Statistical properties of one-dimensional maps

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Physical measures

X topological space;

$T : X \to X$ continuous map, viewed as a discrete time dynamical system;

μ reference measure on X.

Physical or natural measure: an invariant probability measure ν

$$T_* \nu = \nu,$$

such that for a set of initial condition $x_0 \in X$ with positive measure with respect to μ,

$$\frac{1}{n} \left(\delta_{x_0} + \delta_{T(x_0)} + \cdots + \delta_{T^{n-1}(x_0)} \right) \xrightarrow[n \to +\infty]{} \nu,$$

in the weak* topology.

Physical measures

$$\frac{1}{n}\left(\delta_{x_0} + \delta_{T(x_0)} + \cdots + \delta_{T^{n-1}(x_0)}\right) \xrightarrow[n\to+\infty]{} \nu$$

implies that for every continuous

$$\varphi : X \to \mathbb{R},$$

$$\frac{1}{n}\left(\varphi(x_0) + \varphi(T(x_0)) + \cdots + \varphi(T^{n-1}(x_0))\right) \xrightarrow[n\to+\infty]{} \int \varphi d\nu.$$

The time average converges to the space average.

One-dimensional maps

Two types of maps:

`Sufficiently regular interval maps.`
The reference measure will be the Lebesgue measure.

`Complex rational maps acting on the Julia set.`
Usually the reference measure will be a "conformal measure". When the Julia set is the whole Riemann sphere $\overline{\mathbb{C}}$, the reference measure will be the spherical one ("the Lebesgue measure").

Real maps

A non-injective smooth map

$$f : [0,1] \to [0,1]$$

is non-degenerate if:

- it has finitely many critical points:

$$\mathrm{Crit}(f) := \{c \in [0,1] : f'(c) = 0\} < +\infty;$$

- all of its critical points are non-flat: for each $c \in \mathrm{Crit}(f)$ a high derivative of f is non-zero at c.

For $c \in \mathrm{Crit}(f)$ the least integer $\ell_c > 1$ such that $f^{(\ell_c)}(c) \neq 0$ is called the order of c.

Real maps

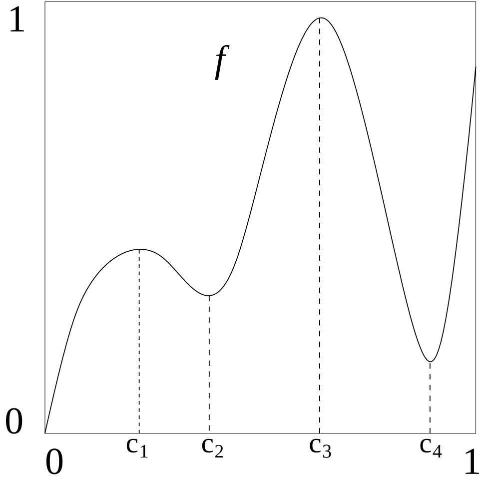


Figure: A non-degenerate smooth map and its critical points

Complex maps

A complex rational map f of degree $d \geq 2$,

$$f(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0} \in \mathbb{C}(z).$$

The JULIA set of f is:

$$J(f) := \overline{\{ \text{ repelling periodic points } \}}.$$

We will view f as a dynamical system acting on its JULIA set

$$f : J(f) \to J(f).$$

Complex maps

When $J(f) = \overline{\mathbb{C}}$, our reference measure will be the spherical measure ("the Lebesgue measure").

When $J(f) \neq \overline{\mathbb{C}}$, our reference measure will be a "conformal measure of minimal exponent".

Given $\alpha > 0$ a Borel measure μ on $J(f)$ is conformal of exponent α if for each Borel set $A \subset J(f)$ on which f is injective,

$$\mu(f(A)) = \int_A |f'|^\alpha d\mu.$$

In good situations there is a unique conformal measure of minimal exponent and this measure is very regular.
We will use it as a reference measure.

One-dimensional maps

Summarizing, the maps we will consider are:

- `Non-degenerate smooth interval maps`. We will assume them to be topologically exact and with all cycles hyperbolic repelling.
  - The reference measure will be the Lebesgue measure on the interval domain.
- `A complex rational map` f, viewed as dynamical system acting on its Julia set $J(f)$.
  - When $J(f) = \overline{\mathbb{C}}$, the reference measure will be spherical measure.
  - When $J(f) \neq \overline{\mathbb{C}}$, the reference measure will be a conformal measure of minimal exponent.

For interval maps there is a natural defintion of Julia set (Martes–de Melo–van Strien). When the Julia set is not an interval, conformal measures of minimal exponent are a natural choice for reference measures.

Absolutely continuous invariant probabilities

For a quadratic interval map an invariant probability measure that is absolutely continuous with respect to the Lebesgue measure is automatically a physical measure (Blokh–Lyubich).

