Recovering a Theorem of Poincaré

Gonzalo Contreras

CIMAT
Guanajuato, Mexico

University of Warwick,
July 11, 2011.
Dedicated to Anthony Manning.

C^2-densely, the 2-sphere has an elliptic closed geodesic.

Gonzalo Contreras
CIMAT, Guanajuato
México

Fernando Oliveira
UFMG, Belo Horizonte
Brasil

M. Herman's memorial issue
Ergodic Theory & Dynamical Systems, 2004.

1905 Tr. AMS. H. Poincaré claims:

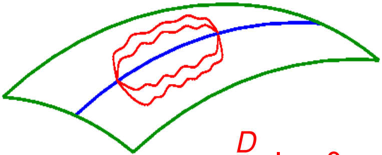
Any convex surface in $\mathbb{R}^3$ has an elliptic or degenerate *simple* closed geodesic.

1979 A.I. Grjuntal: Counterexample.

Geodesics

M riemannian surface.

Geodesic = curve that locally minimizes length.



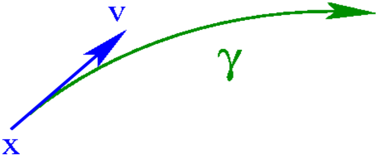
$$\frac{D}{dt}\dot{\gamma} = 0$$

curve with
no aceleration
"inside the surface".

EXAMPLE:

Embedded surface $M \subset \mathbb{R}^3$

$\gamma \subset M$ geodesic $\iff$ $\ddot{\gamma} \perp T_\gamma M$

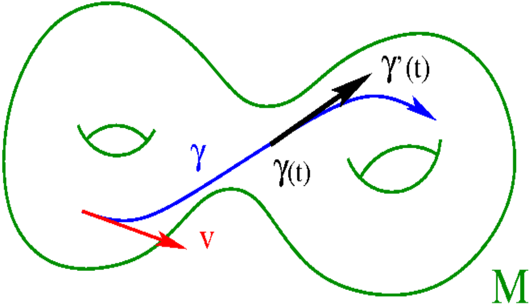


$\forall (x, v) \in TM \quad \exists$ geodesic γ

$$\gamma(0) = x$$

$$\dot{\gamma}(0) = v$$

Geodesic Flow



$$\varphi_t : TM \to TM$$

$$\varphi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$$

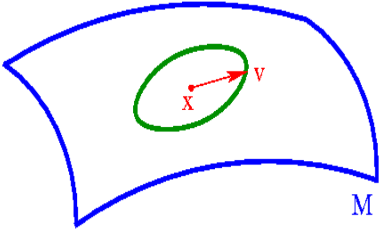
γ geodesic $\implies$ $\|\dot{\gamma}(t)\|$ constant.

$\implies$ unit tangent bundle.

$$SM = \{\, (x, v) \in TM \mid \|v\| = 1 \,\}$$

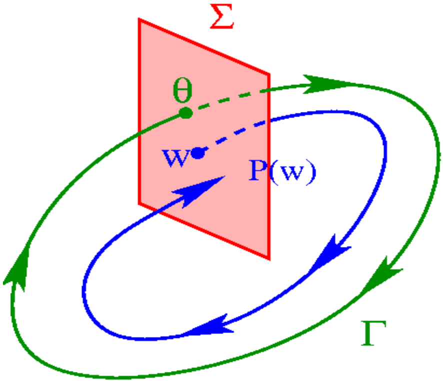
is invariant under $\varphi_t : SM \to SM$.

- On $[\|v\| = a]$, $a \neq 1$, φ_t is a reparametrization of $\varphi_t|_{SM}$.
- $\dim M = 2 \implies \dim SM = 3$.



γ closed geodesic $\longleftrightarrow$ $\Gamma = (\gamma, \dot{\gamma})$ periodic orbit for geodesic flow

1st return map = “Poincaré map”



$P : \Sigma \to \Sigma$

$d_\theta P : T_\theta \Sigma \to T_\theta \Sigma$

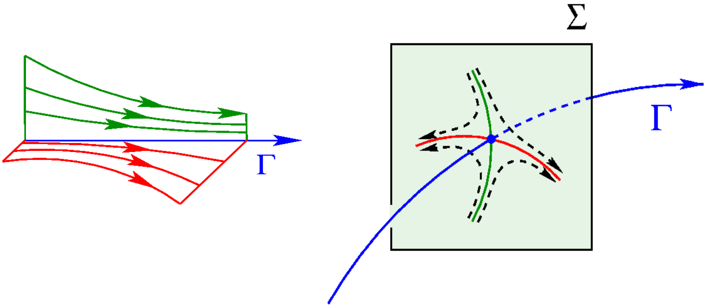
preserves area

$\Longrightarrow$ eigenvalues $\lambda, \frac{1}{\lambda}$

dim $SM = 3$

γ or Γ is *degenerate* $\iff$ $d_\theta P$ has an eigenvalue 1.

hyperbolic $\iff$ $d_\theta P$ has no eigenvalue of modulus 1.



elliptic $\iff$ $d_\theta P$ has eigenvalues of modulus 1.

