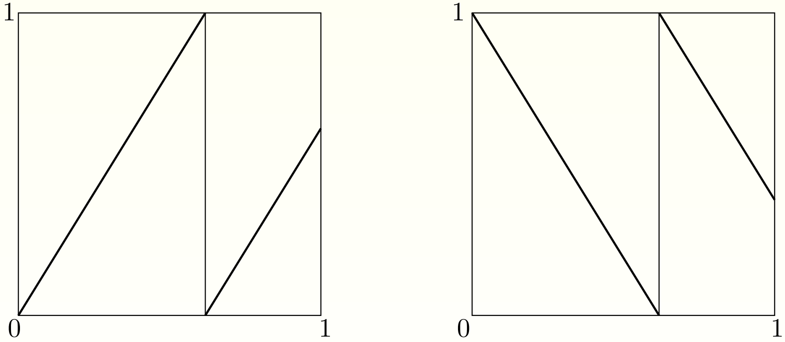

FIG.: β-transformation (left) and $(-\beta)$-transformation (right), $\beta = \frac{1+\sqrt{5}}{2}$.

II. The $(-\beta)$-transformation

Define $T_{-\beta} : (0,1] \to (0,1]$ by

$$T_{-\beta}(x) := -\beta x + \lfloor \beta x \rfloor + 1\,.$$

Let

$$d_{-\beta,1}(x) = \lfloor \beta x \rfloor + 1\,, \quad d_{-\beta,n}(x) = d_{-\beta,1}(T_{-\beta}^{n-1}(x)) \quad \text{for } n \geq 1\,.$$

Then

$$x = \frac{-d_{-\beta,1}}{-\beta} + \frac{-d_{-\beta,2}}{(-\beta)^2} + \frac{-d_{-\beta,3}}{(-\beta)^3} + \frac{-d_{-\beta,4}}{(-\beta)^4} + \cdots\,.$$

Sequence $d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x)\cdots \longrightarrow$ **$(-\beta)$-expansion of x**.

Example : $\beta = \frac{1+\sqrt{5}}{2}$,

$$1 = \frac{-2}{-\beta} + \frac{-1}{(-\beta)^2} + \frac{-1}{(-\beta)^3} + \frac{-1}{(-\beta)^4} + \cdots\,.$$

$\to$ expansion of $1 \longrightarrow 2\overline{1} = 2111....$

III. Remarks about the definition

- **Ito and Sadahiro 2009** : On the interval $[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$:

$$x \mapsto -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor.$$

  $\longrightarrow$ conjugate to our $T_{-\beta}$ through the conjugacy function $\phi(x) = \frac{1}{\beta+1} - x$. So all results can be translated to our case.

- Our definition is one case of generalized β-transformations studied by **Góra 2007** and **Faller 2008 (Ph.D Thesis)**.

IV. Other works on $(-\beta)$-transformations

- P. Ambrož, D. Dombek, Z. Masákova, and E. Pelantová
- A. Bertrand-Mathis
- K. Dajani and C. Kalle
- C. Frougny and A. Lai

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V. Admissible sequence and $(-\beta)$-shift

A sequence $a_1a_2\cdots$ is said **admissible** if $\exists x \in (0,1], d_{-\beta}(x) = a_1a_2\cdots$.

Alternate order : $a_1a_2\cdots \prec b_1b_2\cdots$ if and only if

$$\exists k \geq 1, \quad a_i = b_i \text{ for } i < k \quad \text{and} \quad (-1)^k(b_k - a_k) < 0.$$

Denote $a_1a_2\cdots \preceq b_1b_2\cdots$, if $a_1a_2\cdots \prec b_1b_2\cdots$ or $a_1a_2\cdots = b_1b_2\cdots$.

The **$(-\beta)$-shift** $S_{-\beta}$ on the alphabet $\{1,\ldots,\lfloor\beta\rfloor+1\}$ is the closure of the set of admissible sequences.

