CONCAVITY IMPLIES ATTRACTION

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Strange Nonchaotic Attractors

The main concept of a Strange Nonchaotic Attractor is as follows. We consider a skew product on a space $X = B \times Y$. The space B is the *base* and Y is the *fiber space*; for each $\vartheta \in B$ the set $\{\vartheta\} \times Y$ is the *fiber* over ϑ. The skew product $F : X \to X$ can be written as

$$F(\vartheta, x) = (R(\vartheta), \psi(\vartheta, x)). \tag{1}$$

Here R is a map from the base to itself, and for any $\vartheta \in B$ the map ψ_ϑ, given by $\psi_\vartheta(x) = \psi(\vartheta, x)$ maps the fiber over ϑ to a fiber over $R(\vartheta)$. In most cases Y is a manifold, very often the real line or an interval and the maps ψ_ϑ are smooth. In fact, we are considering convergence in the fibers, so the minimal assumption we should make is that Y is a metric space. No special assumptions on the structures on B or of R are really necessary.

However, if B is just a set without any structure, then we can treat the whole system as a bunch of nonautonomous systems (fibers over full orbits of R), not connected with each other. Therefore one usually requires that B is either a manifold or a topological space (usually compact Hausdorff; often a circle), or a space with finite measure on it.

Attractor:

The main idea that distinguishes this concept of attractor from the usual one is that *we consider only attraction in the fibers.* That is, the distance of a point p from the attractor is measured as the distance of p from the intersection of the attractor and the fiber containing p. Consequently, attraction means that this distance goes to zero. This agrees with the vision of the system as a bunch of nonautonomous systems.

Further requirements for an attractor can be divided into several categories.

Inside fibers: The intersection of an attractor with a fiber should look like an attractor for a usual system. The simplest case is when it is one point. This is the usual case when the fiber is one-dimensional. Then the whole attractor is a graph of a function $\varphi : B \to Y$. In more complicated cases we can require that the intersection of the attractor with each fiber is compact.

Across the fibers: How should the intersection of the attractor with a fiber depend on the fiber? In the most interesting case, when the attractor is a graph of the function φ this translates into a question: what should we assume about φ? We may impose no assumptions; if B is a topological space, we may require φ to be continuous, upper semicontinuous, Borel, etc. If there is a measure on B, we may require that φ is measurable.

Invariance. For the usual dynamical system, an attractor is compact, and therefore it contains the ω-limit set of every point from its basin. Thus, we can replace it, if necessary, by its subset defined as the closure of the union of the ω-limit sets of the points from its basin. This subset is automatically invariant. Thus, the requirement that the attractor is invariant is natural and in a sense, is satisfied automatically. In our case (except in the periodic fibers, which often do not exist or their union has measure zero) the notion of the ω-limit set makes no sense, because the distance between the points is measured only in the same fiber, and the trajectory does not visit the same fiber twice. Thus, there is no special reason to require the attractor to be invariant. However, such requirement may be included for historical reasons.

Nonchaoticity. Usually this property is stated as the negative Lyapunov exponents in the direction of the fiber. When the attractor is a graph of the function φ, this assumption loses its importance. When it is one point (in each fiber) that is attracting, there is no room for any form of chaos. Moreover, the Lyapunov exponent, if it can be defined (we do not have to assume differentiability in the fibers) will be automatically nonpositive.

Strangeness. The only reasonable meaning of this is when there is topology in B and φ is not continuous.

Theorem 1. *Assume that for a skew product* (1) *there is an ergodic invariant measure* μ *for* R *on the base* B*. Then, if the graphs of measurable functions* $\varphi_1, \varphi_2 : B \to Y$ *are both attractors, it follows that* $\varphi_1 = \varphi_2$ μ*-almost everywhere.*

α-concavity and nonautonomous systems

Let f be a continuous real-valued function on a closed interval I of the real line and let $\alpha \geq 0$. The function f will be called *α-concave* if the function f_α, given by

$$f_\alpha(x) = f(x) + \alpha x^2$$

is concave.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of maps from the interval $[0, a]$ to itself such that $f_n(0) = 0$ for every n. Such a sequence will be called *pinched* when there exists an n such that f_n is identically zero. It will be called *equiconcave* if there exists a positive constant β such that each f_n is $\beta\gamma_n$-concave, where γ_n is the supremum of f_n.