For an interval map with several critical points, such a measure is a finite convex combination of physical measures (Ledrappier, Martens, Cai–Li).

A similar result holds for complex maps and for absolutely continuous invariant probabilities with positive Lyapunov exponent (Dobbs).

Physical measures of one-dimensional maps

For a one-dimensional map, absolutely continuous invariant measures are a natural source of physical measures. In contrast, in dimension two and higher, physical measures are rarely absolutely continuous.

> *From now on we will be interested on the existence and statistical properties of absolutely continuous invariant measures of real and complex one-dimensional maps.*

Correlations

f a real or complex one-dimensional map as before;
ν an invariant probability measure for f.

Definition

Given real functions φ, ψ and $n \geq 1$ we put

$$C_n(\varphi, \psi) := \left| \int \varphi \circ f^n \cdot \psi d\nu - \int \varphi d\nu \cdot \int \psi d\nu \right|.$$

Roughly speaking it measures the (lack of) independence of the random variables $\varphi \circ f^n$ and ψ on the probability space defined by ν.

Decay of correlations

f a real or complex one-dimensional map as before;

ν an invariant probability measure for f.

We will say ν is:

- exponentially mixing, if for every pair of HÖLDER continuous real functions φ, ψ the correlations $C_n(\varphi, \psi)$ at least exponentially with n;
- polynomially mixing of exponent $\gamma > 0$, if for every pair of HÖLDER continuous real functions φ, ψ there is a constant $C > 0$ such that for every $n \geq 1$,

$$C_n(\varphi, \psi) \leq C n^{-\gamma};$$

- super-polinomially mixing, if for every $\gamma > 0$ it is polynomially mixing of exponent γ.

Misiurewicz' condition

The first general sufficient condition for the existence of a.c.i.p.s for a non-degenerate smooth map was given by Misiurewicz (1981):

> **Misiurewicz' condition.** *There is $\delta > 0$ such that for every critical value v of f and each $n \geq 1$*
>
> $$dist(f^n(v), \mathrm{Crit}(f)) > \delta.$$

Collet–Eckmann condition

Collet–Ekcmann condition. *There are $\eta > 1$ and $C > 0$ such that for every critical value v of f and every $n \geq 1$*

$$|Df^n(v)| > C\eta^n.$$

In other words, the lower Lyaupunov exponent of each critical value v is positive:

$$\liminf_{n \to +\infty} \frac{1}{n} \ln |Df^n(v)| > 0.$$

Collet–Eckmann (1983) showed this condition implies the existence of an a.c.i.p. for interval maps with one critical point (Nowicki removed some hypotheses):

Collet–Eckmann condition $\Rightarrow$ existence of an a.c.i.p.

Collet–Eckmann condition

Keller–Nowicki (1992) and Young (1992) showed independently that for an interval map with one critical point:

Collet–Eckmann condition

$\Rightarrow$ existence of an exponentially mixing a.c.i.p.

Nowicki–Sands (1998) proved the reverse implication for maps with one critical point:

Collet–Eckmann condition

$\Leftrightarrow$ existence of an exponentially mixing a.c.i.p.

Collet–Eckmann condition

For interval maps with several critical points of the same order Bruin–Luzzatto–van Strien (2003) showed:

Collet–Eckmann condition

$\Rightarrow$ existence of an exponentially mixing a.c.i.p.

The assumption on the critical orders is not necessary (Przytycki–R-L, 2007).

The reverse implication holds replacing the Collet–Eckmann condition for a weaker condition, called the "Topological Collet–Eckmann condition" (Przytycki–R-L, 2007):

Topological Collet–Eckmann condition

$\Leftrightarrow$ existence of an exponentially mixing a.c.i.p.

Summability condition

A weaker sufficient condition for the existence of an a.c.i.p. was given by Nowicki–van Strien (1991) for interval maps with one critical point and by Bruin–van Strien (2001) for maps with several critical points:

Summability condition. *For every $c \in \mathrm{Crit}(f)$,*

$$\sum_{n=1}^{+\infty} \frac{1}{|Df^n(f(c))|^{1/\ell_{\max}}} < +\infty.$$

If furthermore for each critical value v of f the derivative $f^n(v)$ grows super-polynomially with n, Bruin–Luzzatto–van Strien (2003) showed that the a.c.i.p. is super-polynomially mixing.

Large derivatives condition

Theorem (Shen–R-L, arXiv 2010)

Let f be a non-degenerate and topologically exact smooth map. Assume that for each critical value v of f,

$$\lim_{n \to +\infty} |Df^n(v)| = +\infty.$$

Then f has a super-polynomially mixing a.c.i.p.