Define

$$d^*_{-\beta}(0) := \lim_{x\to 0+} d_{-\beta}(x)$$

$$= \begin{cases} \overline{1b_1b_2\cdots b_{q-1}(b_q-1)}, & \text{if } d_{-\beta}(1) = \overline{b_1\cdots b_{q-1}b_q} \quad \text{for some odd } q \\ 1d_{-\beta}(1) & \text{otherwise.} \end{cases}$$

VI. Admissible sequence and $(-\beta)$-shift (continued)

Theorem (Ito-Sadahiro 2009)

A sequence $a_1a_2\cdots$ is admissible if and only if for each $n \geq 1$

$$d^*_{-\beta}(0) \prec a_na_{n+1}\cdots \preceq d_{-\beta}(1).$$

A sequence $a_1a_2\cdots$ is in $S_{-\beta}$ if and only if for each $n \geq 1$

$$d^*_{-\beta}(0) \preceq x_nx_{n+1}\cdots \preceq d_{-\beta}(1).$$

Theorem (Frougny-Lai 2009)

The $(-\beta)$-shift is of **finite type** *if and only if $d_{-\beta}(1)$ is* **purely periodic**.

Theorem (Ito-Sadahiro 2009)

The $(-\beta)$-shift is **sofic** *if and only if $d_{-\beta}(1)$ is* **eventually periodic**.

Theorem (Frougny-Lai 2009)

If β is a **Pisot number**, *then the $(-\beta)$-shift is* **sofic**.

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Dynamical properties of $(-\beta)$-transformation

I. Some notions of dynamical systems

Suppose $T : X \to X$ be a dynamical system.

- **locally eventually onto** : if for every nonempty open subset $U \subset X$, there exists a positive integer n_0 such that for every $f^{n_0}(U) = X$.
- **exactness** : T acting on $(X, \mathcal{B}, \mu)$ is called exact if

  $$\cap_{n=0}^{\infty} T^{-n}\mathcal{B} = \{X, \emptyset\}$$

  or equivalently, for any positive measure set A with $T^n(A) \in \mathcal{B}$ $(n \geq 0)$,

  $$\mu(T^n(A)) \to 1 \ (n \to \infty).$$

- **maximal entropy measure** : the measure attaining the maximum of

  $$\sup\{h_\mu : \mu \text{ invariant}\}.$$

- **intrinsic ergodicity** : the maximal entropy measure is unique.

II. General piecewise monotone transformation

$T : [0,1] \to [0,1]$.

- finite partition of $[0,1]$: $P = \{P_1, \cdots, P_N\}$.
- on each P_i, T is monotonic, Lipschitz continous and $|T'| \geq \rho > 1$.

Lasota-Yorke 1974 : There is an invariant measure $d\mu = hd\lambda$, where $d\lambda$ is the Lebesgue measure and h is a density of bounded variation.

Keller 1978 : The set $\{h \neq 0\}$ is a **finite** union of intervals.

Wagner 1979 : We can decompose $[0,1] = \cup_{i=1}^{s} A_i \cup B$, such that

- on each A_i there is an invariant measure which is equivalent to the Lebesgue measure restricted to A_i
- each A_i can be decomposed as $A_i = \cup_{j=1}^{m_i} A_{ij}$ and T^{m_i} is **exact** on each A_{ij}.
- the set B satisfies $T^{-1}B \subset B$ and $\lim_{n\to\infty} \lambda(T^{-n}B) = 0$.

Hofbauer 1981 : The number of maximal entropy measures is **finite**. If T is topological transitive, then it is **intrinsic ergodic**.

III. Dynamical properties - the $(-\beta)$ case

Ito-Sadahiro 2009 : Let $h_{-\beta}$ be a real-valued function defined on $(0,1]$ by

$$h_{-\beta}(x) = \sum_{n\geq 1,\ T^n_{-\beta}(1)\geq x} \frac{1}{(-\beta)^n}.$$

Then the measure $h_{-\beta}(x)d\lambda$ is an invariant measure of $T_{-\beta}$.

Remark : The density may be zero on some intervals. So the invariant measure is not equivalent to the Lebesgue measure. (**Different to the β case**).

Góra 2007 : for $\beta > \gamma_1 = 1.3247...$ (the smallest Pisot number), the transformation $T_{-\beta}$ is exact and he conjectured that this would hold for all $\beta > 1$.

Faller 2008 : $\beta > \sqrt[3]{2}$, $T_{-\beta}$ admits a unique maximal measure.

IV. How many gaps ?

A question :

For a given β, how many intervals (gaps) on which the density $h_{-\beta}$ equals to 0 ?