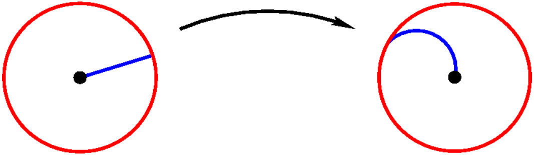
Poincaré on homoclinic points

In 3rd vol. of New Methods of Celestial Mechanics (1899) Poincaré exclaimed, "If one attempts to imagine the figure formed by these two curves and their infinitely many intersections, each of which corresponds to a doubly asymptotic solution, these intersections form something like a lattice or fabric or a net with infinitely tight loops. None of these loops can intersect itself, but it must wind around itself in a very complicated fashion in order to intersect all the other loops of the net infinitely many times. One is struck by the complexity of this figure, which I shall not even attempt to draw. Nothing gives us a better idea of the complicated nature of the three-body problem and the problems of dynamics in general, in which there is no unique integral and in which the Bohlin series diverge."

ELLIPTIC CLOSED GEODESIC:

If it is generic:

Poincaré map is a generic twist map



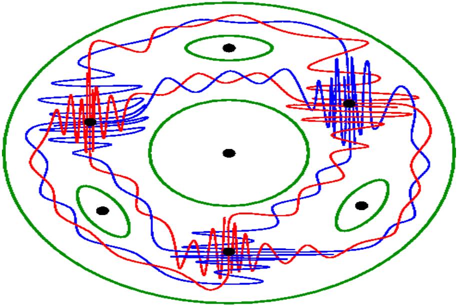
$$(r, \theta) \longmapsto (R, \Theta)$$

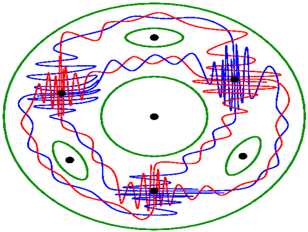
$$\frac{\partial \Theta}{\partial r} > 0$$

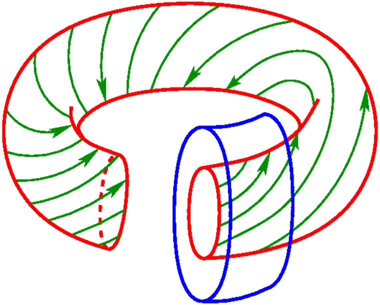
GENERIC TWIST MAP $\Longrightarrow$

1. KAM theorem $\Longrightarrow$
   $\exists$ +ve measure set of invariant circles
   where the Poincaré map is conjugated to a rotation.
2. Between invariant circles

   periodic orbits $\begin{cases} \textit{elliptic} \\ \textit{hyperbolic with } \pitchfork \textit{ intersections} \end{cases}$







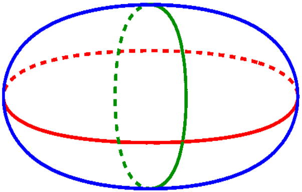
3. Separation of phase space $\Longrightarrow$ non ergodicity

1. Study bifurcations of/by simple closed geodesics & show that

HAS GAPS

elliptic - # hyperbolic = constant.

2. Ellipsoid



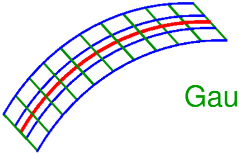
3 simple closed geodesics

2 elliptic
1 hyperbolic $\Big\}$ $2 - 1 = 1 \neq 0$.

THE PROBLEM IN $K \not> 0$: Blue sky catastrophe.
A (simple) closed geodesic may disappear (or appear) when its period $\to +\infty$.

(Skip) Remarks on the bifurcation approach

- Topogonov Thm $\Longrightarrow$ in bounded $K > 0$ length of simple closed geodesics is bounded.
- A geodesic can not touch itself
  $\Longrightarrow$ continuation of simple closed geod. are simple.
- Anosov: Proves that under bifurcations (in $K > 0$) $\#$(simple closed geod.) remains odd.
- $\exists$ metrics on $\mathbb{S}^2$ with simple geodesics with arbitrary large length: large simple closed curve in $\mathbb{R}^2$ + Gauss lemma argument + $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$.



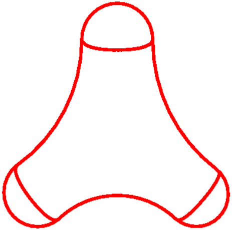
Gauss Lemma

- Blue sky catastrophe $\Longrightarrow$ No way to follow these arguments in $K \not> 0$ case.

Comparison with other theorems

1988 V. Donnay
Burns, Donnay, C^∞

$\exists$ C^∞ riemannian metric on S^2 whose geodesic flow is ergodic and has +ve metric entropy.