Under the same hypotheses the existence was shown by Bruin–Shen–van Strien (2003) for maps with one critical point and by Bruin–R-L–Shen–van Strien (2008) for maps with several critical points.

Large derivatives condition

We also have the following quantitative version.

Theorem (Shen–R-L, arXiv 2010)

For each $\ell > 1$ and $\gamma > 1$ there is $K(\ell, \gamma) > 0$ such that, if f is a non-degenerate and topologically exact smooth map such that the order of each critical point is at most ℓ and such that for all $c \in \mathrm{Crit}(f)$

$$\liminf_{n \to +\infty} |Df^n(f(c))| \geq K(\ell, \gamma),$$

then f has an mixing a.c.i.p., which is polynomially mixing of exponent γ.

Strong summability condition

Let f be a complex rational map.

Graczyk–Smirnov (2009) showed that if for a sufficiently small $\alpha \in (0,1)$ we have for every critical value $v \in J(f)$ of f:

$$\sum_{n=1}^{+\infty} \frac{1}{|Df^n(v)|^\alpha} < +\infty,$$

then there is a unique conformal measure of minimal exponent that this measure is non-atomic, ergodic and of dimension equal to $\mathrm{HD}(J(f))$.

They also showed that, if in addition for every critical value v,

$$\sum_{n=1}^{+\infty} \frac{n}{|Df^n(v)|^\alpha} < +\infty,$$

then there is an a.c.i.p.

Large derivatives condition

Theorem (Shen–R-L, arXiv 2010)

Suppose that f is a non-renormalizable polynomial without neutral cycles such that for every critical value $v \in J(f)$

$$\lim_{n \to +\infty} |Df^n(v)| = +\infty.$$

Then there is a unique conformal measure of minimal exponent that this measure is non-atomic, ergodic and of dimension equal to $HD(J(f))$.
Furthermore there is an super-polynomially mixing a.c.i.p.

Large derivatives condition

- This result applies to the quadratic Fibonacci map (this map fails the Graczyk–Smirnov strong summability condition).
- It also holds for complex rational maps without neutral periodic points such that for every critical value $v \in J(f)$,

$$\sum_{n=1}^{+\infty} \frac{1}{|Df^n(v)|} < +\infty.$$

- For each $p \in (0, \ell_{\max}/(\ell_{\max} - 1))$ the density belongs to the space L^p (for interval maps this follows from the paper by Bruin–R-L–Shen–van Strien).

Large derivatives condition

- For each $\varepsilon > 0$ the conformal measure of minimal exponent μ has the following regularity: for every sufficiently small $\delta > 0$ we have for every $x \in J(f)$

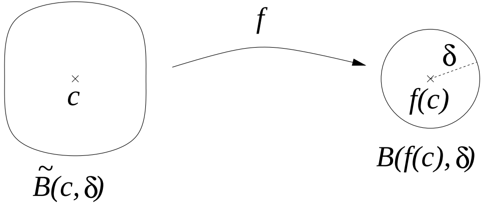
$$\delta^{\mathrm{HD}(J(f))+\varepsilon} \leq \mu(B(x,\delta)) \leq \delta^{\mathrm{HD}(J(f))-\varepsilon}.$$

  The lower bound was shown by Shen–Li (2008).
- Several notions of dimension coincide for $J(f)$.
- The Julia set is locally connected when connected, of Hausdorff dimension strictly less than 2 when different from $\overline{\mathbb{C}}$ and holomorphically removable when f is a polynomial.

Backward contraction

In the previous results, an intermediate condition holds called "Backward contraction".

To define it, for $c \in \text{Crit}(f)$ and $\delta > 0$ let $\widetilde{B}(c, \delta)$ be the connected component of $f^{-1}(B(f(c), \delta))$ containing c.



We define $\text{Crit}'(f) := \text{Crit}(f) \cap J(f)$ if f is complex and $\text{Crit}'(f) := \text{Crit}(f)$ if f is real.

Backward contraction

Definition

For $r > 1$ a real or complex one-dimensional map f is backward contracting with constant r if the following property holds: For every sufficiently small $\delta > 0$, $c \in \mathrm{Crit}'(f)$, every integer $m \geq 1$ and every connected component W of $f^{-m}(\widetilde{B}(c, r\delta))$ we have

$$\mathrm{dist}(W, f(\mathrm{Crit}(f))) < \delta \Rightarrow \mathrm{diam}(W) < \delta.$$

We will say that f is backward contracting if it is backward contracting for every $r > 1$.