When β decreases, the numbers should be like

$$0,\ 1,\ 2,\ 5,\ 10,\ 21,$$

What is the next ?

V. Our results-Notations

For each $n \geq 0$, let γ_n be the positive real number defined by

$$\gamma_n^{g_n+1} = \gamma_n + 1\,, \quad \text{with} \quad g_n = \lfloor 2^{n+2}/3 \rfloor\,.$$

Then

$$2 > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > 1\,.$$

Note that γ_0 is the golden ratio and that γ_1 is the smallest Pisot number.
For each $n \geq 0$ and $1 < \beta < \gamma_n$, set

$$\mathcal{G}_n(\beta) = \left\{G_{m,k}(\beta) \mid 0 \leq m \leq n,\, 0 \leq k < \tfrac{2^{m+1}+(-1)^m}{3}\right\},$$

with open intervals

$$G_{m,k}(\beta) = \begin{cases} \left(T_{-\beta}^{2^{m+1}+k}(1),\ T_{-\beta}^{(2^{m+2}-(-1)^m)/3+k}(1)\right) & \text{if } k \text{ is even,} \\ \left(T_{-\beta}^{(2^{m+2}-(-1)^m)/3+k}(1),\ T_{-\beta}^{2^{m+1}+k}(1)\right) & \text{if } k \text{ is odd.} \end{cases}$$

VI. Our results- Theorems

We call an interval a gap if the density of the invariant measure is zero on it.

Theorem (L-Steiner arXiv 2011)

If $\beta \geq \gamma_0$, then there is no gap. If $\gamma_{n+1} \leq \beta < \gamma_n$, $n \geq 0$, then the set of gaps is $\mathcal{G}_n(\beta)$ which consists of $g_n = \lfloor 2^{n+2}/3 \rfloor$ disjoint non-empty intervals.

Define

$$G(\beta) = \begin{cases} \emptyset & \text{if } \beta \geq \gamma_0\,, \\ \bigcup_{I \in \mathcal{G}_n(\beta)} I & \text{if } \gamma_{n+1} \leq \beta < \gamma_n,\ n \geq 0\,. \end{cases}$$

Theorem (L-Steiner arXiv 2011)

The transformation $T_{-\beta}$ is locally eventually onto on $(0,1] \setminus G(\beta)$,

$$T_{-\beta}^{-1}(G(\beta)) \subset G(\beta) \quad \text{and} \quad \lim_{n\to\infty} \lambda\big(T_{-\beta}^{-n}(G(\beta))\big) = 0\,.$$

VII. Our results-Theorems (continued)

Define a morphism on the symbolic space $\{1,2\}^{\mathbb{N}}$ by

$$\varphi:\ 1 \mapsto 2\,, \quad 2 \mapsto 211\,.$$

Theorem (L-Steiner arXiv 2011)

(1) For every $n \geq 0$, we have $d_{-\gamma_n}(1) = \varphi^n(2\,1^\omega)$. Hence

$$\lim_{\beta\to 1} d_{-\beta}(1) = \lim_{n\to\infty} \varphi^n(2) = 211222112112122\cdots\,.$$

(2) When β tends to 1, the $(-\beta)$ shift $S_{-\beta}$ tends to the substitution dynamical system determined by $211222112112122\cdots$.

Remarks :

(1) Thue-Morse sequence : $0110\ 1001\ 0110....$ $\rightarrow$ $\not{0}\ 110\ 1001\ 0110....$

Then count the numbers of consecutive ones and zeros :

$$\underbrace{11}_{2}\underbrace{0}_{1}\ \underbrace{1}_{1}\underbrace{00}_{2}\ 1\ 0110.....$$

(2) $|\varphi^m(2)| = |\varphi^{m+1}(1)| = g_n + \frac{1-(-1)^n}{2}$ and $|\varphi^m(21)| = 2^{m+1}$.

VIII. Our results-Corollaries

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ is exact.

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ has a unique maximal measure, and hence is intrinsic ergodic.

Corollary

The set of periodic points is dense in $(0,1] \setminus G(\beta)$.