All closed geod. but finite (3) (which are degenerate) are hyperbolic.

NOT KNOWN:

- Donnay's thm in +ve curvature K>0.
- If $\exists$ C^∞ riem. metric on S^2 with all closed geod. hyperbolic.

Theorem

Any riemannian metric on S^2 or $\mathbb{RP}^2$ can be C^2 approximated by a C^∞ metric with an elliptic closed geodesic.

$\Longrightarrow$ $\exists$ open and dense set of riemannian metrics in S^2 or $\mathbb{RP}^2$ (in C^2 topology) whose geodesic flow has an elliptic closed geodesic.

2000 IMPA Michel Herman

- announced this theorem when $K > 0$.
- conjectured it for arbitrary K.

Ballmann, Thorbergsson, Ziller

- Pinching conditions on $K > 0$ to have an elliptic closed geodesic on $\mathbb{S}^n$.

1977 Newhouse Theorem

$H : (M, \omega) \to \mathbb{R}$ smooth hamiltonian on a symplectic manifold.
If the energy level $H^{-1}\{0\}$ is compact
$\Longrightarrow \exists$ C^2 perturbation H_1 of H
s.t. its hamiltonian flow either

- *is Anosov.*
- *has a 1-elliptic closed orbit.*

RMKS:

- Newhouse Thm uses the C^2 Closing Lemma (not known for geodesic flows).
- Corresponds to stability conjecture for hamiltonian flows.
- Main Thm above is a version of Newhouse Thm for geod. flows in $\mathbb{S}^2$ or $\mathbb{RP}^2$ because $\nexists$ Anosov geodesic flow on $\mathbb{S}^2$ [or $\mathbb{RP}^2$].
- Newhouse thm is not known in any other compact mfld.

$\nexists$ Anosov geodesic flow on $\mathbb{S}^2$ [or $\mathbb{RP}^2$].

2 proofs:

1. Anosov flow on $N = T^1\mathbb{S}^2 = \mathbb{RP}^3$
   $\implies \pi_1(N)$ has exponential growth. $(\Rightarrow\Leftarrow)$

2. Anosov geodesic flow for M
   $\implies$ (Klingenberg) $\implies$ No conjugate points
   $\implies \widetilde{M} = \mathbb{R}^n$

   but $\widetilde{\mathbb{S}^2} = \mathbb{S}^2 \not\approx \mathbb{R}^2$ $(\Rightarrow\Leftarrow)$

Klingenberg-Takens-Anosov Theorem

Given a closed geodesic one can perturb the riemannian metric in the C^∞ topology s.t.

1. *does not move the closed geodesic.*
2. *makes any k-jet of the Poincaré map generic.*

Klingenberg-Takens: perturbation for a single periodic orbit.
Anosov: Bumpy metric theorem & $\Longrightarrow$ countable periodic orbits.

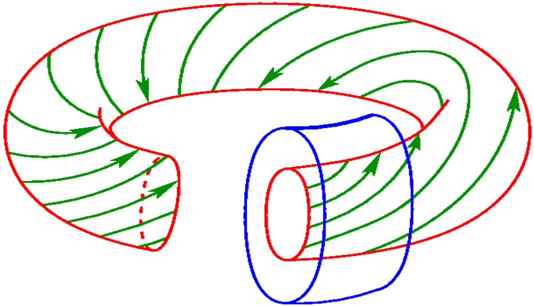
Applications of the Main Theorem

1. Make the Poincaré map of the elliptic geodesic C^4 generic

$\Longrightarrow$ KAM Thm (Moser) $\Longrightarrow$

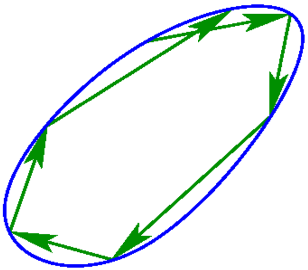
$\exists$ invariant circle which separates the phase space

$\therefore \exists$ C^2-dense set of (C^∞) riemannian metrics on S^2 or $\mathbb{RP}^2$ such that the geodesic flow is not ergodic.



2. Recall Lazutkin:
A billiard map in the interior of a C^∞ embedded curve in $\mathbb{R}^2$ with $+$ve curvature is not ergodic.

In higher dimensions:



Kobachev & Popov:

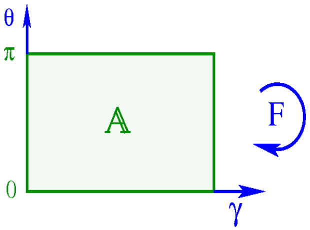
Billiard map in a strictly convex domain in $\mathbb{R}^n$ with C^∞ boundary has a set of positive measure of invariant quasi-epriodic tori provided that the geodesic flow on the boundary has an elliptic periodic geodesic which is k-elementary, $k \geq 5$, (in particular the billiard is not ergodic).

