Some of the Anthony Manning's mathematical achievements

I will recall and discuss briefly the following

[M1] Heather McCluskey, Anthony Manning: Hausdorff dimension for horseshoes, ETDS 3.2 (1983), 251-260 Errata in ETDS 5 (1985).

]M2] Anthony Manning: The dimension of the maximal measure for a polynomial map, Ann. of Math. 119.2 (1984), 425-430

[M3] Anthony Manning: Topological entropy and the first homology group, Dynamical Systems – Warwick 1974 (Proceedings Sympos. Appl. Topology and Dynamical Systems; presented to E. C. Zeeman on his fiftieth birthday), L. N. Math. 468.

I. Hausdorff dimension for horseshoes

Theorem I.1 Let Λ be a basic set C^1 axiom A diffeomorphism $f: M^2 \to M^2$ with $(1,1)$ splitting $T_\Lambda M = E^s \oplus E^u$. Define $\phi(x): W^u(\Lambda) \to R$ by

$$\phi(x) = -\log ||Df_x|_{E^u_x}||.$$

Then $HD(W^u(x) \cap \Lambda)$ is given by the unique $t = \delta$ for which

$$P^u(t) := P_{f|_\Lambda}(t\phi) = 0.$$

δ is independent of $x \in \Lambda$ and depends continuously in the C^1 topology on diffeomorphisms.

Figure: graph of $P^u(t)$.

Axiom A is not needed; it is sufficient if Λ is f-invariant hyperbolic topologically transitive isolated.

R. Bowen in "Hausdorff dimension of quasicircles" did it in dimension 1, expanding, for ϕ Hölder. This (and related) formulas are often called Bowen – McCluskey – Manning.

The upper estimate of HD follows from

$$\operatorname{diam}(I)^t \le \exp tS_n(\phi + \epsilon),$$

where $S_n(\psi)(x) := \sum_{j=0}^{n-1} \psi(f^j(x))$ and I is a component of the f^n-preimage of a unit arc in $W^u(\Lambda)$ (Anthony uses Markov partition).

For this C^1 is enough!

The lower estimate follows from the existence of an ergodic equilibrium measure μ, i.e. such that $h_\mu(f) + \delta \int \phi d\mu = P^u(\delta) = 0$. Then, for W, the set of Birkhoff regular points

$$HD(W) = h_\mu(f)/\chi_\mu(f) \qquad (Ma)$$

where $\chi_\mu(f)$ is the Lyapunov exponent $\int \log ||Df_x|_{E^u_x}||d\mu$.

Anthony did the latter in the paper:

"A relation between exponents, Hausdorff dimension and entropy", ETDS 1 (1981), 451-459.

Higher dimension and non-uniform hyperbolicity

Lai Sang Young:

$$HD(\mu) = h_\mu(f)(1/\lambda_1 - 1/\lambda_2)$$

for f $C^{1+\epsilon}$ diffeomorphism or C^1 Axiom A, and μ f-invariant ergodic hyperbolic measure.

Breakthrough was done by Mañé with a new proof of Pesin formula. This has yielded in dimension 1, that $HD(\mu) = h_\mu(f)/\chi(\mu)$.

Shannon-McMillan-Breiman is substantial
(and Frostman Lemma):

In dimension 1 for $B_n = \mathrm{Comp} f^{-n}(B(f^n(x), r))$

$$\log \mu(B_n)/\log \operatorname{diam}(B_n) \sim \frac{1}{n}\log \mu(B_n)/\frac{1}{n}(-\log |Df^n(x)|)$$

$$\sim h_\mu(f)/\chi_\mu(f).$$

Let me finish this topic with famous L.S. Young, F.Ledrappier's formula:

$$h_\mu(f) = \sum_{i=1,\dots,u} \chi_i^+ \gamma_i$$

where + means we consider only positive Lyapunov exponents (or max with 0), μ is ergodic. γ_i are "dimensions" of μ in directions of E_i (subspaces in Oseledec decomposition). Precisely for the increasing family of related foliations $W^1 \subset W^2 \subset ... \subset W^u$ the authors define

$\gamma_i = \delta_i - \delta_{i-1}$, where
$h_1 = \chi_1\delta_1, h_i - h_{i-1} = \chi_i(\delta_i - \delta_{i-1})$ and
h_i are entropies for conditional measures for W^i.

Pressure (geometric) has been studied in the recent (20) years in 1-dimensional complex and real by Denker, Urbański, Bruin, Rivera-Letelier, Stas Smirnov, myself, ...)

The first zero is hyperbolic dimension, $P(t)$ is real analytic, dimension spectrum for Lyapunov exponents is Legendre transform of pressure etc.

II. The dimension of the maximal measure

Anthony proved the following

Theorem II.1. Suppose that each critical point c of a polynomial $f : C \to C$ (i.e. a point such that $f'(c) = 0$ satisfies $f^n(c) \not\to \infty$ and $c \notin J(f)$ (Julia set). Then the equilibrium distribution m on $J(f)$ has Hausdorff dimension 1.

The assumptions imply that $J(f)$ is connected and f hyperbolic on it.