Theorem 2. *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of monotone maps from the interval* $[0,a]$ *to itself such that* $f_n(0)=0$ *for every* n*. Assume also that this sequence is either pinched or equiconcave. Then for every* $x_0, y_0 \in (0,a]$ *we have*

$$\lim_{n\to\infty} |x_n - y_n| = 0. \tag{2}$$

Assumption on monotonicity can be weakened.

Methods of proof:

Given two points $u, v > 0$ we define

$$\kappa(u,v) := \frac{|v-u|}{\min\{u,v\}}.$$

Lemma 3. *Assume that f is α-concave in the interval $[0,y]$ and $f(0) = 0 < f(y)$. Let $x \in (0,y)$ be such that $0 < f(x) < f(y)$. Then,*

$$\frac{\kappa(f(x), f(y))}{\kappa(x,y)} \leq \frac{f(y)}{f(y) + \alpha y^2}.$$

Nonchaotic Equiconcave Attractors

If $Y = [0, a]$, the family $\{\psi_\vartheta\}_{\vartheta \in B}$ will be called *equiconcave* if there exists a positive constant β such that each ψ_ϑ is $\beta\gamma_\vartheta$-concave, where γ_ϑ is the supremum of ψ_ϑ. Note that now we included the pinched case in the definition of equiconcavity. Indeed, if ψ_ϑ is identically 0 then $\gamma_\vartheta = 0$ and ψ_ϑ is 0-concave.

If $Y = [0, a]$ and the family $\{\psi_\vartheta\}_{\vartheta \in B}$ satisfies $\psi_\vartheta(0) = 0$ for each $\vartheta \in B$ and is equiconcave, then we will call the system (X, F) an *equiconcave skew product*. If additionally all functions ψ_ϑ are monotone, the system will be called a *monotone equiconcave skew product*.

In many standard examples of systems with strange nonchaotic attractors one defines ψ as a product: $\psi(\vartheta, x) = f(x)g(\vartheta)$. In those examples, if f is α-concave for some $\alpha > 0$ then $\{\psi_\vartheta\}_{\vartheta \in B}$ is equiconcave. In particular, if additionally f is monotone, then we get a monotone equiconcave skew product.

A graph of a function $\varphi : B \to [0, a]$ will be called *preinvariant* if for every $\vartheta \in B$ there exists N such that for every $n \geq N$ we have

$$F(R^n(\vartheta), \varphi(R^n(\vartheta))) = (R^{n+1}(\vartheta), \varphi(R^{n+1}(\vartheta))). \tag{3}$$

A point $\vartheta \in B$ will be called *pinching* if there are infinitely many positive integers n such that $\psi_{R^n(\vartheta)}$ is identically equal to 0.

First we do not endow B with any extra structure.

Theorem 4. *Let the system (X, F) with base B and fiber space $[0, a]$ be a monotone equiconcave skew product and let $\varphi : B \to [0, a]$ be a preinvariant function, positive at any point that is not pinching. Then the graph of φ is an attractor with the basin of attraction containing all points whose forward trajectory does not pass through $B \times \{0\}$.*

Theorem 5. *Let the system (X, F) with base B and fiber space $[0, a]$ be a monotone equiconcave skew product. Then there exists a preinvariant function $\varphi : B \to [0, a]$, positive at any point that is not pinching.*

Corollary 6. *Let the system (X, F) with base B and fiber space $[0, a]$ be a monotone equiconcave skew product. Then there exists a function $\varphi : B \to [0, a]$, whose graph is an attractor with the basin of attraction containing all points whose forward trajectory does not pass through $B \times \{0\}$.*

Observe that we cannot always make an attractor invariant. If φ is defined at $R(\vartheta)$, we can try to define $\varphi(\vartheta)$ as $\psi_\vartheta^{-1}(\varphi(R(\vartheta)))$, but it may happen that the image of $[0, a]$ under ψ_ϑ does not contain $\varphi(R(\vartheta))$. It turns out that the problem is deeper, and an invariant attractor which is a graph may not exist.