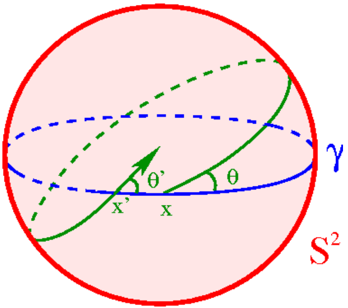
Main Thm $\implies$ For $M \approx \mathbb{S}^2 \subseteq \mathbb{R}^3$, $(n = 3)$,
this condition is C^2 generic.

Case $K > 0$

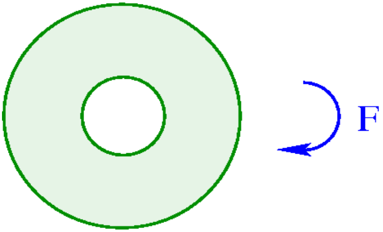
∃ a simple closed geodesic. proof

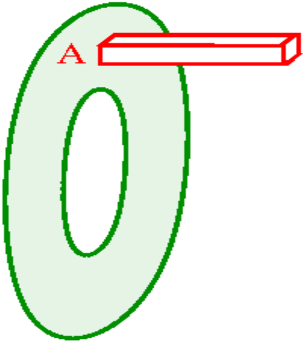
Birkhoff section





$$F(x, \theta) = (x', \theta')$$





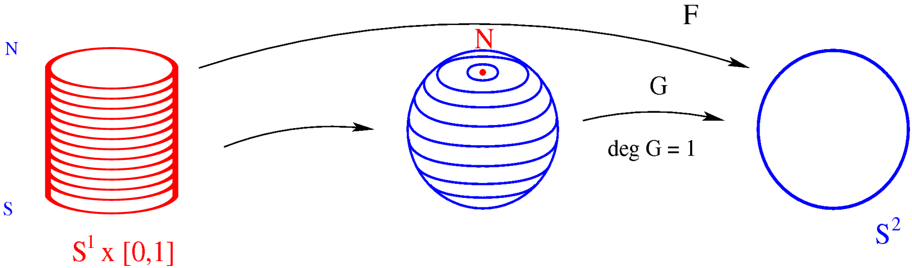
$\mathrm{vol}(\varphi_{[0,\varepsilon]}(A)) = \mathrm{area}(A)$

F : return map is smooth and preserves area.

- $\mathrm{int}(\mathbb{A}) \pitchfork$ geodesic vector field.
- $\partial\mathbb{A} = \Gamma \cup (-\Gamma)$, $\qquad \Gamma = (\gamma, \dot{\gamma})$.
- any orbit $\neq \{\Gamma, -\Gamma\}$ intersects $\mathbb{A}$.
- return times uniformly bounded $0 < T(x,\theta) < T_0$.
- (can extend F to $\partial\mathbb{A}$ by $\theta \mapsto$ 2nd conjugate pt. to θ)

∃ simple closed geodesic on $\mathbb{S}^2$

e.g. Birkhoff minimax closed geodesic.

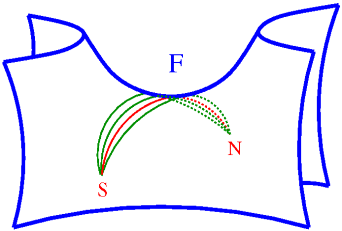


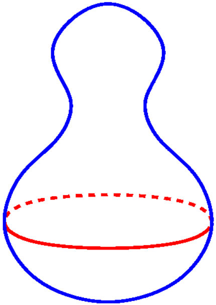
Family of closed curves covering the sphere.

$\gamma : [0,1] \to \mathbb{S}^2$

$E(\gamma) := \int_0^1 |\dot{\gamma}|^2 \, dt$

$c := \inf_F \max_{s \in [0,1]} E(F(\cdot, s)) > 0$





c is a critical value of the energy functional with a critical point γ called the Birkhoff minimax geodesic.

$\mathcal{H}^2(\mathbb{S}^2) := \{\, C^2$ riem. metrics on $\mathbb{S}^2$ without elliptic closed geodesics $\}$

$\mathcal{F}^2(\mathbb{S}^2) := \mathrm{int}_{C^2}(\mathcal{H}^2(\mathbb{S}^2))$

JDG 2002: G. Paternain & G. Contreras

$$\left.\begin{array}{r} g \in \mathcal{F}^2(\mathbb{S}^2) \\ g \in C^4 \end{array}\right\} \Longrightarrow \overline{\mathrm{Per}(g)} \text{ is uniformly hyperbolic.}$$