If f is not polynomial, then the assertion might be false, e.g. for Blaschke product preserving 0, not az^d with $|a|=1$, $HD(m)<1$.

For me the story started in December 1983 with Anthony's visit to Warsaw. At his lecture he suggested that the right measure ν to have $HD(\nu)=1$ on the boundary of the immediate basin of attraction A to a fixed (or periodic) attracting point p is harmonic measure. Very soon I proved this. The same time Lennart Carleson proved the inequality $HD(\omega)<1$ on Cantor sets. My preprint in Mittag-Leffler Institute was 1984.5, Carleson's 1984.4. See [P] F. Przytycki "Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map", Invent. Math. 80 (1985).

The idea was roughly to consider a Riemann mapping $R: D \to A$ so that $R(0)=p$ and to consider $g=R^{-1}\circ f\circ R$ and its holomorphic extension beyond ∂D.

Then $\omega_p=\bar{R}_*(l)$ where l is the length measure on ∂D and $\bar{R}$ the radial limit extension of R l-a.e. Entropies are the same (Beurling). So by (Ma) it suffices to prove

$$\chi_\omega(f)=\chi_l(g).$$

Clearly g' and $f'\circ R$ have the same zero $b_1, b_2, ..., b_{d-1}$. Let $B(z)=\prod_{j=1}^{d-1}\frac{z-b_j}{1-\bar{b}_j z}$. Then

$$\frac{\int\log|g'|dl}{\int\log|f'|dm}=\frac{\int\log|g'|/B dl}{\int\log|f'|\circ\bar{R}/B dl}$$

$$=\frac{\log|g'/B|(0)}{\log|f'|\circ R/|B|(0)}=1.$$

since f and g are conjugate by R. I used the fact that the integrated functions are harmonic on D.

Carleson also relied on the harmonicity.

Manning proved in [M2] that $\chi_m(f) = \log d$ by a different method, making calculation similar to Brolin (1965). Here it is as presented in [P]:

m is a weak* limit of

$$m_n = \frac{1}{d^n} \sum_{y \in f^{-n}(z_0)} \delta_y,$$

where δ_y is the Dirac measure at y and z_0 an arbitrary point in the basin of ∞ close to its boundary. We can assume f is a monic polynomial so we have $f' = d(z - c_1)...(z - c_{d-1})$ for c_j being the critical points.

$$\int \log |z - c_j| dm_n(z) = \frac{1}{d^n} \log \prod_{y \in f^{-n}(z_0)} |y - c|.$$

Notice that for any z we have

$$|f^n(z) - z_0| = \prod_{y \in f^{-n}(z_0)} |y - z|$$

since y are zeros of the left hand side polynomial. Putting $z = c$ we get

$$\int \log |z - c_j| dm_n(z) = \frac{1}{d^n} |f^n(c) - z_0|.$$

If no c escapes to ∞ this expression tends to 0.

In general we get the following [P]

$$HD(m) = \frac{\log d}{\log 2 + \sum_j G(c_j)}.$$

Let me finish with Nikolai Makarov's (1984-5) general result, that for any simply connected domain A in C different from C for harmonic measure (class) ω on ∂A, $HD(\omega)=1$. He, and in dynamics setting me with Anna Zdunik and Mariusz Urbański, explored later finer structures of harmonic measures, via deviations from 0 of the sums

$$\sum_{j=0}^{n}(\log|g'| - \log|f'|\circ \bar{R})(g^j(z)).$$

and Zdunik proved that for a polynomial if f is not Tchebyshev or z^d (up to conformal affine changes of coordinates) and $J(f)$ is connected then $HD(J(f) > 1$.

See also my "On the hyperbolic Hausdorff dimension of the boundary of the basin of attraction ..." Bull Pol. Ac. Sci. Math. 54.1 (2006).

Entropy Conjecture

This is the question (Mike Shub), under what condition on f or on a compact manifold M of dim m, for continuous $f: M \to M$

$$h_{\text{top}}(f) \geq \log spf^*, \qquad (EC)$$

where $f^*: H^*(M,R) \to H^*(M,R)$ is the induced mapping on cohomologies $H^* = \oplus_{i=0}^n H^i$. One can ask also on an f^*-invariant part of H^*.

In general (EC) is not true, consider the south-north pole degree 2 map of S^2. However it is not known for f being C^1.

Anthony proved it in [M3] in H^1. More precisely he proved (Asterisque 50, Warsaw Dynamical Systems Conference, 1977)

Theorem III.1 For all continuous $f : M \to M$

$h_{\mathrm{top}}(f) \geq$ log growth rate of f_* on π_1.

In his proof M is only a compact metric space uniformly locally arcwise connected, with short loops contractible.

Other attempts concerning all continuous maps:

1. Anthony proposed a way to prove (EC) on the subspace in the algebra H^* generated by H^1.

2. Katok Conjecture: EC holds for if the universal cover of M is R^n.

3. I proved a few years ago EC with Marzantowicz for all continuous maps of infranilmanifolds (around 1976 I proved this with Misiurewicz for tori); methods are related to those originated by Anthony.

I must admit that almost all my activities in dynamical systems where inspired by Anthony.

Thank you Anthony !