Example 7. Let $B = \{\vartheta_n\}_{n=-\infty}^{\infty} \cup \{-1, 1\}$, where $\vartheta_n = 1 - \frac{1}{n+1}$ if $n \geq 0$ and $\vartheta_n = -1 - \frac{1}{n}$ if $n < 0$. The map R fixes -1 and 1, and maps ϑ_n to ϑ_{n+1}. We set $a = 1$ and define $\psi_\vartheta(x)$ to be $x(2 - x)$ if $\vartheta \geq 0$ and $x(2 - x)/4$ if $\vartheta < 0$. Assume that the graph of a function $\varphi : B \to [0, 1]$ is an invariant attractor. Since $x(2 - x)/4 \leq x/2$, we have $\varphi(\vartheta_0) \leq \varphi(\vartheta_{-n})/2^n \leq 1/2^n$ for all positive n. Thus, $\varphi(\vartheta_0) = 0$, and consequently, $\varphi(\vartheta_n) = 0$ for all $n > 0$. On the other hand, the trajectory of every $x \in (0, 1]$ under the map $x \mapsto x(2 - x)$ goes to 1, so $\varphi(\vartheta_n) \to 1$ as $n \to \infty$. This is a contradiction, and therefore in this case there is no φ whose graph is an invariant attractor.

Example 7 shows that in general, we cannot count on getting an invariant attractor. However, in this example a preinvariant attracting graph of a continuous function of course exists; just take φ identically equal to 1.

We use the method of constructing of invariant graph described for instance by Keller. Namely, we set $\varphi_n(\vartheta)$ to be equal to the second component of $F^n(\vartheta, a)$. Since F is monotone in the fibers, the sequence $(\varphi_n)_{n=0}^{\infty}$ is decreasing, and therefore convergent pointwise on the whole B. Denote its limit by φ_K.

Note that Example 7 shows that φ_K (which is 0 at all points ϑ_n) may be not an attractor everywhere.

Theorem 8. *Let the system (X, F) with base B and fiber space $[0, a]$ be a monotone equiconcave skew product, and let the base map R be invertible and preserve an ergodic invariant probability measure μ on B. If the function φ_K is positive almost everywhere then its graph is an attractor with the basin of attraction containing the set $Z \times (0, a]$ for some set $Z \subset B$ of full measure μ.*

Example showing problems when R is noninvertible:

In the base B we take the full one-sided shift R on 2 symbols (0 and 1). The fiber space consists of two points (again 0 and 1). We will use the notation $x = (x_0, x_1, x_2, \dots) \in B$. The map $F : B \times \{0,1\} \to B \times \{0,1\}$ is given by

$$F(x, y) = (R(x), x_0).$$

Theorem 9. *For the system described above and an ergodic invariant probability measure μ on B, if there exists a μ-measurable function $\varphi : B \to \{0,1\}$ whose graph is an attractor with the basin of attraction $Z \times \{0,1\}$ for some set $Z \subset B$ of μ measure 1, then the entropy of μ is zero.*

Idea of the proof:

Set $A = \{x \in B : x_0 = \varphi(R(x))\}$. The graph of a function $\varphi : B \to \{0,1\}$ is an attractor if and only if for every $(x, y) \in B \times \{0,1\}$ there is N such that for every $n \geq N$

$$\pi_2(F^n(x,y)) = \varphi(R^n(x)). \tag{4}$$

If $n \geq 1$ then

$$\pi_2(F^n(x,y)) = (R^{n-1}(x))_0,$$

so (4) is equivalent to $R^{n-1}(x) \in A$. We can show that $\mu(A) = 1$.

Let $\xi : B \to B$ be the map that replaces x_0 by $1 - x_0$. By the definition, at most one of the points $x, \xi(x)$ can belong to A. Therefore, $A \cap \xi(A) = \emptyset$. This means that the shift R is one-to-one μ-almost everywhere. Since R has a one-sided generator, this implies that $h_\mu(R) = 0$.

Corollary 10. *For the system described above and an ergodic invariant probability measure* μ *on* B *with positive entropy, there is no Borel function* $\varphi : B \to \{0, 1\}$ *whose graph is an attractor.*