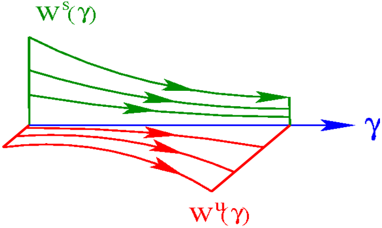
def

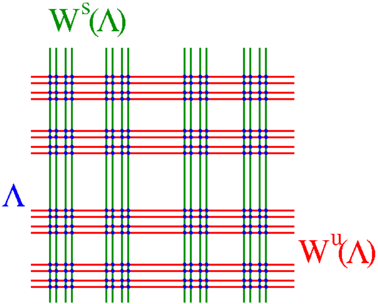
sketch:

- Prove perturb. ("Franks") lemma for geod. flows in dim2.
- The periodic orbits are stably hyperbolic.
- Use Mañé-Liao theory on dominated splittings
  $\Longrightarrow$ $\overline{\mathrm{Per}(g)}$ has dominated splitting.
- Preserves area $\Longrightarrow$ [dom. splitting $\Longrightarrow$ uniform hyp.]

□

Uniform Hyperbolicity





$N = T^1\mathbb{S}^2$

$\phi_t : \Lambda \to \Lambda$ invariant subset, is hyperbolic if

$$T_\Lambda N = E^s \oplus \langle \mathbb{X} \rangle \oplus E^u, \qquad \exists\, C,\ \lambda > 0$$

$$\|d\phi_t|_{E^s}\| < C\, e^{-\lambda t}, \qquad t > 0.$$

$$\|d\phi_t|_{E^u}\| < C\, e^{-\lambda t}, \qquad t > 0.$$

$\mathcal{H}^2(\mathbb{S}^2) := \{\, C^2$ riem. metrics on $\mathbb{S}^2$ without elliptic closed geodesics $\}$

$\mathcal{F}^2(\mathbb{S}^2) := \mathrm{int}_{C^2}(\mathcal{H}^2(\mathbb{S}^2))$

$$\left.\begin{array}{r} g \in \mathcal{F}^2(\mathbb{S}^2) \\ g \in C^4 \end{array}\right\} \overset{\text{JDG 2002}}{\Longrightarrow} \overline{\mathrm{Per}(g)} \text{ is uniformly hyperbolic.}$$

def

- want $\mathcal{F}^2(\mathbb{S}^2) = \emptyset$.
- Assume $\neq \emptyset$, take $g \in \mathcal{F}^2(\mathbb{S}^2)$.
- $\mathcal{F}^2(\mathbb{S}^2)$ is open $\Longrightarrow \begin{cases} \text{can assume } g \in C^\infty, \\ \text{can assume } g \text{ is Kupka-Smale,} \\ \qquad \text{[also in JDG 2002].} \end{cases}$

Bangert + Franks: Any riem. metric on $\mathbb{S}^2$ has ∞-many closed geodesics.

+ Smale Spectral Decomposition Thm.

$\Longrightarrow$ $\overline{\mathrm{Per}(g)}$ contains a non-trivial hyperbolic basic set.

Λ:= homoclinic class = hyperbolic basic set.

$$g \in C^3 \Longrightarrow \quad F : \mathbb{A} \hookleftarrow \text{ is } C^3$$

$$\underset{\text{Bowen}}{\Longrightarrow} \quad \begin{cases} F \text{ Anosov } \Longrightarrow g \text{ Anosov} & (\Rightarrow\Leftarrow) \\ \Lambda \text{ has measure } 0. \end{cases}$$

Poincaré recurrence $\Longrightarrow$ $\mathrm{meas}[W^s(\Lambda) \setminus \Lambda] = 0$
similarly W^u

$\Longrightarrow$ $\mathrm{meas}[W^s(\Lambda) \cap W^u(\Lambda)] = 0.$

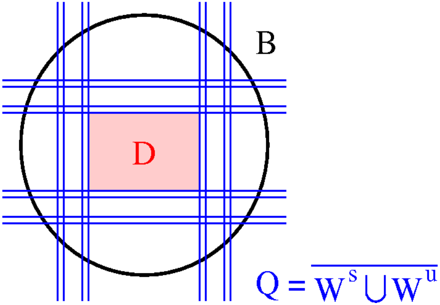
Prove using the hyperbolicity:

B small closed ball, $\mathring{B} \cap \Lambda \neq \emptyset$

$Q := \overline{W^s(\Lambda) \cap W^u(\Lambda)}$

$\Longrightarrow \quad Q \cap B \subset Q.$

Take a "hole" D of Q in B, i.e. :



$D =$ a connected compo. of $\mathbb{A} \setminus Q$ contained in B.