IX. Our results-Proofs

For every word $a_1 \cdots a_n \in \{1,2\}^n$, $n \geq 0$, define the polynomial

$$P_{a_1 \cdots a_n} = (-X)^n + \sum_{k=1}^{n} a_k (-X)^{n-k} \in \mathbb{Z}[X]$$

Lemma

For $1 \leq m < n$, *we have*

$$P_{a_1 \cdots a_n} = (-X)^{n-m} \big(P_{a_1 \cdots a_m} - 1 \big) + P_{a_{m+1} \cdots a_n} \, .$$

For $n \geq 0$ *we have the identities :*

- $X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}$
- $1 - P_{\varphi^n(1)} = X^{\frac{1+(-1)^n}{2}} \prod_{k=0}^{n-1} \big(X^{|\varphi^k(1)|} - 1 \big)$
- $P_{\varphi^n(21)} - P_{\varphi^n(2)} = \big(X^{g_n+1} - X - 1 \big) \prod_{k=0}^{n-1} \big(X^{|\varphi^k(1)|} - 1 \big)$

X. Our results-Proofs-continued

Let $1 < \beta < \gamma_n$, $n \geq 0$. Then the elements of $\mathcal{F}_n(\beta)$ and $\mathcal{G}_n(\beta)$ are intervals of positive length which form a partition of $(0, 1]$. Moreover, we have

1. $d_{-\beta}(1)$ starts with $\varphi^{n+1}(2)$, $T^{|\varphi^{n+1}(2)|}(1) \in F_{n,0}$,
2. $1/\beta$ is an interior point of F_{n,g_n-1},
3. $F_{n,k} = T^k(F_{n,0})$ for all $0 \leq k < g_n$,
4. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n,0}$, $T(F_{n,g_n}) = F_{n+1,0}$, if n is even,
5. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n+1,0}$, $T(F_{n,g_n}) = F_{n,0}$, if n is odd,

XI. Hofbauer's recent work

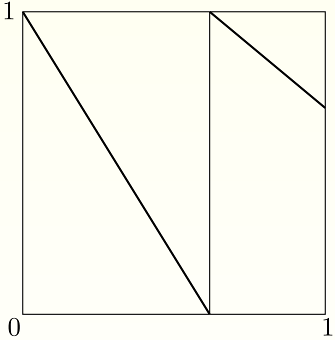

FIG.: Piecewise linear transformation with two different negative slopes.

Theorem (Hofbauer preprint 2011)

We can explicitly construct the non-wandering set which is a union of periodic orbits and some closed intervals.

$(-\beta)$-number VS β-number

I. Definitions

Extend the definition of T_β to 1 by $T_\beta(1) := \beta - \lfloor \beta \rfloor$

- β-number (Parry Number) : number $\beta > 1$ such that the orbit of 1 under T_β is eventually periodic.
- $(-\beta)$-number : number $\beta > 1$ such that the orbit of 1 under $T_{-\beta}$ is eventually periodic.
- Pisot Number : algebraic integer number $\beta > 1$, whose conjugates all have modulus < 1.
- Perron Number : algebraic integer number $\beta > 1$, whose conjugates all have modulus $< \beta$.

II. Results

Schmidt 1980, Bertrand 1977 : All Pisot numbers are β-numbers.

Frougny-Lai 2009 : All Pisot numbers are $(-\beta)$-numbers.

Lind 1984, Denker-Grillenberger-Sigmund 1976 : All β-numbers are Perron numbers.

Solomyak 1994 : All conjugates of a β-number have modulus less than golden number.

Masákova-Pelantová arXiv 2010 : All conjugates of a $(-\beta)$-number have modulus less than 2, so all $(-\beta)$-numbers with modului ≥ 2 are Perron numbers.

Theorem (L-Steiner, arXiv 2011)

All $(-\beta)$-numbers are Perron numbers.

III. Results-Conitnued

Lemma

Let $\beta > 1$ such that $\beta^4 = \beta + 1$, i.e., $\beta \approx 1.2207$. Then $T^{10}_{-\beta}(1) = T^5_{-\beta}(1)$, and $(T^n_\beta(1))_{n\geq 0}$ is aperiodic.

Lemma

Let $\beta > 1$ such that $\beta^7 = \beta^6 + 1$, i.e., $\beta \approx 1.2254$. Then $T^7_\beta(1) = 0$, and $(T^n_{-\beta}(1))_{n\geq 0}$ is aperiodic.