Backward contraction

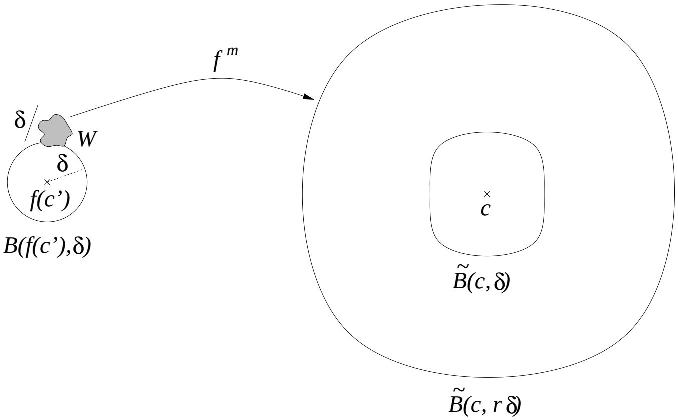


Figure: The backward contraction condition

Backward contraction

A map f is known to be backward contracting if:

- f is a non-degenerate and topologcially exact smooth interval map without neutral periodic points such that for every critical value v
$$\lim_{n \to +\infty} |Df^n(v)| = +\infty;$$
- f is a non-renormalizable complex polynomial without neutral periodic points such that for every critical value $v \in J(f)$
$$\lim_{n \to +\infty} |Df^n(v)| = +\infty;$$
- f is a complex rational map without neutral periodic points and such that for every critical value $v \in J(f)$
$$\sum_{n=1}^{+\infty} \frac{1}{|Df^n(v)|} < +\infty.$$

Large Derivatives

$\Rightarrow$ Backward Contraction

$\Rightarrow$ super-polynomially mixing a.c.i.p.

The Large Derivatives condition is only used to prove Backward Contraction.

For interval maps Shen–Li (2010) showed that Backward Contraction is equivalent to the Large Derivatives condition.

Plan

To prove

Backward Contraction $\Rightarrow$ super-polynomially mixing a.c.i.p.

we will use an inducing scheme (build a YOUNG tower and do a tail estimate), developped by PRZYTYCKI–R-L (2007, 2010).

This is done in several steps.

A. `''Super-polynomial Shrinking of Components''.`

B. `Construct ''nice sets'' at small scales`

C. `Bound the ''badness exponent''`

D. `Tail estimates`

Shrinking of components

Exponential Shrinking of Components:
There are constants $\theta \in (0,1)$, $C > 0$ and $\delta_0 > 0$ such that for every x, $\delta \in (0, \delta_0)$, every integer $m \geq 1$ and every connected component W of $f^{-m}(B(x,\delta))$ we have,

$$\mathrm{diam}(W) \leq C\theta^{-m}.$$

Exponential Shrinking of Components

$\Leftrightarrow$ Topological COLLET–ECKMANN condition

$\Leftrightarrow$ existence of an exponentially mixing a.c.i.p.

Shrinking of components

Polynomial Shrinking of Components of exponent $\beta > 0$:
There are $\delta_0 > 0$ and $C > 0$ such that for every x, $\delta \in (0, \delta_0)$, every integer $m \geq 1$ and every connected component W de $f^{-m}(B(x, \delta))$ we have,

$$\operatorname{diam}(W) \leq Cm^{-\beta}.$$

Super-polynomial Shrinking of Components:
Polynomial Shrinking of Components for each $\beta > 0$.

Shrinking of components

Definition

A map f is expanding away from critical points if for every neighborhood V of $\mathrm{Crit}'(f)$ the map f is uniformly expanding on the maximal invariant set of the complement of V.

Theorem

A backward contracting map that is expanding away from critical points has the Super-polynomial Shrinking of Components property.

Shrinking of components

Sketch of the proof of:

Backward Contraction + expansion away from critical points
$\Rightarrow$ Super-polynomial Shrinking of Components

$\delta > 0$

c a critical point of f in $J(f)$;

$m \geq 1$ an integer;

W a connected component of $f^{-m}(\widetilde{B}(c, \delta))$.

We would like to prove that W is (super-)polynomially small with m.

Using the expansion away from critical points we reduce to the case where W intersects $\widetilde{B}(\mathrm{Crit}(f), \delta)$.

Shrinking of components

$$\nu := \# \left\{ j \in \{0, \ldots, m-1\} \mid \text{ the component of } f^{-(n-j)}(W) \right.$$
$$\left. \text{containing } f^j(W) \text{ intersects } \widetilde{B}(\mathrm{Crit}'(f), \delta) \right\}.$$

There polynomial upper bound for $\mathrm{diam}(W)$ is obtained by combining the following estimates:

- by Backward Contraction $\mathrm{diam}(W)$ is exponentially small with ν;
- by the expansion away from critical points there are $C, \varepsilon > 0$ such that:

$$\mathrm{diam}(W) \leq C \exp\left(-\varepsilon \frac{1}{\ell_{\max}^{\nu}} m \right).$$