- meas$(D) > 0$
- Poincaré recurrence $\quad \exists N > 0 \quad F^N(D) \cap D \neq 0.$

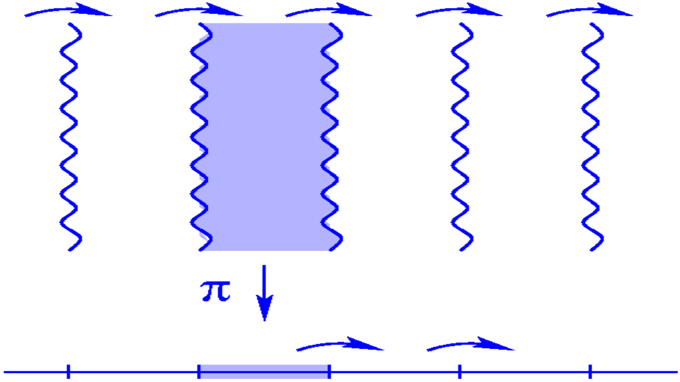
$$\text{but } \left\{ \begin{array}{l} Q \text{ invariant} \\ D \text{ compo. of } \mathbb{A} \setminus Q \end{array} \right\} \Longrightarrow F^N(D) \subset D.$$

Brower Translation Theorem

$f : \mathbb{R}^2 \to \mathbb{R}^2$ homeo. without fixed points

$\Longrightarrow$ it has a "translation domain",

(i.e. it is semiconjugate to a translation in $\mathbb{R}$).



BUT

$F^N(D) = D \approx \mathbb{R}^2$, $\quad F$ preserves finite measure

$\Longrightarrow$ (Poincaré recurrence) $\Longrightarrow$ no translation domain

$\Longrightarrow F^N : D \hookleftarrow$ has fixed pt. x

which is not in $Q = \overline{W^s(\Lambda) \cap W^u(\Lambda)}$.

Uniform hyperbolicity

$\Longrightarrow W^s(\Lambda), W^u(\Lambda)$

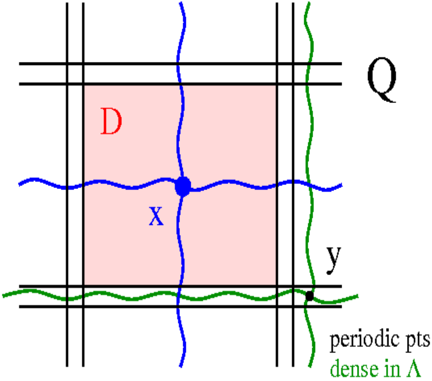
are large.

$\Longrightarrow x \in$ homo. class $\Lambda \cap D$.

$[\ (\Rightarrow\Leftarrow)\ D \subset \mathbb{A} \setminus D\ ]$

$\therefore \quad \mathcal{F}^2(\mathbb{S}^2) = \text{int}_{C^2}\mathcal{H}^2(\mathbb{S}^2) = \emptyset$.

$\blacksquare$

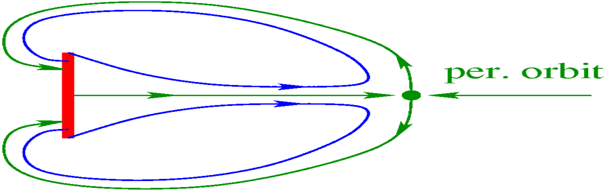


GENERAL CASE

Want the same for local transversal sections.

PROBLEMS:

1. Return time is C^0 only locally: it may tend to ∞.
2. Return map may be discontinuous.

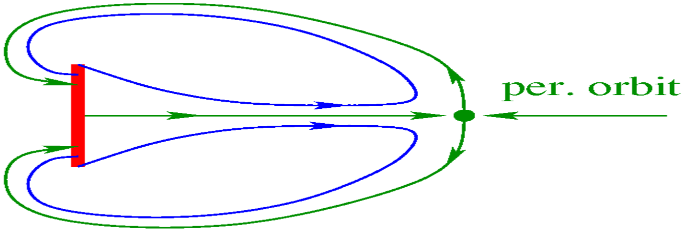


3. Some wandering orbits may tend to some unkown wild strange set.

HOFER - WYSOCKI - ZEHNDER theory will say that

non-returning points can only go to periodic orbits

[i.e. they must be in $W^s(periodic)$]



END OF PART I

HOFER - WYSOCKI - ZEHNDER : Theory for tight contact forms in $\mathbb{S}^3$.

$\dim M = 2n + 1$.

λ: contact 1-form in M: if $\lambda \wedge (d\lambda)^n$ is volume form.

X: Reeb vector field for λ: $\begin{cases} i_X(d\lambda) \equiv 0 \\ \lambda(X) \equiv 1 \end{cases}$

φ_t: Reeb flow preserves λ.

Geodesic case: $\lambda_{(x,v)} = \langle v, dx \rangle_x$ Liouville form on $T^1 M$.

geodesic flow $\equiv$ Reeb flow of λ.

2 KINDS OF CONTACT FORMS IN $\mathbb{S}^3$:

1. Overtwisted.
2. Tight: Canonical contact form in $\mathbb{S}^3$:

$$\eta|_{\mathbb{S}^3} = \frac{1}{2}\left[x\,dy - y\,dx\right]|_{\mathbb{S}^3}$$
$$\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y)$$

θ is tight $\iff$ $\theta = f(x, y) \cdot \eta$

some $f : \mathbb{S}^3 \to \mathbb{R}$.