Theorem (L-Steiner, arXiv 2011)

The set $(-\beta)$-numbers and the set of β-numbers do not include each other.

Questions

I. About the dynamics

- Are the periodic points for the $(-\beta)$-shift uniformly distributed with respect to the unique measure of maximal entropy ?
- Are the invariant measures concentrated on periodic orbits dense in the set of all invariant measures ?
- Characterization of the β such that the corresponding $(-\beta)$-shift satisfies the specification property.
- Characterization of the β such that the corresponding $(-\beta)$-shift is synchronizing.

II. Classification and size

Classical Rényi β case (**Blanchard 1989, Schmeling 1997**) :

- Class C1. simple Parry numbers (S_β is a subshift of finite type)
  $\longrightarrow$ **dense**.
- Class C2. Parry numbers (S_β is a is sofic.)
  $\longrightarrow$ **at most countable**.
- Class C3. (S_β satisfies the specification property)
  $\longrightarrow$ **Lebesgue measure** 0**, Hausdorff dimension** 1.
- Class C4. (S_β is synchronizing)
  $\longrightarrow$ **Lebesgue measure** 0**, Hausdorff dimension** 1.
- Class C5. (the rest)
  $\longrightarrow$ **Lebesgue measure** 1.

Question : What is about the $-\beta$ case ?

III. Univoque set and size

Rényi's β case :

Let $J_\beta := [0, (\lceil\beta\rceil - 1)/(\beta - 1)]$. We are interested in the following set

$$\mathbb{U} := \{(x,\beta) : \beta > 1, x \in J_\beta, x \text{ has exactly one expansion in base } \beta\},$$

and the one dimensional sections :

$$\mathcal{U}_\beta := \{x \in J_\beta : (x,\beta) \in \mathbb{U}\}, \quad \mathcal{U} := \{\beta > 1 : (1,\beta) \in \mathbb{U}\}.$$

- $\mathbb{U}$: $Leb = 0$, $HD = 2$ (**de Vries-Komornik 2010**).
- $\mathcal{U}_\beta$: (**Glendinning-Sidorov 2001**)
  1. $1 < \beta \le (1+\sqrt{5})/2$: two elements ;
  2. $(1+\sqrt{5})/2 < \beta < \beta_{KL}$: countably infinite ;
  3. $\beta_{KL} < \beta \le 2$: positive Hausdorff dimension. (tends 1 when $\beta \to 2$: detailed proof in **Jordan-Shmerkin-Solomyak 2010**)

  Here $\beta_{KL} \approx 1.787$ is the **Komornik-Loreti constant**.
- $\mathcal{U}$: continuum many (**Erdös-Horváth-Joó 1991**), $Leb = 0$, $HD = 1$ (**Daróczy-Kátai 1995**).

Question : What is about the $-\beta$ case ?

IV. Schmidt conjecture

Salem Number : algebraic integer number $\beta > 1$, whose conjugates all have modulus ≤ 1 and at least one $= 1$.

Denote $\mathrm{Per}(\beta)$, $\mathrm{Per}(-\beta)$ the sets of eventually periodic points for T_β and $T_{-\beta}$ respectively.

Schmidt 1980 : If $\mathbb{Q} \cap [0, 1) \subset \mathrm{Per}(\beta)$, then β is either a Pisot number or a Salem number.

Masákova-Pelantová arXiv 2010 : If $\mathbb{Q} \cap (0, 1] \subset \mathrm{Per}(-\beta)$, then β is either a Pisot number or a Salem number.

Conversely,

Bertrand 1977 : If β is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \mathrm{Per}(\beta)$.

Frougny-Lai 2009 : If β is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \mathrm{Per}(-\beta)$.

Schmidt conjecture, 1980

If β is a Salem number, then $\mathbb{Q} \cap (0, 1] \subset \mathrm{Per}(\beta)$.

Question : Schmidt conjecture for $-\beta$ case ?

V. Schmidt conjecture-progress (β case)

Fact : The degree of a Salem number is even and ≥ 4.

Boyd 1989 : If β is a Salem number of degree 4, then the orbit of 1 under T_β is eventually periodic.

Boyd 1996 : Some examples of Salem numbers of degree 6.

"There are also some very large orbits which have been shown to be finite : an example is given for which the preperiod length is 39420662 and the period length is 93218808".