FROM $\mathbb{S}^2$, $\mathbb{RP}^2$ TO $\mathbb{S}^3$

$T\mathbb{S}^2 = \mathbb{RP}^3 \xleftarrow{2\times} \mathbb{S}^3$ double cover

$\mathbb{RP}^2 \xleftarrow{2\times} \mathbb{S}^2$

canonical contact form on $\mathbb{S}^3$ $\Longrightarrow$ Reeb flow = Hopf fibration of $\mathbb{S}^3$ $\xrightarrow{2\times}$ $T^1\mathbb{S}^2$ geod. flow of "round sphere" $K = \text{constant}$

+ all riemannian metrics on $\mathbb{S}^2$

are conformally equivalent (Beltrami eqs.)

$\Longrightarrow$ Liouville forms of any riemannian metric on $\mathbb{S}^2$

lift to tight contact forms on $\mathbb{S}^3$.

Hofer - Wysocki - Zehnder theory

is for "generic" tight contact forms in $\mathbb{S}^3$.

"generic" = all per. orbits non-degenerate

(i.e. no eigenvalue 1).

True for C^∞ generic geodesic flows by Anosov

(Bumpy metric Thm).

Hofer - Wysocki - Zehnder:

- $\exists$ kind of "open book decomposition" of $\mathbb{S}^3$ by "surfaces of section" $\pitchfork$ Reeb flow.
- Each surface $\Sigma \approx \mathbb{S}^2 \setminus \{$ finite points $\}$.
- $\partial\Sigma \subset \{$ finite periodic orbits $\} =$ Biding orbits $=: \mathbb{B}$.
- $d\lambda$-area of each surface is finite.
- $\exists$ finite set of those surfaces ("rigid surfaces") which intersect all orbits except those in $\mathbb{B}$.

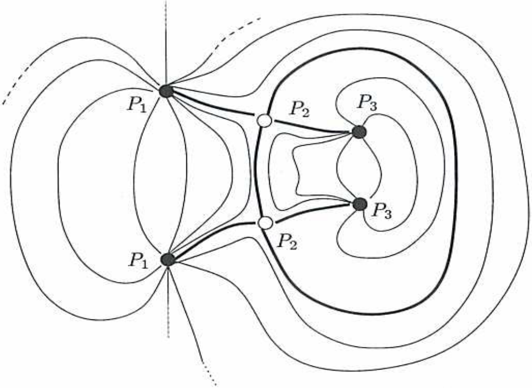


FIGURE 3. Stable finite energy foliation of S^3.

H. HOFER, K. WYSOCKI, AND E. ZEHNDER

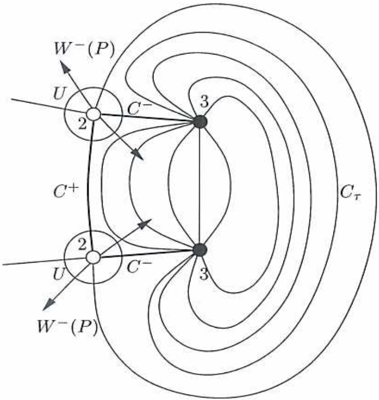
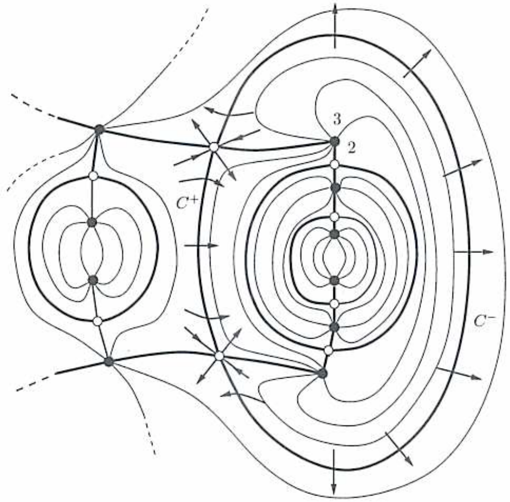


FIGURE 31. A family of surfaces C_τ decomposes into the broken trajectory (C^+, C^-).



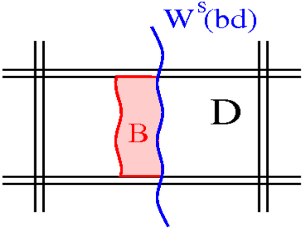
3
2
C^+
C^-

- As before $\overline{\mathrm{Per}(g)}$ uniformly hyperbolic.
- Λ = homoclinic class = hyperbolic basic set.
- Σ finite set of surfaces of section.
- D = small hole in $\Sigma \setminus Q$, $Q = W^s(\Lambda) \cup W^u(\Lambda)$.
  $F : \Sigma \to \Sigma$ return map where well defined.
- Poincaré recurrence $F^N(D) \cap D \neq \emptyset$.

1. If F^N well defined on whole D $\Longrightarrow F^N(D) = D \Longrightarrow$ Brower Translation Thm ....
2. If not $\Longrightarrow$ disconinuity of F^N
   $\Longrightarrow W^s(\text{biding orbits}) \cap D \neq \emptyset$.
   (biding orbits = $\partial\Sigma$)

LEMMA:

A return of an arbitrarily small piece of W^u(biding orbit) must be large [diam $> a$].



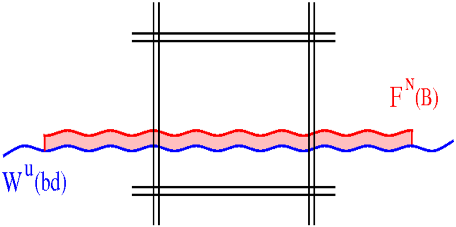
- B a connected compo. of $D \cap [\tau_N < +\infty]$

Since B connected

$B \cap Q = \emptyset$, Q conn., invar.

$Q = W^s(\Lambda) \cup W^u(\Lambda)$

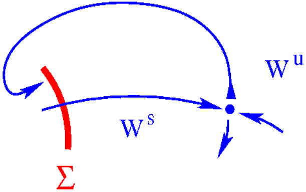
$\Longrightarrow F^N(B) \subset D \quad (\Rightarrow\Leftarrow)$



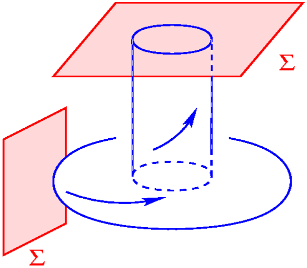
LEMMA: A return of an arbitrarily small piece of W^u(biding orbit) must be large.

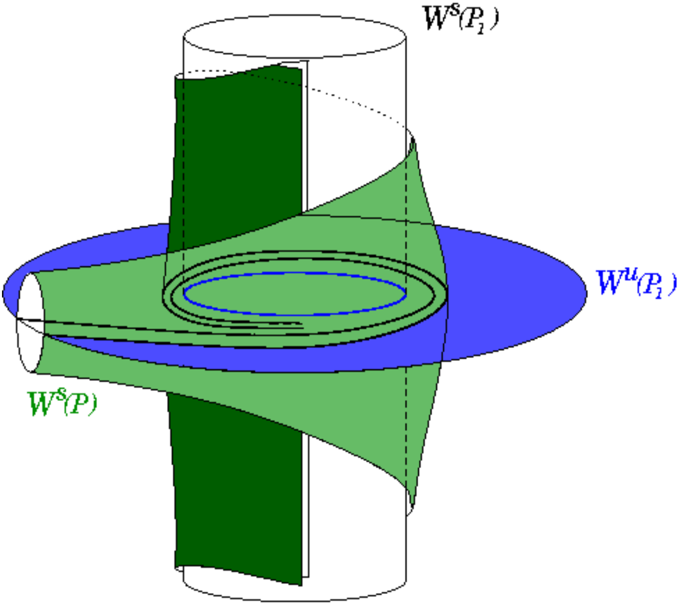
PROOF:

1. Return of W^s is W^u.



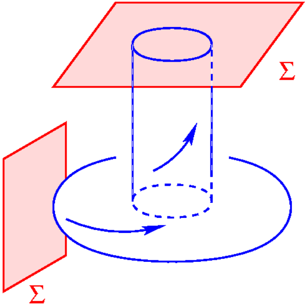
Return of a small transversal to W^s accumulates on whole compo. of W^u

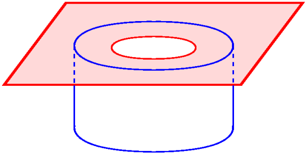



$W^s(P_1)$
$W^u(P_1)$
$W^s(P)$

2. If it is a continuous return.
cases:

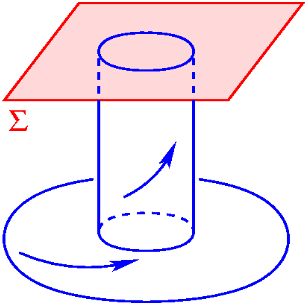
2.a Return circle contains a hole (biding orbit) in $\partial\Sigma$
$\Longrightarrow$ Large.





$\Sigma \approx \mathbb{S}^2 \setminus (\textit{finit. pts.})$

2.b Return does not contain holes of Σ:



Stokes

$$0 = \int_{\text{cylinder}} d\lambda = \underbrace{\int_{\text{per. orbit}} \lambda}_{} - \underbrace{\int_{\text{return to } \Sigma} \lambda}_{}$$

↑ W^u lagrangian or $\left\{ \begin{array}{l} i_X d\lambda = 0 \\ X \in TW^u \end{array} \right\}$

|| period

integral is large
$\Longrightarrow$ return large

3. Following returns always accumulate on a complete continuous return $\Longrightarrow$ Large!